On the Efficiency of Multicast

Piet Van Mieghem, Gerard Hooghiemstra, and Remco van der Hofstad

***Abstract*—The average number of joint hops in a shortest-path multicast tree from a root to m arbitrary chosen group member nodes is studied. A general theory for all graphs, hence including the graph representation of the Internet, is presented which quantifies the multicast reduction in network links compared to m times unicast. For two special types of graphs, the random graph $G_p(N)$ and the k-ary tree, exact and asymptotic results are derived. Comparing these explicit results with previously published Internet measurements [13] indicates that the number of routers in the Internet that can be reached from a root grows exponentially in the number of hops with an effective degree of approximately 3.2.**

***Index Terms*—Efficiency, k-ary tree, multicast, random graph.**

I. Introduction

It is believed that multicast will grow substantially in importance in the near future. Multicast will enable direct marketing, pay TV, movie distribution, automatic update of software releases, and many other services, besides the already known applications such as video conferencing, teleclassing, and electronic games. Although a large number of protocols for multicast has been proposed, as recently reviewed by Ramalho [11] and Almeroth [2], besides the classical group multicast model, new types such as explicit multicast [18] and source-specific multicast [15] are being investigated. These new types are one-to-many and forward IP-packets along the shortest-path source tree.

In this article, we focus on the efficiency or gain of multicast in terms of network resource consumption compared to unicast. Specifically, we concentrate on a one-to-many communication, where a source distributes messages (packets) to m different, uniformly distributed destinations along the shortest path. In unicast, these messages are sent m times from the source to each destination. Hence, unicast uses on average $f_N(m) = mE[H_N]$ link traversals or hops, where $E[H_N]$ is the average number of hops of a message to a uniform location in the graph under consideration containing N nodes. One of the main properties of multicast is that it economizes on the number of link traversals. If we define for multicast $g_N(m)$ to be the average number of hops in the shortest-path tree rooted at a source to m randomly chosen distinct destinations, then, of course, $g_N(m) \leq f_N(m)$. The purpose here is to quantify the multicast efficiency $g_N(m)$. We present general results valid for *all* graphs and more explicit results valid for the random graph, which was proposed as a model for the hopcount in the Internet in [17] and for k-ary graphs [13]. Finally, using the same measurement data as in [13], our analysis indicates that the Internet is an exponentially growing graph (defined in Section VI) with an effective degree of approximately 3.2.

Inspired by a remarkable paper by Phillips, Shenker, and Tangmunarunkit [13], which was in turn triggered by the work of Chuang and Sirbu [5], the present article extends and complements their work. The extension lies in the fact that we present general results for $g_N(m)$ which show that the empirical power law $g_N(m) = E[H_N]m^{0.8}$, coined by Phillips *et al.* the *Chuang–Sirbu scaling law*, might be a reasonable approximation for small m, but cannot be valid for m large (meaning of the same order[1] as the number of Internet routers, i.e., $m = xN$, with $0 < x \leq 1$). This result is illustrated in Figs. 4 and 6. The complementarity refers to the need of considering their many multicast measurements on MBone and Internet, which offer a reality check for the modeling of $g_N(m)$ or for the approximation of Chuang and Sirbu [5]. In view of this reality check, we feel we ought to mention some modeling assumptions also made by Phillips *et al.* and Chuang and Sirbu. First, the multicast process is assumed to deliver packets along the shortest path from source to each of the m destinations. The assumption ignores shared-tree multicast forwarding such as core-based tree (CBT, see RFC2201). As most of the current Internet protocols forward packets based on the (reverse) shortest path, the assumption of shortest-path tree delivery is quite realistic. The second assumption is that the m multicast group member nodes are uniformly chosen out of the total number of nodes N. This assumption has been discussed by Phillips *et al.* They concluded that, if m and N are large, deviations from the uniformity assumption are negligibly small, as also follows from the close agreement with Internet measurements. Also, the recent measurement of Chalmers and Almeroth [4] seem to confirm the validity of the uniformity assumption.

The paper is organized as follows. Section II states and proves the general theorems. In Section III, the empirical Chuang–Sirbu law is discussed. Sections IV and V apply the general theory to random graphs of the class $G_p(N)$, where the existence of links are independent of each others with probability p, and to k-ary trees, respectively. Observations concerning the exponential growth of a graph and a practical method to deduce exponential growth from (measurements of) $g_N(m)$ is presented in Section VI. In Section VII, previously published measurements on Internet are interpreted based

Manuscript received June 29, 2000; revised January 12, 2001 and May 2, 2001; recommended by IEEE/ACM Transactions on Networking Editor E. Biersack.

The authors are with the Faculty of Information Technology and Systems (ITS), Delft University of Technology, 2628 BL Delft, The Netherlands (e-mail: P.VanMieghem@its.tudelft.nl).

Publisher Item Identifier S 1063-6692(01)10554-6.

[1]Here, $f \approx g$ means that f is well approximated by g without any criterion that specifies this approximation, whereas $f \sim g$ for $x \to x_0$ means $\lim_{x \to x_0} f(x)/g(x) = 1$.

on our results. The Appendix contains some mathematical derivations.

II. General Results for $g_N(m)$

Theorem 1: For any connected graph with N nodes

$$m \le g_N(m) \le \frac{Nm}{m+1}. \tag{1}$$

Proof: Clearly, we need at least one edge for each (different) user; therefore, $g_N(m) \ge m$ and the lower bound are attained in a star topology with the source at the center.

We will next show that an upper bound is obtained in a topology on a line. Observe that it is sufficient to consider trees, because multicast only uses shortest paths without cycles. If the tree has not a line topology, then at least one node has degree 3 or the root has degree 2. Take the node closest to the root with this property and cut and paste one of the branches at this node; we paste the branch to a node at the deepest level. Through this procedure, the multicast function $g_N(m)$ stays unaltered or increases. Continuing in this fashion until we reach a line topology gives the claim.

For the line topology, we place the source at the origin and the other nodes at the integers $1, 2, \ldots, N-1$. The edges of the graph are given by $(i, i+1)$, $i = 0, 1, \ldots, N-2$. Observe that $g_N(m) = E[M]$, where M is the maximum of a sample of order m, without replacement, from the integers $1, 2, \ldots, N-1$. Obviously

$$P(M \le k) = \frac{\binom{k}{m}}{\binom{N-1}{m}}, \qquad m \le k \le N-1.$$

Hence

$$\begin{aligned} g_N(m) &= \sum_{k=m}^{N-1} k \, \frac{\binom{k}{m} - \binom{k-1}{m}}{\binom{N-1}{m}} \\ &= \sum_{k=m}^{N-1} k \, \frac{\binom{k-1}{m-1}}{\binom{N-1}{m}} = m \sum_{k=m}^{N-1} \frac{\binom{k}{m}}{\binom{N-1}{m}} \\ &= \frac{mN}{m+1} \sum_{k=m}^{N-1} \frac{\binom{k}{m}}{\binom{N}{m+1}} = \frac{mN}{m+1} \end{aligned}$$

where we used that $\sum_{k=m}^{N-1} \binom{k}{m} / \binom{N}{m+1} = 1$, because it is a sum of probabilities over *all* possible disjoint outcomes. ■

Fig. 1 shows the allowable state space for $g_N(m)$.

Theorem 2: For any connected graph with N nodes, the map $m \mapsto g_N(m)$ is concave and the map $m \mapsto g_N(m)/f_N(m)$ is decreasing.

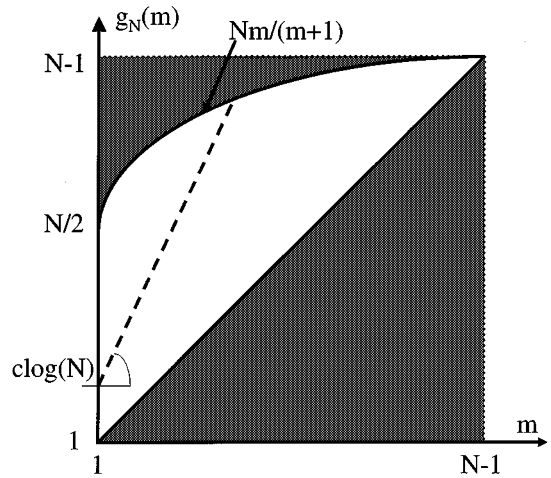

Fig. 1. Allowable region (in white) of $g_N(m)$. Note that for exponentially growing graphs defined in Section VI, $E[H_N] = c \log N$, implying that the allowable region for these graphs is smaller and bounded at the left (in dotted line) by the straight line $m(c \log N)$.

Proof: Define Y_m to be the random variable giving the additional number of hops necessary to reach the mth user when the first $m-1$ users are already connected. Then, we have that

$$E[Y_m] = g_N(m) - g_N(m-1).$$

Moreover, let Y'_m be the random number of additional hops necessary to reach the mth multicast group member, when we discard all extra hops of the $(m-1)$st group member. An example is illustrated in Fig. 2. The random variable Y'_m has the same distribution as Y_{m-1}, because both the $(m-1)$th and the mth group member are chosen uniformly from the remaining $N-m-1$ nodes.[2] Obviously, in general, $Y'_m \neq Y_{m-1}$, but, for each k, $\Pr[Y'_m = k] = \Pr[Y_{m-1} = k]$ and, hence

$$E[Y'_m] = E[Y_{m-1}]. \tag{2}$$

Furthermore, we have by construction that $Y_m \le Y'_m$ with probability 1, implying that

$$E[Y_m] \le E[Y'_m]. \tag{3}$$

Indeed, attaching the mth group member to the reduced tree takes at least as many hops as attaching that same group member to the nonreduced tree because the former is contained in the latter and the extra hops added by the $m-1$ group member can only help us. Combining (2) and (3) immediately gives that

$$\begin{aligned} g_N(m) - g_N(m-1) &= E[Y_m] \le E[Y'_m] \\ &= g_N(m-1) - g_N(m-2). \end{aligned} \tag{4}$$

This is equivalent to concavity of the map $m \mapsto g_N(m)$.

In order to show that $g_N(m)/f_N(m)$ is decreasing, it suffices to show that $m \mapsto g_N(m)/m$ is decreasing, since $f_N(m)$ is

[2] Two discrete random variables X and Y are equal in distribution if $\Pr[X = k] = \Pr[Y = k]$ for all k. For example, if X is the outcome of a throw with a red (fair) die and Y the outcome of a green (fair) die, then X and Y are equal in distribution, but, in general, $X \neq Y$.

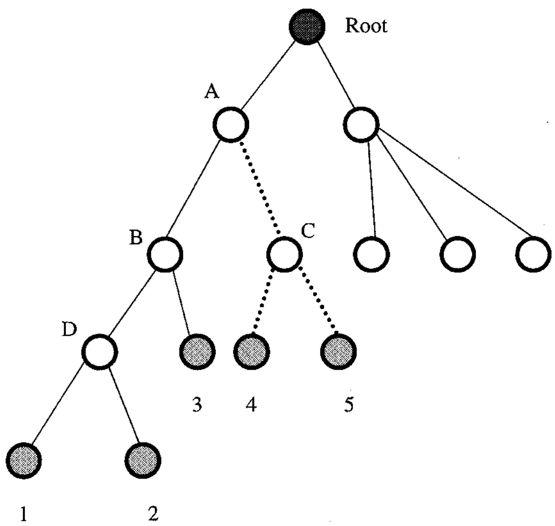

Fig. 2. Multicast session with $m = 5$ group members where $Y_5 = 1$ (namely, link C-5). To construct Y_5', the three dotted lines must be removed, and we observe that $Y_5' = 2$ (A-C-5), which is referred to as the reduced tree. In this example, $Y_5' = Y_4 = 2$ because A-C-4 and A-C-5 both consist of two hops. In general, they are equal *in distribution* because the role of group member 4 and 5 are identical in the reduced tree.

proportional to m. Defining $g_N(0) = 0$, we can write $g_N(m)$ as a telescoping sum.

$$g_N(m) = \sum_{k=1}^{m} \{g_N(k) - g_N(k-1)\}.$$

Then

$$\frac{g_N(m)}{m} = \frac{1}{m}\sum_{k=1}^{m} x_k$$

and

$$\frac{g_N(m-1)}{m-1} = \frac{1}{m-1}\sum_{k=1}^{m-1} x_k$$

where $x_k = g_N(k) - g_N(k-1)$, $k = 1, \ldots, m$. By (4), the sequence x_k is decreasing and, hence

$$\frac{g_N(m)}{m} = \frac{1}{m}\sum_{k=1}^{m} x_k \le \frac{1}{m-1}\sum_{k=1}^{m-1} x_k = \frac{g_N(m-1)}{m-1}.$$

This proves the claim that $m \mapsto g_N(m)/m$ is decreasing. ■

Next, we will give a representation for $g_N(m)$ valid for all graphs. We need the following definition. Let X_i be the number of joint hops that *all* i uniformly chosen and different group members have in common. Then we have the identity:

Theorem 3: For any connected graph with N nodes

$$g_N(m) = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} E[X_i]. \tag{5}$$

Proof: Let $A_1, A_2, \cdots, A_m$ be sets where A_i consists of all edges (hops) that constitute the shortest path from the source to multicast group member i. Denote by $|A_i|$ the number of elements in the set A_i. The multicast group members are chosen uniformly from the set of all nodes except for the root. Hence

$$E[X_1] = E[|A_i|], \qquad \text{for } 1 \le i \le N$$

and

$$E[X_2] = E[|A_i \cap A_j|], \qquad \text{for } 1 \le i < j \le N$$

etc. Now, $g_N(m) = E[|A_1 \cup A_2 \cup \cdots \cup A_m|]$. Since $Q(A) = E[|A|]/\binom{N}{2}$ is a probability measure on the set of all edges, we obtain from the inclusion–exclusion relation [8, Theorem, p. 99] applied to Q and multiplied with $\binom{N}{2}$ afterwards

$$\begin{aligned} g_N(m) &= E[|A_1 \cup A_2 \cup \cdots \cup A_m|] \\ &= \sum_{i=1}^{m} E[|A_i|] - \sum_{i<j} E[|A_i \cap A_j|] \\ &\quad + \cdots + (-1)^{m-1} E[|A_1 \cap A_2 \cap \cdots \cap A_m|] \\ &= mE[X_1] - \binom{m}{2} E[X_2] + \cdots \\ &\quad + (-1)^{m-1} \binom{m}{m} E[X_m]. \end{aligned}$$

This gives the statement of the theorem. ■

Note that

$$g_N(1) = f_N(1) = E[X_1] = E[H_N]$$

so that the decrease in average hops (or "gain") by using multicast over unicast is precisely

$$g_N(m) - f_N(m) = \sum_{i=2}^{m} \binom{m}{i} (-1)^{i-1} E[X_i].$$

However, computing $E[X_i]$ for general graphs is a highly nontrivial exercise.

Corollary 4: For any connected graph with N nodes

$$E[X_m] = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} g_N(i). \tag{6}$$

The corollary is a direct consequence of the inversion formula for the binomial [12, Ch. 2]. Alternatively, in view of the Gregory–Newton interpolation formula [10, Ch. 4, Sec. 2] for

$$g_N(m) = \sum_{i=1}^{\infty} \binom{m}{i} \Delta^i g_N(0)$$

we can write $E[X_i] = (-1)^{i-1}\Delta^i g_N(0)$ where Δ is the difference operator, $\Delta f(0) = f(1) - f(0)$.

III. The Chuang–Sirbu Law

Let us consider the Chuang–Sirbu scaling law, $g_N(m) \approx E[H_N]m^{0.8}$ in more detail.

Corollary 5: For any connected graph, the multicast efficiency $g_N(m)$ is bounded by

$$\frac{f_N(m)}{g_N(m)} \le E[H_N] \tag{7}$$

where $E[H_N]$ is the average number of hops in unicast.

Proof: We give two demonstrations.

1) From $g_N(N-1) = N-1$ (all nodes, source plus $N-1$ destinations, of the graph are spanned by a tree consisting of

$N-1$ links) and the monotonicity of $m \mapsto g_N(m)/f_N(m)$ (see Theorem 2), we obtain

$$\frac{g_N(m)}{f_N(m)} \geq \frac{g_N(N-1)}{f_N(N-1)} = \frac{N-1}{(N-1)E[H_N]} = \frac{1}{E[H_N]}.$$

2) Alternatively, Theorem 1 indicates that $g_N(m) \geq m$, which, with the identity $f_N(m) = mE[H_N]$, immediately leads to (7). ∎

Corollary 5 means that for any *connected* graph, including the graph describing the Internet, the ratio of unicast over multicast efficiency is bounded by the expected hopcount in unicast ($m = 1$). Corollary 5 implies that the empirical law of Chuang–Sirbu cannot hold true for all $m \leq N$. Indeed, if $g_N(m) = E[H_N]m^{0.8}$, we obtain from the inequality (7) and the identity $f_N(m) = mE[H_N]$, that $m^{0.2} \leq E[H_N]$. Write $m = xN$ for a fixed $0 < x < 1$ and x independent of N. Hence, we have shown that:

Corollary 6: For all graphs satisfying the condition that $E[H_N]/N^{0.2} \to 0$, for large N, the empirical Chuang–Sirbu law does not hold in the region $m = xN$ with $0 < x \leq 1$ and sufficiently large N.

The most realistic graph models for the Internet (see [13, Sec. 4.2]) assume that $E[H_N] \approx c \log N$, since this implies that the number of routers that can be reached from any starting destination grows exponentially with the number of hops. For these realistic graphs, Corollary 6 states that empirical Chuang–Sirbu law does not hold for all m. On the other hand, there are more regular graphs (such as a d-lattice, where $E[H_N] \simeq (d/3)N^{1/d}$) with $E[H_N] \sim N^{0.2+\epsilon}$ (and $\epsilon > 0$) for which the mathematical condition $m^{0.2} \leq E[H_N]$ is satisfied for all m and N. However, these classes of graphs, in contrast to random graphs, are not leading to realistic shortest-path trees, as shown in [17].

For the random graph $G_p(N)$, we know from [17], for large N

$$f_N(m) \sim m(\log N + \gamma - 1)$$

where γ is Euler's constant. Below, in Theoreom 7, we prove that for the random graph $G_p(N)$ and for large N and m

$$g_N(m) \sim \frac{mN}{N-m} \log\left(\frac{N}{m}\right) - \frac{1}{2}. \tag{8}$$

The above scaling explains the empirical Chuang–Sirbu law for $G_p(N)$: for m small with respect to N, the graphs of $(\log N + \gamma - 1)m^{0.8}$ and $(mN/N-m)\log(N/m) - (1/2)$ look very alike in a log–log plot as illustrated in Fig. 3.

For small to moderate values of m, g_N (as observed for Internet-like topologies in [13]) is very close to a straight line in a log–log plot. This "power law behavior" implies that $\log g_N(m) \approx \log E(H_N) + \beta(N) \log m$, which is a first-order Taylor expansion of $\log g_N(m)$ in $\log m$. It further suggests to compute[3] as effective power exponent $\beta(N)$

$$\beta(N) = \left.\frac{d \log g_N(m)}{d \log m}\right|_{m=1}. \tag{9}$$

[3]Although (5) only has meaning for integer m, analytic continuation to a complex variable is possible and, hence, differentiation can be defined.

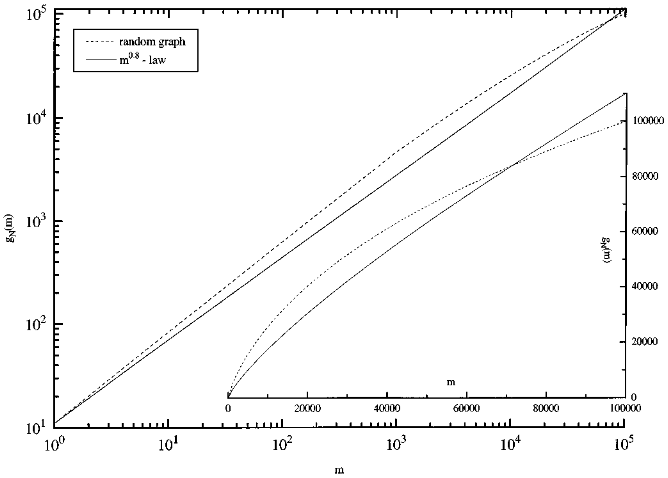

Fig. 3. The Chuang–Sirbu power law versus the exact results for the random graph with $N = 10^5$ on a log–log scale. The insert shows the same data on a linear scale.

Only for a straight line, the differential operator can be replaced by the difference operator such that $\beta(N) \equiv \beta^*(N)$, where

$$\beta^*(N) = \frac{\log \dfrac{g_N(2)}{E[H_N]}}{\log 2}. \tag{10}$$

In general, for small m, the effective power exponent (9) is not a constant 0.8 as in the Chuang–Sirbu law, but is dependent on N. Finally, since $g_N(m)$ is concave by Theorem 2, $\beta(N)$ is the maximum possible value for an effective exponent. A direct consequence of Theorem 1 is that the effective power exponent $\beta(N) \in [1/2, 1]$. From recent Internet measurements, Chalmers and Almeroth [4] found that $0.66 \leq \beta(N) \leq 0.7$.

In summary, many properties in nature seem linear on an insensitive log–log scale (see, e.g., [7]). However, deriving from these plots simple and attractive power laws for complicated matter seems a little oversimplified. Many recent articles devote attention to power law behavior but most of them [9], [4] seem prudent: just recall the immense interest (or hype?) a few years ago in the long-range and self-similar nature of Internet traffic and the relation to the "simple" power law with only the Hurst parameter (comparable to $\beta(N)$ here) in the exponent.

IV. The Random Graph

There exists an astonishingly large amount of literature on properties of random graphs. We refer to Bollobas [3] and to [17] for additional references. The class of random graphs denoted by $G_p(N)$ consists of all graphs with N nodes in which the edges (or links) are chosen independently and with probability p. Although random graphs are *not* modeling realistic network topologies well, computations in $G_p(N)$ of the shortest path from a source to an arbitrary destination result in a remarkably good model of the hopcount (i.e., the number of links) from that source to a destination, as demonstrated theoretically in [16] and verified with Internet measurements in [17]. The explanation for the quality of a random graph with exponentially distributed link weights can be understood when reasoning from the source node on. The view of the source node is a shortest-path *tree.* In [16], [17], we find that the shortest-path problem in $G_p(N)$ with exponentially distributed link weights can be reformulated into a Markov discovery process with an associated uniform recursive tree [14]. The uniform recursive tree thus seems a quite natural shortest-path tree as seen by the source node. For this uniform tree, the corresponding multicast gain is computed in this section.

We further found in [17] that 1) other topologies and 2) other link weight distributions do not fit the Internet data so well, which led us to suggest that the random graph model is a reasonable model for shortest-path behavior. Moreover (see [6]), the resulting hopcount distribution (13) possesses the remarkable property of almost sure behavior, which implies a high degree of robustness. Finally, from a modeling perspective, even though the model describes reality less accurately, the main benefit of the random graphs lies in the fact that it provides relatively simple analytic results and first-order estimates of difficult phenomena that are unlikely to be obtained from more sophisticated models.

In this section, we confine to the random graphs of the class $G_p(N)$ with independent identically and exponentially distributed link weights w with mean $E[w] = 1$ and where $\Pr[w \le x] = 1 - e^{-x}$, $x > 0$. Previously [16], [17], we have shown that the hopcount of the shortest path for *almost all* connected graphs of $G_p(N)$ and independent of the link density p can be computed asymptotically. In summary, we find that $E[H_N] = (N/N-1)(\psi(N)+\gamma) - 1$, where $\psi(x)$ is the digamma function [1, Sec. 6.3] or, for large N

$$E[H_N] \sim \log N + \gamma - 1 \tag{11}$$

$$\mathrm{Var}[H_N] \sim \log N + \gamma - \pi^2/6 \tag{12}$$

$$\Pr[H_N = k] = \frac{(1+o(1))}{N} \sum_{m=0}^{k} c_{m+1} \frac{\ln^{k-m} N}{(k-m)!} \tag{13}$$

where c_m are the Taylor coefficients of $1/\Gamma(z)$ listed in [1, 6.1.34].

Theorem 7: For the class of random graphs $G_p(N)$ with independent, identically and exponentially distributed link weights

$$g_N(m) = mN\left(\frac{\psi(N) - \psi(m)}{N-m}\right) - 1 \tag{14}$$

where $\psi(x)$ is the digamma function.

Proof of Theorem 7: See Appendix A.

We observe that $g_N(N-1) = N-1$ and that, since $\psi(1) = -\gamma$, $g_N(1) = N/(N-1)(\psi(N)+\gamma) - 1 = E[H_N]$.

Using the asymptotic properties of the digamma function ψ, we obtain (8) as an excellent approximation for large N (and all m) or, in normalized form with $m = xN$ and $0 < x < 1$

$$\frac{g_N(xN) + 0.5}{N} \sim \frac{x \log x}{x - 1}. \tag{15}$$

The normalized Chuang–Sirbu law is $g_N(xN)/N = (E[H_N]/N^{0.2})x^{0.8}$. It is interesting to note that the Chuang–Sirbu law is "best" if $E[H_N]/N^{0.2} = 1$, since then both endpoints $x = 0$ and $x = 1$ coincide with (15). This optimum is achieved when $N \approx 250\,000$, which is close to the current estimated number of routers of the Internet (as deduced from measurements of the hopcount in [17]). This observation may explain the fairly good correspondence (on a less sensitive log–log scale) with Internet measurements. At the same time, it shows that for a growing Internet, the fit of the Chuang–Sirbu law will deteriorate.

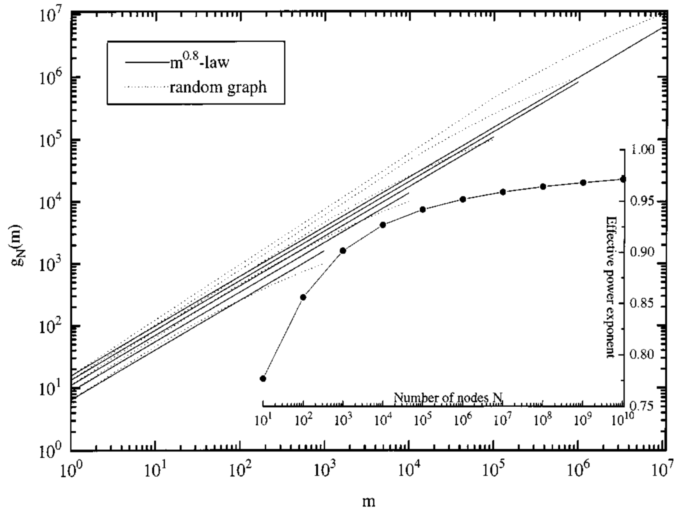

Fig. 4. Multicast efficiency for $N = 10^j$ with $j = 3, 4, \ldots, 7$. The endpoint of each curve $g_N(N-1) = N-1$ determines N. The insert shows the effective power exponent versus N.

Fig. 4 compares $(\log N + \gamma - 1)m^{0.8}$ and $(mN/N - m)\log(N/m) - 1/2$ for various values of N on a log–log scale. For $N \ge 10^6$, the Chuang–Sirbu law underestimates $g_N(m)$ for all m. The effective power exponent $\beta(N)$ as defined in (9) for the random graph is

$$\beta(N) = \frac{N\left(\psi(N) + \gamma - \frac{\pi^2}{6} + \frac{\pi^2}{6N}\right)}{(N-1)\left(\psi(N) + (\gamma - 1) + \frac{1}{N}\right)}$$

while, according to (9)

$$\beta^*(N) = 1 + \log_2\left[\frac{(N-1)(\psi(N) + \gamma - 3/2 + 1/N)}{(N-2)(\psi(N) + \gamma - 1 + 1/N)}\right].$$

The difference $\beta(N) - \beta^*(N)$ monotonously decreases and is largest, 0.048 at $N = 3$ while 0.008 31 at $N = 10^5$ and 0.0037 at $N = 10^{10}$. This effective power exponent $\beta(N)$ is drawn in the insert of Fig. 4, which shows that $\beta(N)$ is increasing and not a constant close to 0.8. More interestingly, for large N, we find with (11) and (12) that $\beta(N) \sim var[H_N]/E[H_N]$ and that $\lim_{N\to\infty} \beta(N) = 1$. In [17], the ratio $var[H_N]/E[H_N]$ pops up naturally as the extreme value index of the distribution of the link weights in a topology. Since measurements of the hopcount in Internet indicate that $var[H_N]/E[H_N] \approx 1$, this index strongly favors the model of the hopcount based on shortest paths in $G_p(N)$, although random graphs do *not* model the Internet topology well.

Thus, if the number of nodes in the Internet is still growing and well modeled by the random graph, we suggest, *only for*

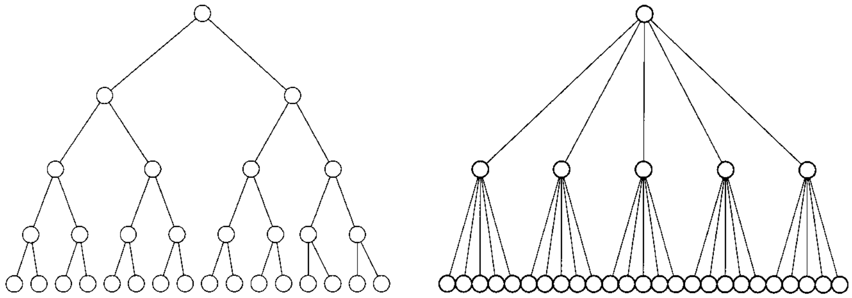

Fig. 5. The left-hand side tree ($k = 2$) has $N = 31$ and $D = 4$, while the right-hand side ($k = 5$) has $N = 31$ and $D = 2$.

small to moderate values of m, to consider as a power law approximation

$$g_N(m) \approx E[H_N] m^{var[H_N]/E[H_N]}$$

instead of the Chuang–Sirbu law.

V. k-ARY TREES

Let us consider, as in [13], the k-ary tree of depth[4] D with the source at the root of the tree and m receivers at randomly chosen nodes (see Fig. 5). In a k-ary tree, the total number of nodes satisfies

$$N = 1 + k + k^2 + \cdots + k^D = \frac{k^{D+1} - 1}{k - 1} \tag{16}$$

so that $N \sim k^D$.

Theorem 8: For the k-ary tree

$$g_{N,k}(m) = N - 1 - \sum_{j=0}^{D-1} k^{D-j} \frac{\binom{N-1-\frac{k^{j+1}-1}{k-1}}{m}}{\binom{N-1}{m}}. \tag{17}$$

Proof of Theorem 8: See Appendix B.

Unfortunately, the j summation seems difficult to express in closed form. Observe that $g_N(N-1) = N-1$, because all binomials vanish. The sum extends over all levels $j \le D-1$, for which the remaining number of nodes in the lower levels l (i.e., $D \ge l > j$) is larger than m nodes. In some sense, we may regard (17) as an (exact) expansion around $m = N-1$. Explicitly

$$\begin{aligned} g_{N,k}(m) =& N - 1 - k^D \left(1 - \frac{m}{N-1}\right) \\ & - k^{D-1} \prod_{q=0}^{k} \left(1 - \frac{m}{N-1-q}\right) \\ & - \sum_{j=2}^{D-1} k^{D-j} \prod_{q=0}^{(k^{j+1}-1)/(k-1)-1} \left(1 - \frac{m}{N-1-q}\right) \end{aligned} \tag{18}$$

[4]The depth D is equal to the number of hops from the root to a node at the leaves.

which shows that $g_{N,k}(m)$ is a polynomial in m of degree $\le (N-1)/k$. Moreover, the terms in the j-sum rapidly decrease; their ratio equals

$$\frac{\prod_{q=(k^j-1)/(k-1)}^{(k^{j+1}-1)/(k-1)-1} \left(1 - \frac{m}{N-1-q}\right)}{k} < \frac{1}{k} \left(1 - \frac{m}{N - 1 - \frac{k^j - 1}{k-1}}\right)^{k^j} \ll 1.$$

Fig. 6 indicates that (17), although derived subject to (16), also seems valid when

$$D = \left\lfloor \frac{\log[1 + N(k-1)]}{\log k} - 1 \right\rfloor$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. This suggests that the deepest level D need not be filled completely to count k^D nodes and that (17) may extend to "incomplete" k-ary trees. As further observed from Fig. 6, $g_{N,k}(m)$ is monotonously decreasing in k.

Conjecture 9: The map $k \mapsto g_{N,k}(m)$ is decreasing in $k \in [1, N-1]$.

After rewriting (18) with $k^D = N - ((N-1)/k)$, we find for $N - 1 - k \le m \le N - 1$

$$g_{N,k}(m) = \frac{mN}{N-1} + \frac{N-1}{k}\left(1 - \frac{m}{N-1}\right) - 1$$

which is clearly monotonously decreasing in k and independent of k for $m = N-1$. From the explicit expression (19) for $g_{N,k}(1)$ and (21) for $g_{N,k}(2)$, the corollary is verified for $m = 1$ and $m = 2$. Although verified numerically, a rigorous proof for all m and N is difficult. Intuitively, Conjecture 9 can be understood from Fig. 5. Both the $k = 2$ and $k = 5$ tree have an equal number of nodes. We observe that the deeper D (or the smaller k), the more overlap is possible, hence, the larger $g_{N,k}(m)$.

Theorem 1 can also be deduced from (17). The lower bound is attained in a star topology where $k = N-1$, $D = 1$, and

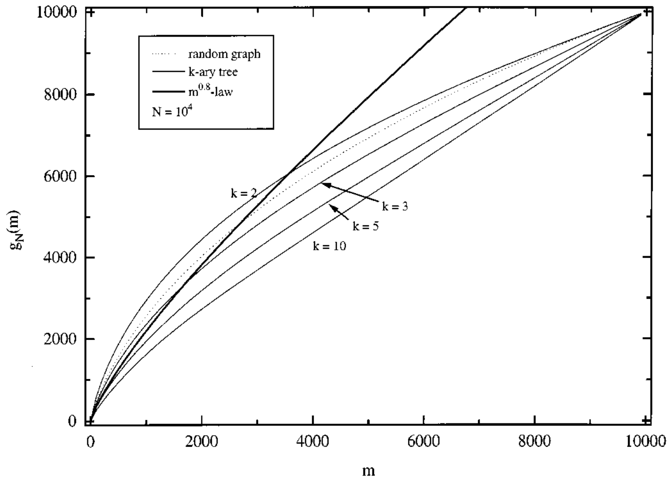

Fig. 6. Multicast function $g_N(m)$ computed for the k-ary tree with four values of k, the random graph (with "effective" $k_{rg} = e = 2.718\ldots$), and the Chuang–Sirbu power law for $N = 10^4$ on a linear scale where the prefactor $E[H_N]$ is given by (11).

$E[H_N] = 1$. The upper bound is attained in a line topology where $k = 1$, $D = N-1$, and $E[H_N] = N/2$. Further, for real values of $k \in [1, N-1]$, the set of curves specified by (17) covers the total allowable state space of $g_{N,k}(m)$, as shown in Fig. 1. This suggests to consider (17) for estimating k in real topologies (see Section VI).

The asymptotic form for $N \to \infty$ is deduced from (17) as follows. If N is large

$$\binom{N-1}{m} \sim \frac{(N-1)^m}{m!}$$

and similarly

$$\binom{N-1-\frac{k^{j+1}-1}{k-1}}{m} \sim \frac{\left(N-1-\frac{k^{j+1}-1}{k-1}\right)^m}{m!}.$$

Then, using (16)

$$\begin{aligned} g_{N,k}(m) &\sim N-1-\sum_{j=0}^{D-1} k^{D-j}\left(1-\frac{k^{j+1}-1}{k^{D+1}-1}\right)^m \\ &= \sum_{j=1}^{D} k^j\left[1-\left(1-\frac{k^{D-j+1}-1}{k^{D+1}-1}\right)^m\right]. \end{aligned}$$

This is somewhat deviating from [13, eq. (21)], that, written in our notation, is

$$g_{N,k}(m) \sim \sum_{j=1}^{D} k^j\left[1-\left(1-\frac{k^{D-j+1}-1}{k^{D+1}-k}\right)^m\right].$$

The difference lies in the term $k^{D+1}-k$ in [13, eq. (21)] instead of $k^{D+1}-1$. As the complexity of this asymptotic result is not much lower than that of the exact (17), no additional insight is gained.

Since $g_N(1) = E[H_N]$, the average hopcount in a k-ary tree follows from (17) as

$$\begin{aligned} E[H_N] &= N-1-\sum_{j=0}^{D-1} k^{D-j}\frac{N-1-\frac{k^{j+1}-1}{k-1}}{N-1} \\ &= \frac{1}{N-1}\sum_{j=0}^{D-1} k^{D-j}\,\frac{k^{j+1}-1}{k-1} \\ &= \frac{ND}{N-1}+\frac{D}{(N-1)(k-1)}-\frac{1}{k-1}. \end{aligned} \tag{19}$$

For large N, we find with

$$\begin{aligned} D &= \left\lfloor \frac{\log[1+N(k-1)]}{\log k} - 1 \right\rfloor \\ &\sim \log_k N + \log_k(1-1/k) + O(1/N) \end{aligned}$$

that

$$E[H_N] = \log_k N + \log_k(1-1/k) - \frac{1}{k-1} + O\left(\frac{\log_k N}{N}\right). \tag{20}$$

Since

$$\begin{aligned} g_{N,k}(2) &= N-1-\sum_{j=0}^{D-1}\frac{\left(N-1-\frac{k^{j+1}-1}{k-1}\right)}{k^{j-D}(N-1)} \\ &\quad\times\frac{\left(N-2-\frac{k^{j+1}-1}{k-1}\right)}{N-2} \\ &= \frac{(2N-3)DN}{(N-1)(N-2)}+\frac{(2N-3)D}{(N-1)(N-2)(k-1)} \\ &\quad-\frac{(2N-3)}{(N-2)(k-1)}-\frac{N(N-1-2D)}{(N-1)(N-2)(k-1)}+ \\ &\quad-\frac{(N-1-2D)}{(N-1)(N-2)(k-1)^2}-\frac{1}{(N-2)(k-1)^2} \end{aligned} \tag{21}$$

or, for large N

$$g_{N,k}(2) \sim 2D - \frac{3}{k-1} + O\left(\frac{\log_k N}{N}\right)$$

the effective power exponent $\beta^*(N)$, as defined in (10), equals for the k-ary tree and large N

$$\begin{aligned} &\beta^*(N) \\ &\quad\sim \frac{\log\left[\dfrac{2D-\frac{3}{k-1}}{D-\frac{1}{k-1}}\right]}{\log 2} \\ &\quad= 1+\log_2\left[1-\frac{1}{2(k-1)\left(\log_k N+\log_k\left(1-\frac{1}{k}\right)-\frac{1}{k-1}\right)}\right] \\ &\quad= 1+\log_2\left[1-\frac{1}{2(k-1)E[H_N]}\right] \\ &\quad\sim 1-\frac{1}{(\log 4)(k-1)E[H_N]}. \end{aligned} \tag{22}$$

Comparing (20) with the average hopcount in the random graph (11) shows equality to first order if $k_{rg} = e$. Moreover, both the second-order terms $\gamma - 1 = -0.42$ and $\log(1 - 1/e) - 1/(e-1) = -1.04$ are $O(1)$ and independent of N.

VI. Effective Nodal Degree and Exponential Growth of a Graph

Let us denote by Q_j the number of nodes at precisely j hops from a source in a shortest-path tree. If we take $Q_0 = 1$ to include the source, then $\sum_{j=0}^{N-1} Q_j = N$. The usual definition of exponential growth of a graph states that a tree grows exponentially in the number of nodes with degree k if $\lim_{j\to\infty}(Q_j)^{1/j} = k$ or, equivalently, $Q_j \sim k^j$, for large j. The fundamental problem with this definition is that it only holds for infinite graphs $N = \infty$. For real (finite) graphs, there must exist a $j = l$ for which the sequence $Q_l, Q_{l+1}, \ldots, Q_{N-1}$ ceases to grow because of $\sum_{j=0}^{N-1} Q_j = N < \infty$. This boundary effect complicates the definition of exponential growth in finite graphs. The second complication is that even in the finite set $Q_0, Q_1, \ldots, Q_l$ not necessarily all Q_j with $0 \le j \le l$ need to obey $Q_j \sim k^j$, but "enough" should. Without the limit concept, we cannot specify the precise conditions of exponential growth in a finite shortest-path tree. If we assume in finite graphs that $Q_j \sim k^j$ for $j \le l$, then $\sum_{j=0}^{l} Q_j = \alpha N$ with $0 < \alpha < 1$. Indeed, for $k > 1$, the highest hopcount level l possesses by far the most nodes since $(k^{l+1} - 1)/(k - 1) \approx k^l$ which cannot be larger than a fraction αN of the total number of nodes. Thus, $k^l \approx \alpha N$ from which $l \approx \log_k N$. The relation for the average hopcount[5] (20) indicates that $l \approx E[H_N]$. The argument also shows that only very few levels around $l \approx E[H_N]$ play a role in the determination of exponential growth. These considerations invite us to propose a definition which takes the size of the graph more naturally into account.

By extending k to real numbers in (20), the parameter k can be interpreted as an effective nodal degree

$$k = \lim_{N\to\infty} \exp\left[\frac{\log N}{E[H_N]}\right]. \tag{23}$$

In a k-ary tree (where k is an integer), the parameter k precisely equals the outdegree and, apart from the source node, $k+1$ is the nodal degree. Hence, (23) reflects the average number of "new" nodes (the outdegree) that can be reached from a node in one hop. If, for $\epsilon > 0$ and large N, the average hopcount $E[H_N] = O(\log^{1+\epsilon} N)$, the graph is not exponentially growing ($k = 1$), whereas if $E[H_N] = O(\log^{1-\epsilon} N)$, the graph is superexponentially growing ($k = \infty$). An example of a nonexponentially growing graph is the regular d-lattice, while a tree that expands at each level with growing k, thus the root has k_1 children, these each have $k_2 > k_1$ children and so on, is an example of a superexponentially growing graph. A value of $k < 1$ would indicate that the graph is not connected. The k-ary tree can be regarded as the most regular, exponentially growing tree, whereas the uniform tree (the shortest-path tree in a random graph $G_p(N)$ with exponentially distributed link weights) is an example of a highly nonregular, exponentially growing tree with $k = e$. One may expect that realistic shortest-path trees lie in between these extremes.

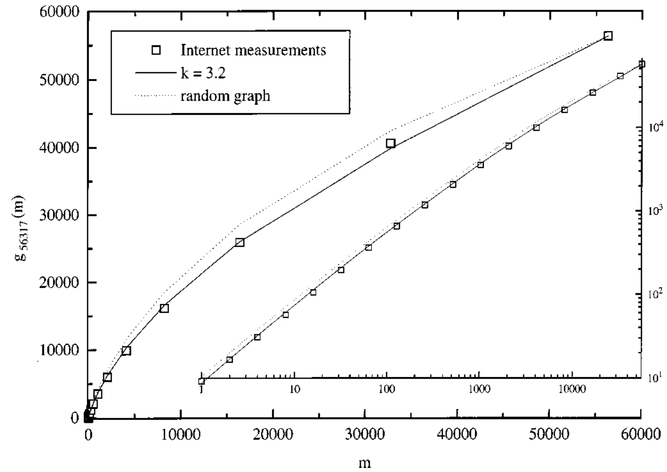

Fig. 7. Internet measurements [13, Fig. 1b] where $N = 56\,317$ and $g_{56\,317}(m)$, computed for the k-ary tree with $k = 3.2$ and for the random graph on a linear scale. The insert shows the same data on a log–log plot.

As a practical method, we propose that *if for a graph* $g_N(m) \le g_{N,\underline{k}}(m)$ [*specified in* (17)] *for all values of* m *and some* $\underline{k} > 1$, *then the graph grows exponentially with effective degree at least* $\underline{k}$. Since the whole state space of $g_N(m)$ can be covered by the family $g_{N,k}(m)$ (for real $k \in [1, N-1]$), all possible outcomes that any graph may produce in the g-domain, can be bounded from above by a $g_{N,\underline{k}}$-curve where $\underline{k}$ is the smallest value for which $g_N(m) \le g_{N,\underline{k}}(m)$. Conjecture 9 states that the map $k \mapsto g_{N,k}(m)$ is increasing which indicates that such a $\underline{k}$ exists. The definition (23) further suggests that from the measurements in g-domain on the shortest-path tree of the source, the growth of that tree is at least $\underline{k}$.

VII. Measurements of $g_N(m)$

Precisely the same data as in [13, Fig. 1b] has been fitted[6] with (17) yielding for the MBone $k_{\text{MBone}} = 4.2$ and for the Internet $k_{\text{Internet}} = 3.2$. For the Internet, the difference between measurement and fit is hardly visible on a linear or on a logarithmic plot as illustrated in Fig. 7. Using (23), we can compute the value of k for any graph. From the measurement data $g_{56\,317}(1) = 9.217$ and $g_{4179}(1) = 9.856$ for Internet and MBone, respectively, and $g_N(1) = E[H_N]$, (23) gives the values $k^u_{\text{Internet}} = 3.27$, $k^u_{\text{MBone}} = 2.32$, where the superscript u refers to the unicast hopcount ($m = 1$). The exact formula (19) leads to slightly smaller values $k^{u*}_{\text{Internet}} = 2.95$, $k^{u*}_{\text{MBone}} = 2.03$. Although the effective nodal degree is related at first glance to the average degree, defined by $d_a = 2E/N$ and where E is the number of links in the graph, these k-values have little in common with the reported [13] average degree of the graph, $d_{a;Internet} = 2.7$ and $d_{a;MBone} = 4.1$. In fact, for any graph with E links and N nodes containing no cycles, we have that $E = N - 1$ and, hence, $d_a = 2 - (2/N)$. Thus, for large N, the deviation of the average degree from 2 can be interpreted as a measure of the number of cycles in the graph.

[5] In general, for any graph holds that $E[H_N] = (1/N)\sum_{j=1}^{N-1} jQ_j$.

[6] The value of k is the minimizer of $\sum_{m=1}^{N-1}(g_N^*(m) - g_{N,k}(m))^2$ where $g_N^*(m)$ are the Internet measurements and with $g_{N,k}(m)$ given by (17).

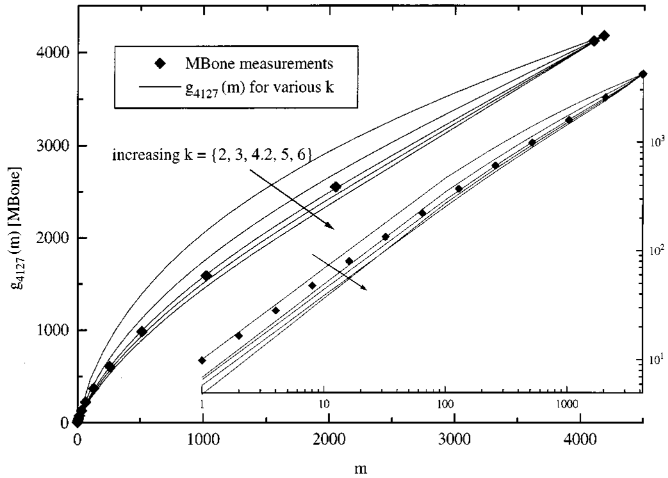

Fig. 8. MBone measurement from [13] where $N = 4179$ and $g_{4179;k}(m)$ computed for the k-ary tree for various values of k. The upper curve corresponds to $k = 2$; the curves decrease monotonuously for increasing k. The best fit corresponds to $k = 4.2$. The insert shows the same data on a log–log scale.

However, it is easy to produce graphs that are not exponentially increasing, but that have an arbitrarily large average degree as N grows large. On the other hand, a distribution of the outdegree at each level of the tree relates better to k. Chalmers and Almeroth [4, Figs. 8–11] present measurements of average degree per level. Their values agree in magnitude with the k-values we found from the data of [13].

Based on the measurement data, the two differently computed k-values agree for the Internet, which seems to indicate that the Internet is exponentially growing with effective degree approximately 3.2. Also the quality of the fit on both linear and log–log scale in Fig. 7 over the whole m-range is persuasive. This is an important consequence, since it is difficult to judge whether a graph is exponentially increasing without knowing its precise topological structure. The above considerations clearly indicate that this fact cannot be decided upon the information of the average degree solely. In contrast to the Internet data, the Mbone data is not so well fitted, as illustrated in Fig. 8. The insert on a log–log scale shows that the MBone shifts gradually when m increases toward higher k-curves. The discrepancy by almost a factor 2 between k^u_{MBone} derived from the unicast hopcount at $m = 1$ and k_{MBone} fitted from the entire m-range of the multicast gain $g_{4179}(m)$ may be explained by the abundant use of IP tunnels in the MBone [2]. Tunnels may be viewed as an overlay tree: they shortcut branches in the shortest-path tree and decrease the possible overlap in paths which diminishes $g_{4179}(m)$. The more group members are subscribed, the more tunnels seem to effect the structure of the shortest-path tree. Chalmers and Almeroth [4] also report differences in the unicast hopcount and multicast hopcount and assign the origin to tunnels in the multicast architecture, but also hint to the possible influence of policy routing (as a deviating factor from shortest-path routing) in interdomain multicast routing. Although the Mbone is a connected subgraph of the Internet, exponential growth in the Internet does not necessarily imply exponential growth in the MBone. However, the practical method in previous section is applicable with $\underline{k} = 2$ which suggests exponential growth in the MBone, although we cannot determine the precise growth rate k_{MBone} as for the Internet.

To understand the close relation of the Internet to tree-like graphs, note that any shortest path started from a single source will give rise to a graph containing no cycles, i.e., a tree. This is because we will always take the shortest path along any cycle, and disregard the links that are not used. Hence, even though the k-ary tree is clearly not an accurate model for the Internet topology as a graph, it might be a good model for that portion of the Internet used by shortest-path routing from a single source.

VIII. Conclusion

In this paper, general results are presented on the multicast efficiency $g_N(m)$, which are valid for *all* connected graphs and where m is the number of multicast group members. Using these general theorems, we show that the so-called Chuang–Sirbu power law, $g_N(m) = E[H_N]m^{0.8}$, cannot generally hold if $E[H_N]$ grows logarithmically in N and the number of multicast group members m is of the same order as the number of nodes N in the graph. Moreover, we define the effective power exponent $\beta(N)$ and show that, in general, $\beta(N)$ is not a constant equal to 0.8.

We have also derived exact and asymptotic expressions for g_N in the case of random graphs of the class $G_p(N)$ and for k-ary trees. These expressions generalize previous results obtained in [13]. They confirm that *only* for small and moderate values of the number m of multicast group members the Chuang–Sirbu law is a reasonable approximation for g_N. Based on computations for the random graph, we find that the Chuang–Sirbu law is 1) best for $N \approx 10^5$, but 2) degrades for $N \geq 10^6$. In addition, the analysis for random graphs suggests, *only for small to moderate values of* m, to consider $g_N(m) \approx E[H_N]m^{var[H_N]/E[H_N]}$ instead of the Chuang–Sirbu law because the effective power exponent $\beta(N) \approx var[H_N]/E[H_N]$ is not constant, but slowly increases from 0.71 at $N = 2$ toward 1 as $N \to \infty$. Our proposed power law depends on the size N, which is important since the Internet is still in evolution. A similar expression can be deduced from our computations on the k-ary tree where $\beta(N)$ is replaced by $\beta^*(N)$ in (22).

Finally, previously reported Internet measurements have been fitted with the exact expression of g_N for the k-ary tree. Based on these measurement data, our analysis seems to indicate that the Internet is growing exponentially with an effective degree approximately $k = 3.2$. As far as we are aware, this is the first time the exponential growth of the Internet has been quantified.

Appendix A
Proof of Theorem 7

Before embarking with the proof of Theorem 7, we first proof the following lemma.

Lemma 10: For $a > b$,

$$S(a, b) = \sum_{k=1}^{b} \frac{(a-k)!}{(b-k)!} \frac{1}{k} = \frac{a!}{b!} [\psi(a+1) - \psi(a-b+1)]$$

and

$$S(b, b) = \sum_{k=1}^{b} \frac{1}{k} = \psi(b+1) + \gamma.$$

Proof: We start by writing

$$\begin{aligned} S(a, b) &= \sum_{k=1}^{b} \frac{(a-k) \cdots (b-k+1)}{k} \\ &= a \sum_{k=1}^{b} \frac{(a-1-k) \cdots (b-k+1)}{k} \\ &\quad - \sum_{k=1}^{b} (a-1-k) \cdots (b-k+1). \end{aligned}$$

Since $(a-1-k) \cdots (b-k+1) = (a-b-1)!\binom{a-1-k}{b-k}$ and by the recurrence for the binomial

$$\sum_{k=1}^{b} \binom{a-1-k}{b-k} = \binom{a-1}{b-1}$$

we have that

$$S(a, b) = aS(a-1, b) - \frac{1}{a-b} \frac{(a-1)!}{(b-1)!}.$$

After p iterations, we have

$$\begin{aligned} S(a, b) &= a(a-1) \cdots (a-p+1) S(a-p, b) \\ &\quad - \frac{a!}{(b-1)!} \sum_{j=0}^{p-1} \frac{1}{(a-j)(a-j-b)} \end{aligned}$$

and, if $p = a - b$, the recursions stops with result

$$\begin{aligned} S(a, b) &= \frac{a!}{b!} \sum_{k=1}^{b} \frac{1}{k} - \frac{a!}{(b-1)!} \sum_{j=0}^{a-b-1} \frac{1}{(a-j)(a-j-b)} \\ &= \frac{a!}{b!} \sum_{k=1}^{b} \frac{1}{k} - \frac{a!}{b!} \left(\sum_{k=1}^{a-b} \frac{1}{k} - \sum_{k=b+1}^{a} \frac{1}{k} \right) \\ &= \frac{a!}{b!} \left(\sum_{k=1}^{a} \frac{1}{k} - \sum_{k=1}^{a-b} \frac{1}{k} \right) \end{aligned}$$

from which the lemma follows. ■

Proof of Theorem 7: We will investigate $E[X_i] = E[X_i^{(N)}]$ on the uniform tree with N nodes. Here $E[X_i]$ is the number of joint hops in a multicast shortest-path tree from the root to i uniformly chosen nodes in the uniform tree and where all the group member nodes are different from the root. Let $E[\tilde{X}_i]$ be the same quantity where we allow the group member nodes to be the root. Then

$$E\left[\tilde{X}_i\right] = \frac{N-i}{N} E[X_i]$$

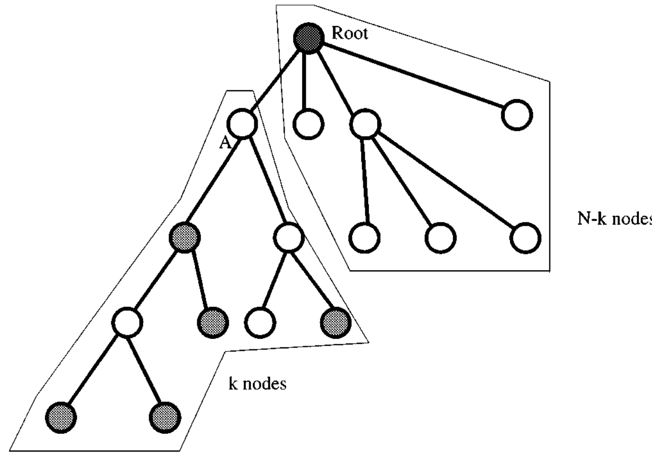

Fig. 9. The two contributing clusters leading to the $E[\tilde{X}_i^{(N)}]$ recursion.

since there are i possibilities each with probability $1/N$ that one of the nodes equals the root, in which case $X_i = 0$.

The average number of joint hops $E[\tilde{X}_i]$ is deduced from Fig. 9, where two clusters are shown each with, respectively, k and $N-k$ nodes. The first cluster with k nodes does not possess the root (dark shaded), but it contains the i multicast group members (light shaded). There is already at least one joint hop because the link between the root and node A, which can be viewed as the root of the first cluster, and is used by all i group members lying in the first cluster. Given the size k of the first cluster, the probability that all i uniformly chosen group members belong to the first cluster equals $(k(k-1) \cdots (k-i+1))/(N(N-1) \cdots (N-i+1))$, because the probability that the first group member belongs to that cluster which is k/N, the probability that the second group member also belongs to the first cluster which is $(k-1)/(N-1)$ and so on. Since the size of the first cluster connected to the root is uniform in between 1 and $N-1$, the probability that the size is k equals $1/(N-1)$. When all i nodes are in that first cluster of size k, X_i is at least 1, and the problem restarts, but with N replaced by k and A being the root. Hence, if all i group members belong to the first cluster, the average number of joint hops is

$$\frac{1}{N-1} \sum_{k=1}^{N-1} \frac{k(k-1) \cdots (k-i+1)}{N(N-1) \cdots (N-i+1)} \left(1 + E\left[\tilde{X}_i^{(k)}\right]\right)$$

because we must sum over all possible sizes for the first cluster. If *not* all i group member nodes are in the first cluster, the group member nodes are divided over the two clusters. But, in that case, we have no joint overlaps or $X_i = 0$. Thus, if not all i group members nodes are in the first cluster, the only way that there are possible joint overlaps ($X_i > 0$), is that all i group member nodes are in the second cluster. However, by removing the first cluster, we are left again with a uniform recursive tree of size $N-k$. The average number of joint hops in this case is

$$\begin{aligned} &\frac{1}{N-1} \sum_{k=1}^{N-1} \frac{(N-k)(N-k-1) \cdots (N-k-i+1)}{N(N-1) \cdots (N-i+1)} \\ &\times E\left[\tilde{X}_i^{(N-k)}\right]. \end{aligned}$$

Adding both contributions results in the recursion formula

$$E\left[\tilde{X}_i^{(N)}\right] = \frac{1}{N-1}\sum_{k=1}^{N-1}\frac{k(k-1)\cdots(k-i+1)}{N(N-1)\cdots(N-i+1)} \times\left(1+2E\left[\tilde{X}_i^{(k)}\right]\right). \quad (24)$$

We next write

$$\alpha_i^{(N)} = N(N-1)\cdots(N-i+1)E\left[\tilde{X}_i^{(N)}\right] = \frac{N!}{(N-i)!}E\left[\tilde{X}_i^{(N)}\right]$$

then the above recurrence equation (24) turns into

$$\alpha_i^{(N)} = \frac{1}{N-1}\sum_{k=1}^{N-1}[k(k-1)\cdots(k-i+1)+2\alpha_i^{(k)}] = \frac{1}{N-1}\sum_{k=i}^{N-1}k(k-1)\cdots(k-i+1)+\frac{2}{N-1}\sum_{k=1}^{N-1}\alpha_i^{(k)}.$$

Subtracting

$$(N-1)\alpha_i^{(N)}-(N-2)\alpha_i^{(N-1)} = \frac{(N-1)!}{(N-i-1)!}+2\alpha_i^{(N-1)}$$

from which we obtain

$$\frac{\alpha_i^{(N)}}{N} = \frac{(N-2)!}{N(N-i-1)!}+\frac{\alpha_i^{(N-1)}}{N-1}. \quad (25)$$

Iterating (25) gives

$$\frac{\alpha_i^{(N)}}{N} = \sum_{j=0}^{k-1}\frac{(N-2-j)!}{(N-j)(N-i-1-j)!}+\frac{\alpha_i^{(N-k)}}{N-k}.$$

Since $\alpha_i^{(i)} = E\left[\tilde{X}_i^{(i)}\right] = 0$, because the root is then always one of the group member nodes, we finally obtain

$$\alpha_i^{(N)} = N\sum_{j=0}^{N-i-1}\frac{(N-2-j)!}{(N-j)(N-i-1-j)!} = N\sum_{k=i+1}^{N}\frac{(k-2)!}{k(k-i-1)!}. \quad (26)$$

It can be shown that, for large N

$$\alpha_i^{(N)} \sim \frac{N}{i-1}\frac{(N-2)!}{(N-i-1)!}.$$

Because

$$E[X_i^{(N)}] = \frac{N}{N-i}E\left[\tilde{X}_i^{(N)}\right] = \frac{(N-i-1)!}{(N-1)!}\alpha_i^{(N)}$$

we have that

$$E\left[X_i^{(N)}\right] = \frac{(N-i-1)!N}{(N-1)!}\sum_{k=i+1}^{N}\frac{(k-2)!}{k(k-i-1)!}. \quad (27)$$

and, for large N

$$E\left[X_i^{(N)}\right] \sim \frac{1}{i-1}\frac{N}{(N-1)} \sim \frac{1}{i-1}.$$

Invoking Theorem 3, the average number of multicast hops for m uniformly chosen, distinct group members is

$$g_N(m) = \sum_{i=1}^{m}\binom{m}{i}(-1)^{i-1}\frac{(N-i-1)!N}{(N-1)!} \times\sum_{k=2}^{N}\frac{(k-2)!}{k(k-i-1)!} = \frac{-N}{(N-1)!}\sum_{s=0}^{N-2}\frac{(N-2-s)!}{N-s} \times\sum_{i=1}^{m}\binom{m}{i}\frac{(-1)^i(N-i-1)!}{(N-i-1-s)!}.$$

The i-summation can be executed as follows. Consider

$$x^{N-1}(1-1/x)^m = \sum_{i=0}^{m}\binom{m}{i}(-1)^i x^{N-i-1}.$$

Differentiating s times yields

$$\sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{(N-i-1)!}{(N-i-1-s)!}x^{N-i-s-1} = \frac{d^s}{dx^s}[x^{N-1-m}(x-1)^m].$$

Expanding the right-hand side R around $x=1$ gives

$$R = \frac{d^s}{dx^s}[x^{N-1-m}(x-1)^m] = \sum_{k=0}^{\infty}\binom{N-1-m}{k}\frac{d^s}{dx^s}(x-1)^{k+m} = \sum_{k=0}^{\infty}\binom{N-1-m}{k}\frac{(k+m)!(x-1)^{k+m-s}}{(k+m-s)!}.$$

Evaluation at $x = 1$ only leads to a nonzero contribution if $k + m - s = 0$. Hence

$$\sum_{i=1}^{m} \binom{m}{i} (-1)^i \frac{(N-i-1)!}{(N-i-1-s)!} = \binom{N-1-m}{s-m} s! - \frac{(N-1)!}{(N-1-s)!}$$

and

$$\begin{aligned} g_N(m) &= \frac{-N(N-1-m)!}{(N-1)!} \sum_{s=0}^{N-2} \frac{s!}{(N-s)(s-m)!(N-1-s)} \\ &\quad + N \sum_{s=0}^{N-2} \frac{1}{(N-s)(N-1-s)} \\ &= \frac{-N(N-1-m)!}{(N-1)!} \left[\sum_{s=m}^{N-2} \frac{s!}{(s-m)!(N-1-s)} \right. \\ &\quad \left. - \sum_{s=m}^{N-2} \frac{s!}{(s-m)!(N-s)} \right] + N \left[\sum_{k=1}^{N-1} \frac{1}{k} - \sum_{k=2}^{N} \frac{1}{k} \right] \\ &= \frac{-N(N-1-m)!}{(N-1)!} \left[\sum_{k=1}^{N-m-1} \frac{(N-k-1)!}{(N-k-1-m)!k} \right. \\ &\quad \left. - \sum_{k=2}^{N-m} \frac{(N-k)!}{(N-k-m)!k} \right] + N - 1. \end{aligned}$$

Rewrite the first summation F as

$$\begin{aligned} F &= \sum_{k=1}^{N-m-1} \frac{(N-k-1)!}{(N-k-1-m)!k} \\ &= \frac{(N-2)!}{(N-2-m)!} + \sum_{k=2}^{N-m} \frac{(N-k-1)!(N-k-m)}{(N-k-m)!k} \\ &= \frac{(N-2)!}{(N-2-m)!} + \sum_{k=2}^{N-m} \frac{(N-k)!}{(N-k-m)!k} \\ &\quad - m \sum_{k=2}^{N-m} \frac{(N-k-1)!}{(N-k-m)!k}. \end{aligned}$$

Then

$$\begin{aligned} g_N(m) &= \frac{-N(N-1-m)!}{(N-1)!} \left[\frac{(N-2)!}{(N-2-m)!} \right. \\ &\quad \left. - m \sum_{k=2}^{N-m} \frac{(N-k-1)!}{k(N-k-m)!} \right] + N - 1 \\ &= \frac{N(m-1)+1}{(N-1)} \\ &\quad + \frac{mN(N-1-m)!}{(N-1)!} \sum_{k=2}^{N-m} \frac{(N-k-1)!}{k(N-k-m)!} \\ &= -1 + \frac{mN(N-1-m)!}{(N-1)!} \sum_{k=1}^{N-m} \frac{(N-k-1)!}{k(N-k-m)!}. \end{aligned}$$

Using Lemma 10

$$\sum_{k=1}^{N-m} \frac{(N-k-1)!}{(N-k-m)!} \frac{1}{k} = \frac{(N-1)!}{(N-m)!} [\psi(N) - \psi(m)]. \quad (28)$$

finally leads to (14).

APPENDIX B
PROOF OF THEOREM 8

Let $\tilde{X}_i$ be the number of joint hops for i *different* multicast group members (we allow the root to be a user in which case $\tilde{X}_i = 0$), then $\Pr[\tilde{X}_i \geq 1]$ is the probability that all group members belong to the same cluster connected to the root. Due to the structure of the k-ary tree, this probability $\Pr[\tilde{X}_i \geq 1]$ equals k times the probability P that all group members belong to the *first* cluster connected to the root. Thus

$$\begin{aligned} \Pr\left[\tilde{X}_i \geq 1\right] &= k \cdot P \\ &= k \frac{\binom{(N-1)/k}{i}}{\binom{N}{i}} \\ &= k \frac{\binom{1+k+\cdots+k^{D-1}}{i}}{\binom{1+k+\cdots+k^{D}}{i}}. \end{aligned} \quad (29)$$

By self-similarity of k-ary trees, we obtain

$$\begin{aligned} \Pr\left[\tilde{X}_i \geq 2 | \tilde{X}_i \geq 1\right] &= p_i^{(D-1)} \\ &= k \frac{\binom{1+k+\cdots+k^{D-2}}{i}}{\binom{1+k+\cdots+k^{D-1}}{i}} \end{aligned}$$

because each cluster extending from the root is itself a k-ary tree of depth $D-1$. In general, we have $\Pr[\tilde{X}_i \geq j] = \Pr[\tilde{X}_i \geq j | \tilde{X}_i \geq j-1] \Pr[\tilde{X}_i \geq j-1]$. Hence, by iteration

$$\Pr\left[\tilde{X}_i \geq j\right] = \prod_{n=D-j+1}^{D} p_i^{(n)}, \quad j = 1, 2, \ldots, D-1. \quad (30)$$

Note that for $i \geq 2$ the probability $\Pr[\tilde{X}_i \geq D] = 0$, because if $\tilde{X}_i = D$ some destinations must be identical. From the well-known identity that $E[\tilde{X}_i] = \sum_{j \geq 1} \Pr[\tilde{X} \geq j]$, we obtain for $i \geq 2$

$$\begin{aligned} E\left[\tilde{X}_i\right] &= \sum_{j=1}^{D-1} \prod_{n=D-j+1}^{D} p_i^{(n)} = \sum_{j=1}^{D-1} \frac{k^j \binom{1+\cdots+k^{D-j}}{i}}{\binom{1+\cdots+k^{D}}{i}} \\ &= \sum_{j=1}^{D-1} \frac{k^{D-j} \binom{1+\cdots+k^{j}}{i}}{\binom{1+\cdots+k^{D}}{i}}. \end{aligned} \quad (31)$$

Since $E[X_i] = (N/(N-i))E[\tilde{X}_i]$, we find

$$E[X_i] = \frac{N}{N-i}\sum_{j=1}^{D-1}\frac{k^{D-j}\binom{1+\cdots+k^j}{i}}{\binom{1+\cdots+k^D}{i}}, \qquad i \geq 2. \tag{32}$$

For the value of $E[\tilde{X}_1]$ and $E[X_1]$, we find

$$\begin{aligned} E\left[\tilde{X}_1\right] &= \frac{1}{N}\sum_{j=1}^{D} k^j(1+\cdots+k^{D-j}) \\ &= \frac{1}{N(k-1)}\{Dk^{D+1} - (N-1)\} \end{aligned}$$

and

$$\begin{aligned} E[X_1] &= \frac{1}{N-1}\sum_{j=1}^{D} k^j(1+\cdots+k^{D-j}) \\ &= \frac{N}{N-1}\sum_{j=1}^{D-1}\frac{k^{D-j}\binom{1+\cdots+k^j}{1}}{\binom{1+\cdots+k^D}{1}} + \frac{k^D}{N-1}. \end{aligned}$$

Invoking Theorem 3 yields

$$g_{N,k}(m) = \frac{mk^D}{N-1} - \sum_{i=1}^{m}\binom{m}{i}(-1)^i\frac{N}{N-i}\sum_{j=1}^{D-1} \times \frac{k^{D-j}\binom{1+\cdots+k^j}{i}}{\binom{1+\cdots+k^D}{i}}.$$

Writing $A_j = (k^{j+1}-1)/(k-1)$ and reversing the i- and j-summation yields using (16)

$$g_{N,k}(m) = \frac{mk^D}{N-1} - N\sum_{j=1}^{D-1}k^{D-j}\frac{A_j!}{N!}\sum_{i=1}^{m}\binom{m}{i}(-1)^i \times \frac{(N-i-1)!}{(A_j-i)!}.$$

Concentrating on the inner sum with lower sum bound $i=0$, denoted as S_j, and substituting $k = m-i$, we have

$$S_j = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{\Gamma(N-m+k)}{\Gamma(A_j-m+k+1)}.$$

Invoking the Taylor series of the hypergeometric function [1, 15.1.1]

$$F(a,\, b;\, c;\, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n.$$

$m!S_j$ is the coefficient in z^m of the Cauchy product of

$$(1-z)^m = \sum_{k=0}^{\infty}\binom{m}{k}(-1)^k z^k$$

and

$$\frac{\Gamma(N-m)}{\Gamma(A_j-m+1)}F(1,\, N-m;\, A_j-m+1;\, z) = \sum_{k=0}^{\infty}\frac{\Gamma(N-m+k)}{\Gamma(A_j-m+1+k)}z^k.$$

Hence

$$S_j = \frac{1}{m!}\frac{\Gamma(N-m)}{\Gamma(A_j-m+1)} \times \frac{d^m}{dz^m}\left[(1-z)^m F(1,\, N-m;\, A_j-m+1;\, z)\right]\Big|_{z=0}.$$

Invoking the differentiation formula [1, 15.2.7], denoted by DF

$$\begin{aligned} DF &= \frac{d^m}{dz^m}\left[(1-z)^{a+m-1}F(a,\, b;\, c;\, z)\right] \\ &= \frac{(-1)^m\Gamma(a+m)\Gamma(c-b+m)\Gamma(c)}{\Gamma(a)\Gamma(c-b)\Gamma(c+m)} \\ &\quad \times (1-z)^{a-1}F(a+m,\, b;\, c+m;\, z) \end{aligned}$$

we have, since $a=1$ and $F(a,\, b;\, c;\, 0) = 1$

$$S_j = \frac{(-1)^m\Gamma(N-m)\Gamma(A_j+1-N+m)}{\Gamma(A_j+1-N)\Gamma(A_j+1)}.$$

Thus

$$\begin{aligned} g_{N,k}(m) &= \frac{mk^D}{N-1} - N\sum_{j=1}^{D-1}k^{D-j}\frac{A_j!}{N!} \\ &\quad \times \left(\frac{(-1)^m(N-m-1)!(A_j-N+m)!}{(A_j-N)!A_j!} - \frac{(N-1)!}{A_j!}\right) \\ &= \frac{mk^D}{N-1} + \sum_{j=1}^{D-1}k^{D-j} \\ &\quad + \frac{(-1)^{m-1}(N-m-1)!}{(N-1)!}\sum_{j=1}^{D-1}k^{D-j}\frac{(A_j-N+m)!}{(A_j-N)!} \end{aligned}$$

from which (17) is immediate.

ACKNOWLEDGMENT

The authors would like to thank R. van den Berg (CWI, The Netherlands) for his input to Theorem 2. They also thank J. Chuang, M. Sirbu, G. Philips, S. Shenker, and H. Tangmunarunkit for sending the data in [13, Fig. 1b]. Finally, they are grateful to one of the reviewers for pointing to the alternative proof (B) in Corollary 5.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1968.

[2] K. C. Almeroth, "The evolution of multicast: From the MBone to inter-domain multicast to Internet2 deployment," *IEEE Network, Special Issue on Multicasting*, vol. 14, pp. 10–20, Jan./Feb. 2000.

[3] B. Bollobas, *Random Graphs*. Boston, MA: Academic, 1985.

[4] R. C. Chalmers and K. C. Almeroth, "Modeling the branching characteristics and efficiency gains in global multicast trees," in *IEEE INFOCOM*, Apr. 2001, pp. 449–458.

[5] J. Chuang and M. Sirbu, "Pricing multicast communication: A cost-based approach," presented at the INET, 1998.

[6] R. van der Hofstad, G. Hooghiemstra, and P. Van Mieghem, "On the covariance of the level sizes in recursive tree," Random Structures and Algorithms, 2002, to be published.
[7] M. Faloutsos, P. Faloutsos, and C. Faloutsos, "On power-law relationships of the internet topology," in *Proc. ACM SIGCOMM*, Cambridge, MA, 1999, pp. 251–262.
[8] W. Feller, *An Introduction to Probability Theory and its Applications*, 3rd ed. New York: Wiley, 1968, vol. 1.
[9] R. Govindan and H. Tangmunarunkit, "Heuristics for internet map discovery," in *IEEE INFOCOM*, 2000, pp. 1371–1380.
[10] C. Lanczos, *Applied Analysis*. New York: Dover, 1988.
[11] M. Ramalho, "Intra-and inter-domain multicast routing protocols: A survey and taxonomy," *IEEE Commun. Surveys*, 2000.
[12] J. Riordan, *Combinatorial Identities*. New York: Wiley, 1968.
[13] G. Phillips, S. Shenker, and H. Tangmunarunkit. Scaling of multicast trees: Comments on the Chuang–Sirbu scaling law. presented at ACM SIGCOMM'99. [Online]. Available: http://www.acm.org/sigcomm/sigcomm99/papers/session2-1.html
[14] R. T. Smythe and H. M. Mahmoud, "A survey of recursive trees," *Theor. Probability Mathemat. Statist.*, vol. 51, pp. 1–27, 1995.
[15] H. Holbrook and B. Cain. Source-Specific Multicast for IP. [Online]. Available: http://www.ietf.org/internet-drafts/draft-holbrook-ssm-arch-02.txt
[16] R. van der Hofstad, G. Hooghiemstra, and P. Van Mieghem, "First-passage percolation on the random graph," *Probability in the Engineering and Informational Sciences (PEIS)*, vol. 15, pp. 225–237, 2001.
[17] P. Van Mieghem, G. Hooghiemstra, and R. van der Hofstad. A Scaling Law for the Hopcount, Report 2000125. Delft University of Technology, Delft, The Netherlands. [Online]. Available: http://wwwtvs.et.tudelft.nl/people/piet/telconference.html
[18] R. Boivie, N. Feldman, Y. Imai, W. Livens, D. Ooms, and O. Paridaens. Explicit Multicast (Xcast) Basic Specification. [Online]. Available: http://www.ietf.org/internet-drafts/draft-ooms-xcast-basic-spec-02.txt

Piet Van Mieghem received the Master and Ph.D. degrees in electrical engineering from the K.U.Leuven, Belgium, in 1987 and 1991, respectively.

He was with the Interuniversity Micro Electronic Center (IMEC) from 1987 to 1991. He was a Visiting Scientist at the Massachusetts Institute of Technology (MIT), Cambridge, Department of Electrical Engineering from 1992 to 1993. From 1993 to 1998, he was with Alcatel Corporate Research Center, Antwerp, Belgium, where he gained experience in performance analysis of ATM systems and network architectural concepts of both ATM networks (PNNI) and the Internet. Currently, he is a full Professor at Delft University of Technology, Delft, The Netherlands, with a chair in telecommunication networks. The main theme of his research is evolution of the Internet architecture toward a broadband and QoS-aware network.

Gerard Hooghiemstra was born on November 16, 1949. He studied mathematics at the University of Amsterdam, The Netherlands, and received the Ph.D. degree from the University of Utrecht, The Netherlands.

Having received the doctoral degree, he was a teacher for a short period and then joined the Department of Technical Mathematics and Computer Science, Delft University, The Netherlands, where he became an Associate Professor in 1986. During the academic year 1985–1986, he was a Visiting Professor at the University of British Columbia, Vancouver, BC, Canada. He has published about 40 journal papers in the field of probability and statistics. His research interests include stochastic processes, queueing theory, distributional limit theorems and extreme value theory.

Remco van der Hofstad was born in Eindhoven, The Netherlands, on May 3, 1971. He received the M.Sc. degree with honors from the Department of Mathematics, University of Utrecht, The Netherlands, in 1993, and the Ph.D. degree in 1997 on research in statistical physics, also from the University of Utrecht.

In 1998, he was with McMaster University, Hamilton, Canada, and worked at Microsoft Research, Redmond, WA. In 1998, he became an Assistant Professor with Delft University of Technology, The Netherlands. His research is in applications of probability, mostly in statistical physics and in electrical engineering. In statistical physics, his research is focused on the mathematical investigation of polymers and percolation models. At Delft, he started with research on the modeling of the Internet as a (random) graph, and on improving the performance of wireless communication systems using code division multiple access techniques.