The Laplacian Spectrum of Complex Networks

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***Abstract*— The set of all eigenvalues of a characteristic matrix of a graph, also referred to as the spectrum, is a well-known topology retrieval method. In this paper, we study the spectrum of the Laplacian matrix of an observable part of the Internet graph at the IP-level, extracted from traceroute measurements performed via RIPE NCC and PlanetLab. In order to investigate the factors influencing the Laplacian spectrum of the observed graphs, we study the following complex network models: the random graph of Erdős-Rényi, the small-world of Watts and Strogatz and the scale-free graph, derived from a Havel-Hakimi power-law degree sequence. Along with these complex network models, we also study the corresponding Minimum Spanning Tree (MST). Extensive simulations show that the Laplacian spectra of complex network models differ substantially from the spectra of the observed graphs. However, the Laplacian spectra of the MST in the Erdős-Rényi random graph with uniformly distributed link weights does bear resemblance to it. Furthermore, we discuss an extensive set of topological characteristics extracted from the Laplacian spectra of the observed real-world graphs as well as from complex network models.**

I. Introduction

Complex networks describe a wide range of systems in nature and society. Traditionally, the topology of a complex network has been modeled as the Erdős-Rényi random graph. However, the growing observation that real-world networks do not follow the prediction of random graphs has prompted many researchers to propose other models, such as small-world [21] and scale-free [2] network. Besides the modeling, considerable attention has been given to the problem of capturing and characterizing, in quantitative terms, the topological properties of complex networks(e.g. [3], [7], [22]). In particular, important information on the topological properties of a graph can be extracted from the eigenvalues of the associated adjacency, Laplacian or any other type of matrix. The eigenvalues of the adjacency matrix were much more investigated in the past than the eigenvalues of the Laplacian matrix: see e.g. [5], [6] for books on the eigenvalues of the adjacency matrix and e.g. [14], [15] for surveys on the eigenvalues of the Laplacian matrix. Nevertheless, we believe that for the Laplacian matrix, as already proved for its natural complement the adjacency matrix, many valuable topological properties can be deduced from its spectrum. It is the aim of this paper to show where this belief comes from by offering a detailed Laplacian spectrum analysis of generic complex network models.

Significant research efforts have recently been conducted in the spectral analysis of the Internet topology (e.g. [10]). Our paper contributes to this research by analyzing an observable part of the Internet topology, extracted from the traceroute measurement performed via RIPE and PlanetLab. In order to investigate the factors influencing the Laplacian spectrum of the observed graphs, we study generic complex network models: the random graph of Erdős-Rényi, the small-world graph of Watts and Strogats and the scale-free graph derived from a Havel-Hakimi power-law degree sequence. Along with these complex network models, we also study the corresponding Minimum Spanning Tree (MST). The application of the Laplacian spectrum analysis reveals that the observed Internet topology differs substantially from that of generic models but it does bear resemblance with the MST structure in the Erdős-Rényi random graph with uniformly distributed link weights. This observation is in contrast to results found in the literature, where it is overwhelmingly shown that the Internet topology belongs to the class of scale-free graphs. Nevertheless, this observation is interesting because this part of the Internet is responsible for carrying transport and, therefore, only this part is observable or measurable.

The paper is organized as follows. Section II presents the Laplacian spectra of the observed IP-level Internet graphs. Section III offers the Laplacian spectrum analysis of models used to describe complex network topology: the random graph of Erdős-Rényi in III-A, the small-world graph of Watts and Strogats in III-B, and the scale-free graph, derived from a Havel-Hakimi power-law degree sequence in III-C. Section IV summarizes our main results on the Laplacian spectra of both complex network models as well as observed real-world graphs.

II. Spectra of the Internet Graphs

Let G be a graph, and let $\mathcal{N}$ and $\mathcal{L}$ denote the node set and the link set, consisting of $N = |\mathcal{N}|$ nodes and $L = |\mathcal{L}|$, respectively. The Laplacian matrix of a graph G with N nodes is an $N \times N$ matrix $Q = \Delta - A$ where $\Delta = diag(D_i)$, D_i is the degree of the node $i \in \mathcal{N}$ and A is the adjacency matrix of G. The eigenvalues of Q are called the Laplacian eigenvalues. The Laplacian eigenvalues are all real and nonnegative [15]: they are contained in the interval $[0, \min\{N, 2D_{\max}\}]$, where $D_{\max}$ is the maximum degree of G. The set of all N Laplacian eigenvalues $\lambda_N = 0 \leq \lambda_{N-1} \leq ... \leq \lambda_1$ is called the Laplacian spectrum of a graph G. The second smallest eigenvalue is $\lambda_{N-1} \geq 0$, but equal to zero only if a graph is disconnected. Thus, the multiplicity of 0 as an eigenvalue of Q is equal to the number of components of G [8].

We have calculated the spectrum of the Laplacian matrix of an observable part of the Internet graph, extracted from the traceroute measurements performed via RIPE NCC [19] and PlanetLab [18]. Hence, the resulting graphs are observed Internet graphs at the IP-level because the traceroute utility returns the list of IP-addresses of routers along the path from a source to a destination. In fact, a graph obtained form traceroute measurements is an approximation of the Internet graph at the router-level, which again is the union of shortest paths between each pair of a small group of routers. This explains why such graph is denoted as the overlay graph on top of the actual Internet topology. Hence, the RIPE NCC measurements, executed on September 18th 2004, have resulted in a graph consisting of 4058 nodes and 6151 links and the PlanetLab experiments, executed on November 10th 2004, in a graph with 4214 nodes and 6998 links.

Figure 1 shows the degree distribution and Figure 2 the Laplacian spectrum of the observed graphs. In spite of two different sources of traceroute measurements, the Laplacian spectrum stays almost the same: both Laplacian spectra contain a peak at $\lambda = 2$, which, most likely, is due to the majority of nodes with degree 2. Besides, the Laplacian spectra contain smaller peaks at $\lambda = 1$ and $\lambda = 3$, although the first one only appears in the spectrum of the graph observed via RIPE. The peak at $\lambda = 3$ possibly also originates from a significant amount of nodes with the corresponding degree, while the peak at $\lambda = 1$ surely does not, since the graph observed via RIPE does not contain nodes with degree 1. Given that the peak at $\lambda = 2$, most likely originates from the majority of nodes with degree 2, the question to be answered is, whether also a specific spectral behavior (e.g. the peak at $\lambda = 3$) comes from the majority of nodes with the corresponding degree. To answer this question, we need to inspect whether the Laplacian spectra of generic complex network models are suitable to describe the underlying structure of the graphs under consideration. In particular, we need to inspect to what extent the basic topological structures, such as a path, cycle and a tree, are responsible for a specific

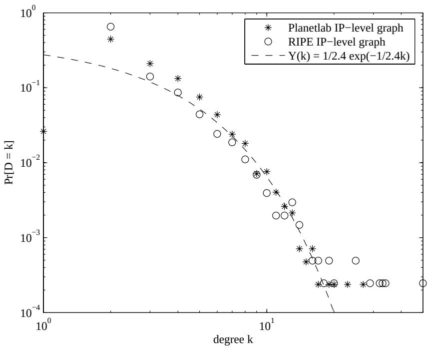

Fig. 1. The degree distribution of an observable part of the IP-level Internet graph, performed via RIPE and Planetlab.

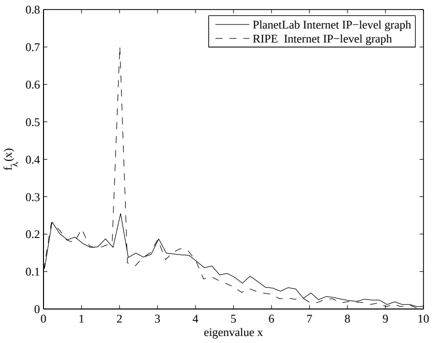

Fig. 2. The Laplacian spectrum of an observable part of the IP-level Internet graph, performed via RIPE and Planetlab.

spectral behavior.

III. Spectra of Complex Network Models

We have performed a comprehensive set of simulations to compare the Laplacian spectrum of the two observed IP-level Internet graphs to the spectra of generic complex network models [1]. Prior to analyzing the Laplacian spectra, we define and briefly discuss simulation models.

A. Random Graph of Erdős-Rényi

In this set of simulations we consider both realizations of the Erdős-Rényi random graph (for details see [4]), $G_p(N)$ and $G(N, L)$, with $N = 50, 100, 200$ and 400. For $G_p(N)$, the probability[1] of having a link between any two nodes (link probability p) is $p \geq p_c$ and $p_c = \frac{\log N}{N}$, so that the total number of links on average is equal to $pL_{\max}$. For $G(N, L)$, the number of links L in a graph is precisely equal to $p\binom{N}{2}$. In particular, in both realizations of the random graph, the corresponding link probability p is equal to $p = p_c\alpha$, where α ranges from 1 to 10. Furthermore, for each combination of N and p (for $G_p(N)$) or N and $L = p\binom{N}{2}$ (for $G(N, L)$), we have simulated 10^4 independent configurations of the random graph. For each independent configuration, the set of N eigenvalues of the Laplacian matrix has been computed, leading eventually to the Laplacian spectrum, created by picking at random one out of N eigenvalues.

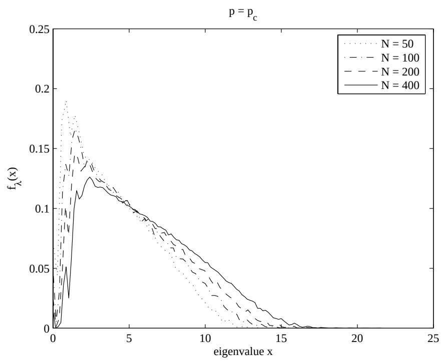

Fig. 3. The Laplacian spectrum of the Erdős-Rényi random graph with $N = 50, 100, 200, 400$ and $p = p_c$.

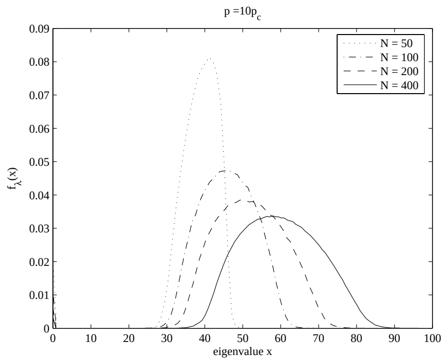

Fig. 4. The Laplacian spectrum of the Erdős-Rényi random graph with $N = 50, 100, 200, 400$ and $p = 10p_c$.

Figures 3, 4 show the Laplacian spectrum of $G_p(N)$ for the link probability $p = p_c$ and $p = 10p_c$, and for the increasing number of nodes N. The Laplacian spectrum of $G(N, L)$ with fixed number of links, i.e. $L = p_c\binom{N}{2}$ and $L = 10p_c\binom{N}{2}$, and for increasing N turns out to be indistinguishable from the spectrum of $G_p(N)$. Therefore, we further consider only the spectrum of $G_p(N)$. At the critical threshold probability $p = p_c$, there exists random graphs that are not connected. If $\lambda_{N-i+1} = 0$ and $\lambda_{N-i} \neq 0$ of Q, then a graph G has exactly i components. Therefore, by inspecting that $\lambda_{N-1} \neq 0$, we have considered only connected Erdős-Rényi random graphs.

With $p = p_c$, the spectrum is skewed with the main bulk pointing towards the small eigenvalues. Such behavior of a Laplacian spectrum is often found in cases where the topology has a tree-like structure. An extreme case of such type of structure is the star $K_{1,N-1}$, where the eigenvalues are N, 0 and 1 (with multiplicity $N-2$). In order to examine this in more detail, we plot the spectrum of the MST, found in each of 10^4 independent configurations of $G_p(N)$. Figure 5 shows that the spectrum of the MST in $G_p(N)$ is indeed highly skewed with the main bulk pointing towards the small eigenvalues. In particular, the underlying tree with degree 1 nodes is responsible for the peak at $\lambda = 1$. The spectrum of sparse $G_p(N)$ shows a similar behavior at small eigenvalues (see Figure 3), what can be interpreted as the structure that is mainly determined by the underlying tree. Such behavior is more obvious at smaller N, since the larger graph size cause an increase in the link density. More important is that the maximum λ_{N-1} of a tree on $N \geq 3$ is 1 and $\lambda_{N-1} = 1$ if and only if the underlying graph is the star $K_{1,N-1}$. At the other extreme, the minimum λ_{N-1} occurs at the path P_N, namely $\lambda_{N-1}(P_N) = 2\left[1 - \cos\left(\frac{\pi}{N}\right)\right]$. Thus, roughly speaking, λ_{N-1} decreases as the diameter increases [9]: for the MST in

[1] The value of the link probability p above which a random graph almost surely becomes connected tends, for large N, to $p_c \sim \frac{\log N}{N}$ (for details see [13]).

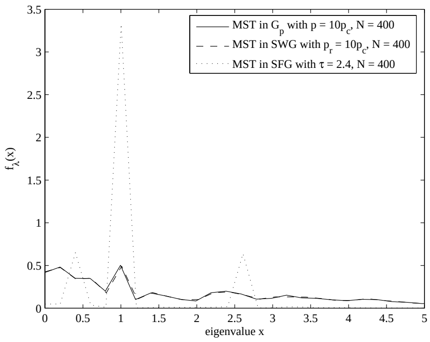

Fig. 5. The Laplacian spectrum of the MST in the Erdős-Rényi random graph, the Watts and Strogatz small-world graph and the Havel-Hakimi scale-free graph, all with $N = 400$.

$G_{p_c}(N)$, $\lambda_{N-1} \ll 1$ while for sparse $G_p(N)$, $\lambda_{N-1} < 1$, implying that the underlying tree-like structure of a sparse $G_p(N)$ has a small diameter.

With $p = 10p_c$, the spectrum has a bell shape (see Figure 4), centered around the mean nodal degree $E[D] = p(N-1)$. Moreover, for fixed $p = 10p_c$, the high peak becomes smaller while the bell shape becomes wider, representing that, for increasing N, the spectrum variance is in agreement with the Wigner's Semicircle law [20]. In fact, the spectrum is pointing to uncorrelated randomness what is a characteristic property of an Erdős-Rényi random graph [20]. Hence, the Laplacian spectra are indicating that, for the increasing link density, the underlying structure of $G_p(N)$ graphs transforms from a tree-like structure with a small diameter into a more homogeneous graph where the degree is closely centered around the mean degree.

B. Small-World Graph of Watts and Strogatz

In this set of simulations we consider exclusively the Watts and Strogatz small-world graph [22], built on the ring lattice $C(N,k)$ with $N = 50, 100, 200$ and 400. For each graph size N, every node is connected to its first $2k$ neighbors (k on either side). In order to have a sparse but connected graph, we have considered $N \gg 2k \gg \ln N$ in the following ring lattice graphs: $C(50,4)$, $C(100,8)$, $C(200,16)$, $C(400,32)$. The small-world model is then created by moving, with probability p_r, one end of each link to a new location chosen uniformly in the ring lattice, except that no double links or self-edges are allowed. The rewiring probability p_r equals the link probability in the random graph $G_p(N)$: it starts from $p_r = \frac{\log N}{N}$ and ends with $p_r = \frac{10 \log N}{N}$. Furthermore, for each combination of the graph size N, the neighbor size k and the rewiring probability p_r, we have simulated 10^4 independent configurations of the Watts and Strogatz small-work graph, leading eventually to the Laplacian spectrum by picking at random one out of N eigenvalues.

For the small rewiring probability $p_r = 0$ the Watts and Strogatz small-world graph is regular and also periodical. Because of the highly ordered structure, we see in Figure 6 that for small p_r the spectrum is highly skewed with the bulk towards the high eigenvalues, distributed around the mean nodal degree, which, irrespective of p_r equals $E[D] = 2k$. The spectrum of the two-dimensional lattice graph with $N \times N$ nodes aims to illuminate this effect. The Laplacian spectrum of the two-dimensional lattice is the sum of two path graphs P_N whose eigenvalues are $\lambda_i(P_N) = 2 - 2\cos(\pi i/N), i = 1, 2, ..., N$. Consequently, the spectrum of the two-dimensional lattice converges to a pointy shape with a peak centered around the mean nodal degree, which for $N \to \infty$, converges to 4. The same tendency is observable in the Watts and Strogatz small-world graph: in Figure 6, the bulk part, centered around the mean nodal degree, together with remaining peaks means that the graph is still highly regular and periodical. In fact, the Laplacian spectrum of the ring lattice $C(N,k)$ with N nodes and $2k$ neighbors comprises the eigenvalues $\lambda_i(C(N,k)) = 2k - \left(\frac{\sin(\frac{\pi}{N}(i-1)(2k+1))}{\sin(\frac{\pi}{N}(i-1))} - 1 \right), i = 1, 2, ..., N$. Hence, upon increasing k-regularity, the bulk part of the spectrum shifts towards the mean nodal degree, similar to the Laplacian spectrum of the Erdős-Rényi random graph. In order to examine this in more detail, we have calculated the fraction between the largest and the second smallest Laplacian eigenvalue. The fraction in the small-world graph with $p_r = \frac{\log N}{N}$ and $N = 400$ is approximately 4 times larger than the fraction in the small-world with $p_r = \frac{10 \log N}{N}$, indicating that the entire Laplacian spectrum of the small-world graph shifts towards λ_1. This transition of the bulk spectrum is known as the spectral phase transition phenomenon [17].

C. Scale-Free Graph derived from a Havel-Hakimi Power-Law Degree Sequence

In this set of simulations we consider a scale-free graph, which, for a given degree sequence, constructs a graph with a power-law degree distribution. Havel [12] and Hakimi

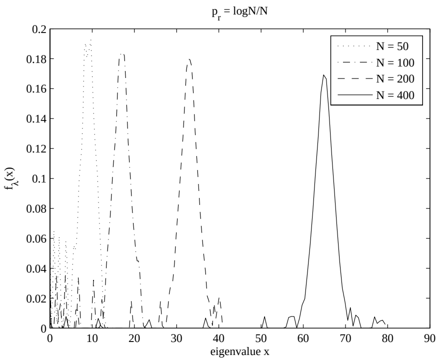

Fig. 6. The Laplacian spectrum of the Watts and Stogatz small-world graph with $N = 50, 100, 200, 400$ and $p_r = \frac{\log N}{N}$.

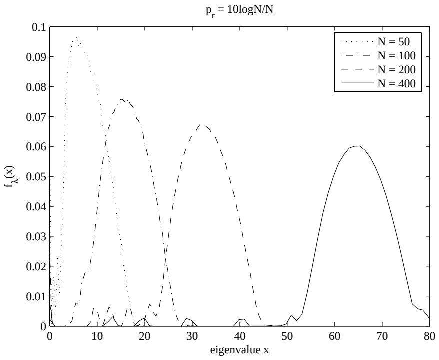

Fig. 7. The Laplacian spectrum of the Watts and Stogatz small-world graph with $N = 50, 100, 200, 400$ and $p_r = \frac{10 \log N}{N}$.

[11] proposed an algorithm that allows us to determine which sequences of nonnegative integers are degree sequences of graphs. In other words, in the limit of large N, this model, as shown in Figure 8, will have degree distribution with a power-law tail, $\Pr[D_i = k] \approx ck^{-\tau}$, where $c \approx (\zeta(\tau))^{-1}$ and the exponent τ typically lies in the interval between 2 and 3. In order to have a graph, which is in agreement with the real-world networks [16], we have used the exponent $\tau = 2.4$. Then, for each combination of the graph size N and the exponent τ, we have simulated 10^4 independent configurations of the Havel-Hakimi scale-free graph, leading eventually to the Laplacian spectrum by picking at random one out of N eigenvalues.

As shown in Figure 9, the spectrum of the Havel-Hakimi scale-free graph is completely different from the spectra of the other two complex network models. Because of the highly centralized structure, the spectrum in Figure 9 is skewed with the bulk towards the small eigenvalues. Recall that the Laplacian spectrum of the star $K_{1,N-1}$ is N, 0 and 1 (with multiplicity $N-2$). Consequently, the spectrum is indicating that an underlying structure of this type of the scale-free graph is a star-like structure with few highly connected nodes: although peaks at $\lambda = 2$ and $\lambda = 3$ have vanished, the MST found in the Havel-Hakimi scale-free exhibits a visually similar spectrum (see Figure 5). This means that most likely peaks in a spectrum, exemplified here with the peak at $\lambda = 1$, are due to the majority of nodes with the corresponding degrees.

Moreover, for the connected graph, the product of the non-zero Laplacian eigenvalues equals N times the number of Spanning Trees (ST) found in the corresponding graph [15]. From the simulation results we have found that the number of ST in sparse $G_p(N)$ is much higher than the number found in the Havel-Hakimi scale-free graph. In addition, we have found that the sum of the eigenvalues in $G_p(N)$ that equals the sum of the degrees, i.e. $\sum_{i=1}^{N} \lambda_i = \sum_i D_i$, is about double the sum of the eigenvalues found in the Havel-Hakimi scale-free graph. Also, the largest Laplacian eigenvalue [15], which is bounded by $\left[\frac{N}{N-1} D_{\max}, 2D_{\max}\right]$, grows approximately with N. Hence, the structure of this type of a scale-free graph is highly concentrated around nodes with very large nodal degrees.

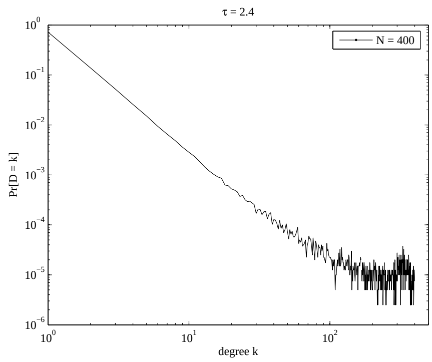

Fig. 8. The degree distribution of the Havel-Hakimi scale-free graph with $N = 400$ and $\tau = 2.4$.

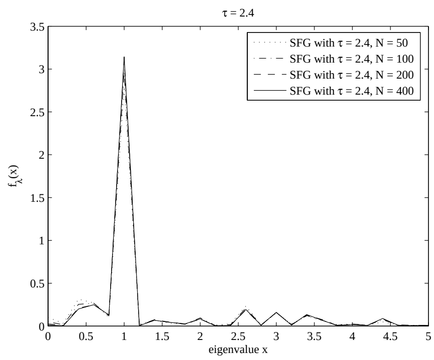

Fig. 9. The Laplacian spectrum of the Havel-Hakimi scale-free graph with $N = 50, 100, 200, 400$ and $\tau = 2.4$.

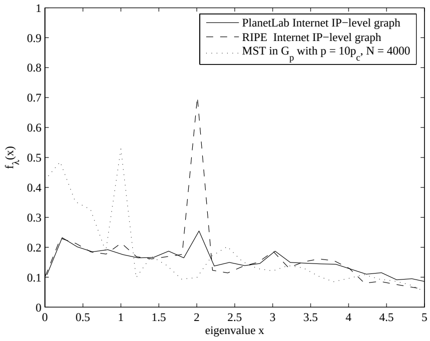

Fig. 10. The Laplacian spectra of the two observed IP-level Internet graphs along with MST in the Erdős-Rényi random graph with $N = 4000$ and $p = p_c$.

IV. Conclusion

In this paper, we have presented the Laplacian spectrum of an observable part of the Internet IP-level topology, which was extracted from traceroute measurements performed via RIPE and PlanetLab. In order to investigate the factors influencing the Laplacian spectrum of the two observed graphs, we presented the following complex network models: the random graph of Erdős-Rényi, the small-world of Watts and Strogatz and the scale-free graph derived from a Havel-Hakimi power-law degree sequence. Along with these three complex network models, we also presented the corresponding Minimum Spanning Tree (MST).

Extensive simulations show that the Laplacian spectra of the observed Internet graphs differ substantially from the spectra of generic complex networks models. Also, we found that the Erdős-Rényi random and the Watts and Strogatz small-world graph show a similar spectral behavior, which differs considerably from that of the scale-free graph, derived from a Havel-Hakimi power-law degree sequence. Despite this discrepancy, Figure 10 illustrates that the spectrum of the MST in the Erdős-Rényi random graph with uniformly distributed link weights does bear resemblance to the spectra of the observed graphs. In fact, the peak at $\lambda = 1$ in the Laplacian spectrum of the MST in the Erdős-Rényi random graph is mainly to due to the simple tree structure where the majority of nodes has degree 1. The Laplacian spectrum of the observed graphs seems to give support to this conjecture, since the peak at some particulair eigenvalue (e.g. the peak at $\lambda = 2$) most likely originates from the majority of nodes with the corresponding degree. This resemblance in spectra could be due to the fact that the observed part of the Internet graph is a subgraph of the complete observable path, just as the MST is.

V. Acknowledgements

We would like to thank Xiaoming Zhou for providing us with the measurements data. This research is funded by Next Generation Infrastructures (www.nginfra.nl).

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