Abstract

We consider the partitioning of m-dimensional lattice graphs using Fiedler's approach [1], that requires the determination of the eigenvector belonging to the second smallest eigenvalue of the Laplacian. We examine the general m-dimensional lattice and, in particular, the special cases: the 1-dimensional path graph P_N and the 2-dimensional lattice graph. We determine the size of the clusters and the number of links, which are cut by this partitioning as a function of Fiedler's threshold α.

Fiedler's Clustering on m–dimensional Lattice Graphs

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1   Introduction

There are many methods and approaches for graph partitioning. Here, we shall focus only on Fiedler's approach to clustering, which theoretically determines the relation between the size of the obtained clusters and the number of links that are cut by this partitioning as a function of a threshold α and of graph properties such as the number of nodes and links. When applying Fiedler's beautiful results [1] to the Laplacian matrix Q of a graph, the eigenvector belonging to the second smallest eigenvalue, known as the algebraic connectivity, needs to be computed. We apply Fiedler's approach to the m-dimensional lattice graph and determine the cluster size as a function of the threshold α. Following the notation in [2], a graph G consists of a set $\mathcal{N}$ of $N = |\mathcal{N}|$ nodes and a set $\mathcal{L}$ of $L = |\mathcal{L}|$ links. We denote by $x_i = \begin{bmatrix} (x_i)_1 & (x_i)_2 & ... & (x_i)_N \end{bmatrix}^T$ the eigenvector of the $N \times N$ symmetric Laplacian Q belonging to the eigenvalue μ_i. Since eigenvectors of a symmetric matrix are orthogonal, we normalize the eigenvectors of Q by requiring that

$$\|x_i\|_2^2 = x_i^T x_i = 1 \text{ for each } i = 1, ..., N \tag{1}$$

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The last condition ensures that the eigenvector is unique. The eigenvalues of the Laplacian are nonnegative with at least one eigenvalue equal to zero [1] and they can be ordered as $0 = \mu_N \le \mu_{N-1} \le ... \le \mu_1$. If the graph is connected, then $\mu_{N-1} > 0$ and the components of the correspondent eigenvector x_{N-1} determine the Fiedler partitioning with respect to the threshold α: the set of nodes $\mathcal{M} = \left\{ j \in \mathcal{N} : (x_{N-1})_j \ge \alpha \right\}$ defines the first (connected) cluster and the set $\mathcal{N} \backslash \mathcal{M}$ determines the second (connected) cluster. Our interest concerns the size (or the number of nodes) of the obtained clusters and the number of links that will be cut by Fiedler's partitioning. The end points of those links are nodes in two separate clusters. We denote by $c(G)$ the number of links in G that will be cut by this partitioning. Furthermore, we define the "ratio of cut links"

$$r(G) = \frac{c(G)}{L} \tag{2}$$

where $L = |\mathcal{L}|$ is the total number of links in the graph. Clearly, $0 \le r(G) \le 1$.

2 The path and lattice graphs

In this Section, we examine the effect of Fiedler's clustering on the lattice graph. We will start with the 1-dimensional path graph P_N on N nodes and containing $N-1$ links or hops, which we subsequently will generalize to m dimensions. Finally, we will apply the results to a 2-dimensional lattice.

2.1 A path P_N of $(N-1)$ hops

In [3], the Laplacian eigenvalues (as well as the eigenvectors) of the path P_N are derived as $\mu_{N-m}(P_N) = 2\left(1 - \cos\left(\frac{\pi m}{N}\right)\right)$ for $m = 0, 1, 2, ..., N-1$. The second smallest eigenvalue of the Laplacian is

$$\mu_{N-1}(P_N) = 2\left(1 - \cos\left(\frac{\pi}{N}\right)\right) = 4\sin^2\left(\frac{\pi}{2N}\right)$$

and the corresponding Laplacian eigenvector components are [3]

$$(x_{N-1})_j = \sqrt{\frac{2}{N}} \cos\frac{\pi}{2N}(2j - 1)$$

where $1 \leq j \leq N$. The corresponding Fiedler partitioning rule for the components of the eigenvector x_{N-1} with respect to the threshold α is

$$(x_{N-1})_j \geq \alpha$$

Clustering into two separate, non-empty sets of nodes will exist if and only if $|\alpha| \leq \sqrt{\frac{2}{N}}$. Because $\cos \frac{\pi}{2N}(2j-1)$ decreases with j, the nodes labeled by j will belong to the first cluster provided

$$j \leq \left[\frac{1}{2} + \frac{N}{\pi} \arccos \left(\alpha \sqrt{\frac{N}{2}} \right) \right] \tag{3}$$

Relation (3) shows for $\alpha = 0$ that one half of the nodes will belong to both clusters. In all cases only one link will be cut, thus $c(P_N) = 1$.

2.2 The general m dimensional lattice

We consider the m-dimensional lattice $\mathcal{C}_m = \mathrm{La}_{(z_1+1) \times (z_2+1) \times \ldots \times (z_m+1)}$ with lengths $z_1, z_2, \ldots, z_m$ in each dimension, respectively, and where at each lattice point with integer coordinates a node is placed that is connected to its nearest neighbors whose coordinates only differ by 1 in only 1 components. The total number of nodes in $\mathcal{C}_m$ is $N = (z_1 + 1) \times (z_2 + 1) \times \ldots \times (z_m + 1)$. The lattice graph is a Cartesian product [7] of m path graphs, denoted by $\mathcal{C}_m = P_{(z_1+1)} \square P_{(z_2+1)} \square \ldots \square P_{(z_m+1)}$. According to [3, 4, 5, 6], the eigenvalues of $\mathcal{C}_m$ can be written as a sum of one combination of eigenvalues of path graphs and the corresponding eigenvector is the Kronecker product of the corresponding eigenvectors of the same path graphs,

$$\begin{aligned} \mu_{i_1 i_2 \ldots i_N} (\mathcal{C}_m) &= \sum_{j=1}^{m} \mu_{i_j} \left(P_{(z_j+1)} \right) \\ x_{i_1 i_2 \ldots i_m} (\mathcal{C}_m) &= x_{i_1} \left(P_{(z_j+1)} \right) \otimes x_{i_2} \left(P_{(z_2+1)} \right) \otimes \ldots \otimes x_{i_m} \left(P_{(z_m+1)} \right) \end{aligned} \tag{4}$$

where $i_j \in \{1, 2, \ldots, z_j + 1\}$ for each $j \in \{1, 2, \ldots, m\}$. Without loss of generality we can assume that $z_1 \leq z_2 \leq \ldots \leq z_m$. In Section 2.1, we obtained the Laplacian eigenvalues of the path on N nodes and for the second smallest eigenvalues μ_{N-1} of $P_{(z_1+1)}, P_{(z_2+1)}, \ldots, P_{(z_m+1)}$, we have that

$$\mu_{z_1} \left(P_{(z_j+1)} \right) \geq \mu_{z_2} \left(P_{(z_j+1)} \right) \geq \ldots \geq \mu_{z_m} \left(P_{(z_j+1)} \right)$$

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Substituted into (4), the second smallest Laplacian eigenvalue of $\mathcal{C}_m$ is obtained for $i_j = z_j + 1$, $j \in \{1, 2, \ldots, m-1\}$ and $i_m = z_m$. Since $\mu_N = 0$ or, equivalently, $\mu_{z_1+1}\left(P_{(z_j+1)}\right) = \mu_{z_2+1}\left(P_{(z_j+1)}\right) = \ldots = \mu_{z_{m-1}+1}\left(P_{(z_j+1)}\right) = 0$, we find that

$$\mu_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right) = \mu_{z_m}\left(P_{(z_j+1)}\right) = 2\left(1 - \cos\left(\frac{\pi}{z_m+1}\right)\right)$$

From (4), the corresponding eigenvector is

$$x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right) = x_{z_1+1}\left(P_{(z_1+1)}\right) \otimes x_{z_2+1}\left(P_{(z_2+1)}\right) \otimes \ldots \otimes x_{z_m}\left(P_{(z_m+1)}\right)$$

To shorten the notation, we define $s = (z_1 + 1)(z_2 + 1)\ldots(z_{m-1} + 1)$ and

$$t = \left[\frac{1}{2} + \frac{z_m + 1}{\pi}\arccos\left(\alpha\sqrt{\frac{s(z_m+1)}{2}}\right)\right]$$

All components of $x_{z_i+1}\left(P_{(z_i+1)}\right) = \frac{1}{\sqrt{z_i+1}}$ for $i \in \{1, 2, \ldots, m-1\}$ are equal, so their final result is Kronecker product of a vector with all equal components and $y = x_{z_m}\left(P_{(z_m+1)}\right)$. Hence, we have

$$x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right) = K\left[\underbrace{\begin{array}{cccc} y & y & \ldots & y \end{array}}_{s \text{ times}}\right]^T \tag{5}$$

After proper normalization using (1), we obtain $K = \sqrt{\frac{2}{s(z_m+1)}}$ (see Appendix A.1). According to (5), $x_{z_m}\left(P_{(z_m+1)}\right)$ occurs $(z_1 + 1)\ldots(z_{m-1} + 1)$ times in $x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right)$. This last result illustrates that every component of $x_{(z_1+1)(z_2+1)\ldots z_m}\left(\mathcal{C}_m\right)$ repeats periodically after $(z_m + 1)$ next components, such that the Fiedler partitioning condition reads

$$\sqrt{\frac{2}{s(z_m+1)}}\cos\frac{(2j-1)\pi}{2(z_m+1)} \geq \alpha \tag{6}$$

only for $j = 1, 2, \ldots, (z_m + 1)$. Thus, clustering of the m-dimensional lattice $\mathcal{C}_m$ into two non-empty subsets exists if and only if $|\alpha| \leq \sqrt{\frac{2}{s(z_m+1)}}$ in which case $j \leq t$. Because every component periodically repeats after $(z_m + 1)$ components, the final condition for the node labeled by j to belong to the first cluster is $j \mod (z_m + 1) \in \{1, 2, \ldots, t\}$. Hence, those nodes are

$$j \in \{w + v | \forall w = 0, \ldots (s-1)(z_m+1) \text{ and } \forall v = 1, \ldots t\} \tag{7}$$

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This means that the number of nodes in the first cluster equals $s \cdot t$ and that in the second clusters equals $s \cdot (z_m + 1 - t)$. The $(m-1)$-dimensional hyperplane divides the m-dimensional lattice $\mathcal{C}_m$ into two clusters. Let us consider the links that will be cut by this partitioning. Those links are

$$w + t \leftrightarrow w + (t+1), \text{ for } w = 0, \ldots (s-1)(z_m + 1)$$

Hence, the number of cut links is

$$c(\mathcal{C}_m) = s = \prod_{i=1}^{m-1} (z_i + 1) \tag{8}$$

Finally, the total number of links in the general lattice $\mathcal{C}_m$ is specified in

Lemma 1 *The number of links in the $\mathcal{C}_m = La_{(z_1+1) \times (z_2+1) \times \ldots \times (z_m+1)}$ is*

$$L = \left[\prod_{i=1}^{m} (z_i + 1) \right] \sum_{i=1}^{m} \frac{z_i}{z_i + 1}$$

Proof: See Appendix A.2. □

Using (4), the ordering $z_1 \leq z_2 \leq \ldots \leq z_m$ and Lemma 1, the "ratio of cut links" is

$$r(\mathcal{C}_m) = \frac{1}{(z_m + 1) \sum_{i=1}^{m} \frac{z_i}{z_i+1}}$$

For the most common case of $\alpha = 0$ in (7), both clusters have almost the same number of nodes. For a 3-dimensional lattice $La_{(z_1+1) \times (z_2+1) \times (z_3+1)}$, a plane divides $La_{(z_1+1) \times (z_2+1) \times (z_3+1)}$ into two clusters with the same number of links. Figure 1 is an example for $m = 2$, in which $z_1 = 6$ and $z_2 = 7$ and Fiedler's partitioning for $\alpha = \frac{1}{20}$ In this case $c(La_{7 \times 8}) = 7$ and $L = z_1(z_2 + 1) + z_2(z_1 + 1) = 97$. Hence $r(La_{7 \times 8}) = \frac{c(La_{7 \times 8})}{L} = \frac{7}{97} \approx 7.22\%$ of all links will be cut by Fiedler's partitioning.

3   Conclusion

We have applied Fiedler's partitioning algorithm to an m-dimensional lattice $La_{(z_1+1) \times (z_2+1) \times \ldots \times (z_m+1)}$ and have calculated the size of the two clusters, the number of links that are cut by this partitioning and the percentage of cut links as a function of the Fiedler threshold α and the characteristic dimensions of the lattice. In the most common case of $\alpha = 0$, both clusters have equal sizes. The number of cut links does not depend on α.
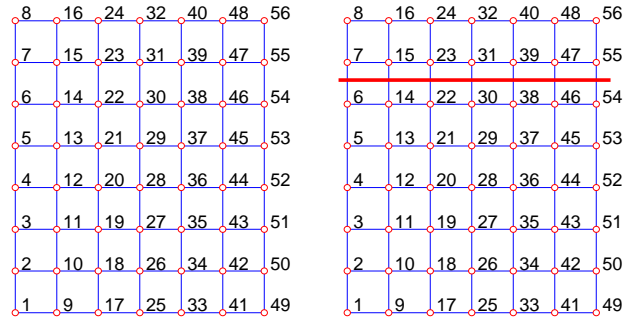
Figure 1: The two-dimensional lattice $La_{7 \times 8}$ for $\alpha = \frac{1}{20}$.

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A   Appendix

A.1   The normalization coefficient of $\mathcal{C}_m$

According to (1), we normalize the eigenvector of $\mathcal{C}_m$ as

$$\sum_{j=1}^{(z_1+1)\times(z_2+1)\times...\times(z_m+1)} \left(x_{(z_1+1)\times(z_2+1)\times...\times z_m}\left(\mathcal{C}_m\right)\right)^2_j = 1$$

which is equivalent to determining K such that

$$\underbrace{\sum_{j=0}^{z_m} \left(x_{(z_1+1)(z_2+1)...z_m}\left(\mathcal{C}_m\right)\right)^2_j = 1}_{(z_1+1)(z_2+1)...(z_{m-1}+1)\ \text{times}}$$

Hence,

$$\underbrace{\sum_{j=0}^{z_m} K^2 \cos^2 \frac{(2j+1)\pi}{2\left(Z_m+1\right)} = 1}_{(z_1+1)(z_2+1)...(z_{m-1}+1)\ \text{times}}$$

and, explicitly,

$$K^2 \left(z_1+1\right)\left(z_2+1\right)\ldots\left(z_{m-1}+1\right)\sum_{j=0}^{z_m}\cos^2\frac{(2j+1)\pi}{2\left(z_m+1\right)} = 1$$

Now, since

$$\sum_{j=0}^{z_m}\cos^2\frac{(2j+1)\pi}{2\left(z_m+1\right)} = \sum_{j=0}^{z_m}\frac{1+\cos\left(\frac{(2j+1)\pi}{z_m+1}\right)}{2}$$

$$= \frac{z_m+1}{2} + \mathrm{Re}\left\{\sum_{j=0}^{z_m}\left(e^{\frac{(2j+1)\pi}{z_m+1}}\right)^j\right\}$$

$$= \frac{z_m+1}{2} + \mathrm{Re}\left\{e^{\frac{\pi}{Z_m+1}i}\sum_{j=0}^{z_m}\left(e^{\frac{2\pi}{Z_m+1}i}\right)^j\right\}$$

$$= \frac{z_m+1}{2} + \mathrm{Re}\left\{e^{\frac{\pi}{Z_m+1}i}\frac{e^{\frac{2\pi\cdot i}{Z_m+1}(Z_m+1)}-1}{e^{\frac{2\pi\cdot i}{Z_m+1}}-1}\right\}$$

$$= \frac{z_m+1}{2} + \mathrm{Re}\left\{e^{\frac{\pi}{Z_m+1}i}\frac{1-1}{e^{\frac{2\pi\cdot i}{Z_m+1}}-1}\right\} = \frac{z_m+1}{2}$$

we find that

$$K = \sqrt{\frac{2}{\left(z_1+1\right)\left(z_2+1\right)\ldots\left(z_{m-1}+1\right)\left(z_m+1\right)}}$$

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A.2 Proof of the Lemma 1

We will prove the lemma by induction. Let the number of links in the k-dimensional lattice $La_{(z_1+1)\times(z_2+1)\times...\times(z_k+1)}$ be $l(z_1, z_2, \ldots, z_k)$.

1) For $k = 1$, we have a path graph P_{z_1+1} and its number of links is $L = l(z_1) = z_1 = (z_1 + 1)\frac{z_1}{z_1+1}$.

2) Let us assume that the lemma holds for k-dimensional lattices. We consider the $(k + 1)$-dimensional lattice $La_{(z_1+1)\times(z_2+1)\times...\times(z_{k+1}+1)}$, that is constructed from k different k-dimensional lattices ($La_{(z_{i_1}+1)\times(z_{i_2}+1)\times...\times(z_{i_k}+1)}$, where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, (k + 1)\}$) in the following way. We position a total of $(z_{i_{k+1}} + 1)$ such k-dimensional lattices $La_{(z_{i_1}+1)\times(z_{i_2}+1)\times...\times(z_{i_k}+1)}$ in next to each other in the direction of i_{k+1} dimension. In this way, every link is counted k-times in all of the dimensions. Intuitively, this construction is easier to imagine in three dimensions, where the 3-dimensional lattice $La_{(z_1+1)\times(z_2+1)\times(z_3+1)}$ is constructed by $(z_3 + 1)$ consecutive 2-dimensional $La_{(z_1+1)\times(z_2+1)}$ planes that are positioned next to each other in the direction of the third dimension, $(z_2 + 1)$ consecutive 2-dimensional $La_{(z_1+1)\times(z_3+1)}$ planes that are positioned next to each other in the direction of the second dimension and, finally, $(z_1 + 1)$ consecutive 2-dimensional La $L_{(z_2+1)\times(z_3+1)}$ planes that are positioned next to each other in the direction of the first dimension. All links in this process are counted twice. Returning to the k-dimensional case, we thus deduce that

$$l(z_1, z_2, \ldots, z_{k+1}) = \frac{1}{k} \sum_{i=1}^{k+1} (z_i + 1) l(z_{j_1}, z_{j_2}, \ldots, z_{j_k})$$

where $j_w \neq i$ for each $i = 1, 2, \ldots, (k + 1)$ and $w = 1, 2, \ldots, k$. Introducing the induction hypothesis for k-dimension lattices, we obtain

$$l(z_1, z_2, \ldots, z_{k+1}) = \frac{1}{k} \sum_{i=1}^{k+1} (z_i + 1) \prod_{j=1, j \neq i}^{k+1} (z_j + 1) \sum_{j=1, j \neq i}^{k+1} \frac{z_j}{z_j + 1}$$

$$= \frac{1}{k} \prod_{j=1}^{k+1} (z_j + 1) \sum_{i=1}^{k+1} \sum_{j=1, j \neq i}^{k+1} \frac{z_j}{z_j + 1}$$

$$= \frac{1}{k} \prod_{j=1}^{k+1} (z_j + 1) k \sum_{i=1}^{k+1} \frac{z_j}{z_j + 1}$$

which illustrates that the induction hypothesis is true for $(k + 1)$, and consequently it is true for each dimension $m \geq 1$.