Weight of a link in a shortest path tree and the Dedekind Eta function

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Abstract

The weight of a randomly chosen link in the shortest path tree on the complete graph with exponential i.i.d. link weights is studied. The corresponding exact probability generating function and the asymptotic law are derived. As a remarkable coincidence, this asymptotic law is precisely the same as the distribution of the cost of one "job" in the random assignment problem. We also show that the asymptotic (scaled) maximum interattachment time to that shortest path tree, which is a uniform recursive tree, equals the square of the Dedekind Eta function, a central function in modular forms, elliptic functions and the theory of partitions.

1 Introduction

Here and as motivated in [21, Section 16.1], we assign i.i.d. exponentially distributed weights with unit mean to links in the complete graph K_{N+1} with $N+1$ nodes. The shortest path between two nodes is the path for which the sum of the link weights is minimal. The shortest path tree (SPT) rooted at a random node to the N other nodes is the union of the shortest paths from that node to all N different nodes in the graph. The precise physical meaning of the link weight is irrelevant here – it can refer to distance, delay, monetary cost, etc. –, as long as the weights are additive. The confinement to the complete graph and i.i.d. exponential link weights has resulted in a shortest path tree that is a *uniform recursive tree (URT)* as outlined in Section 2. To a good approximation, the URT is also the shortest path tree in connected Erdös-Rényi random graphs (or dense homogeneous graphs) with i.i.d. regular[1] weights [21, Section 16.1]. Extending the shortest path tree problem to non-homogeneous networks or non-i.i.d. link weights seems exceedingly difficult, which justifies to extend the URT-model as far as possible.

Items (e.g. packets, information) in real-world networks are most often transported along shortest paths. Inferring the whole network topology (see e.g. [23]) via measurements based on transport is inherently biased, because mainly shortest path links are observed. Here, we show that the distribution of a uniformly chosen or random link weight in a shortest path tree is significantly different than that of

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[1] A *regular* distribution has a power series around zero.

a random link in the complete network. In particular, the weight of a random link in the URT and the maximum of the interattachment times to the URT are studied. We found that both are close to each other, at least in average and variance. The asymptotic laws of both are derived. The asymptotic (scaled) maximum interattachment time to the URT equals, apart from an exponential factor, the square of the Dedekind Eta function. The Dedekind Eta function plays an important role in modular forms, elliptic functions and in number theory, in particular, in the theory of partitions (see e.g. [3, 15, 4]). This observation complements Wästlund's excitement [11] about the frequent appearance of the Riemann Zeta function (at integer values) in asymptotic results of the URT. In addition to those shown in Section 2 and [21, Chapter 16], the pdf in (23) and the "Dedekind distribution" in (25) join the list of limit distributions, other than a Gaussian, that characterize properties of this shortest path tree.

There is even a more remarkable fact[2]: my exotically looking asymptotic pdf (23) of the weight of an arbitrary link in the URT was earlier derived by Aldous [2] in the context of the random assignment problem. The similarity of the shortest path and the random assignment problem has been explored by Wästlund in [25].

The paper is outlined as follows. Section 2 introduces the SPT and shows that it is a URT. Since this paper builds upon and complements properties of the shortest path tree in a complete graph with i.i.d. exponential link weights, we briefly overview earlier results on both the shortest path (in Section 2.1) and the shortest path tree (in Section 2.2). A brief discussion on the random assignment problem is incorporated in Section 2.3. After this short review in Section 2, we derive in Section 3 the probability generating function (pgf) of the weight of a random link. The exact pgf in any finite URT of size $N + 1$ is stated in Theorem 1, from which the mean weight of a random link is derived (in three ways) in Section 3.1 and the corresponding variance in Section 3.2. Section 3.3 focuses on the asymptotic law of the weight of a random link in infinitely large URTs. Theorem 2 provides the asymptotic pgf, that can be expressed in terms of the Hurwitz Zeta function (18), that generalizes the Riemann Zeta function. The corresponding pdf is derived in Corollary 1. Any moment in (22) of the asymptotic scaled weight of a random link in the URT, that elegantly follows from properties of the Hurwitz Zeta function, is proportional to the Riemann Zeta function at integer values, again illustrating the intriguing pervasiveness of the Riemann Zeta function.

Section 4 concentrates on the distribution of the minimum, but mainly of the maximum of the interattachment times to the URT. As shown in (1), the weight of a random link can be written as a sum of interattachment times. Theorem 3 gives the asymptotic law of the maximum interattachment time to the URT, while Section 5 expresses this asymptotic law in terms of the Dedekind function. Section 5.1 relates the coefficients of the Taylor series of the asymptotic maximum interattachment time to the URT to number theoretic functions, such as the partition function and the divisor function. The sequel of Section 5 invokes the remarkably powerful modular transforms to compute the asymptotic random variable's moments (in Section 5.2) and to derive (in Section 5.3) fast converging series for the pgf of the maximum interattachment time. The appendix contains analytic proofs.

[2] Just before the publication of this article, Johan Wästlund has informed me about the curious relation between the shortest path and the random assignment problem. Fortunately, the editor of RSA has still given me the opportunity to incorporate Section 2.3 that outlines the connection with the random assignment problem.

2 The URT as shortest path tree

In [21, 18], we have rephrased the shortest path problem between two random nodes in the complete graph K_{N+1} with exponential link weights as a Markov discovery process, that starts the path searching process at the source. The discovery process is a continuous-time Markov chain with $N+1$ states. Each state n represents the n already discovered nodes (including the source node). If, at some stage in the Markov discovery process, n nodes are discovered, then the next node is reached with rate $n\,(N+1-n)$, which is the transition rate in the continuous time Markov chain. Since the discovering of nodes at each stage only increases n, the Markov discovery process is a pure birth process with birth rate $n\,(N+1-n)$. We call τ_n the interattachment time between the inclusion of the n-th and $(n+1)$-th node to the SPT for $n=1,\ldots,N$. By the memoryless property of the exponential distribution and the symmetry of the complete graph, a new node is added uniformly to an already discovered node. Hence, the resulting SPT to all nodes is exactly a uniform recursive tree (URT). A URT of size $N+1$ is a random tree rooted at some source node and where at each stage a new node is attached uniformly to one of the existing nodes until the total number of nodes is equal to $N+1$. As proved in [18] for large N, the URT is also asymptotically the SPT in the class of (connected) Erdös-Rényi random graphs with i.i.d. exponential link weights. We believe that, when the number N of nodes grows large, the URT is the asymptotic SPT in a larger class of graphs.

The interattachment time τ_n is exponentially distributed with parameter $n(N+1-n)$ as follows from the theory of Markov processes and the discovery time to the k-th discovered node from a source node (root) equals

$$v_k = \sum_{n=1}^{k} \tau_n \tag{1}$$

where $\tau_1, \tau_2, \cdots, \tau_k$ are independent, exponentially distributed random variables with parameter $n(N+1-n)$ with $1 \leq n \leq k$. The discovery time v_k also equals the length or total weight from the root towards the k-th attached (discovered) node.

2.1 Overview of results on the shortest path

The weight W_{N+1} of a shortest path from a root to a random node in K_{N+1} with i.i.d. exponential link weights with mean 1 can be represented as

$$W_{N+1} = \sum_{k=1}^{N} v_k 1_{\{\text{node } k \text{ is the end node of the shortest path}\}}$$

where it is understood that the end node is different from the root. The pgf $\varphi_{W_{N+1}}(z) = E\left[e^{-zW_{N+1}}\right]$ of the weight W_{N+1} of the shortest path is derived [21, Chapter 16] from (1) as

$$\varphi_{W_{N+1}}(z) = \frac{1}{N} \sum_{k=1}^{N} \prod_{n=1}^{k} \frac{n(N+1-n)}{z+n(N+1-n)} \tag{2}$$

The pgf $\varphi_{H_{N+1}}(z) = E\left[z^{H_{N+1}}\right]$ of hopcount H_{N+1}, the number of links in the URT, is a classical result, also derived in [21, Chapter 16],

$$\varphi_{H_N+1}(z) = \frac{1}{(N+1)!} \prod_{k=1}^{N} (z+k) = \frac{\Gamma(N+1+z)}{\Gamma(N+2)\Gamma(z+1)}$$

The joint generating function $E[s^{H_{N+1}}e^{-tW_{N+1}}]$ of the hopcount H_{N+1} and the weight W_{N+1} of a shortest path is computed in [8] as

$$E[s^{H_{N+1}}e^{-tW_{N+1}}] = \frac{1}{N}\sum_{k=1}^{N}\left(\prod_{n=1}^{k}\frac{n(N+1-n)}{t+n(N+1-n)}\right)\frac{\Gamma(k+s)}{k!\Gamma(s)} \tag{3}$$

In the anycast problem, the shortest path from a root to a set of m uniformly chosen nodes is computed [22]. In the special case $m=1$, we have shown that the asymptotic law of the weight of a shortest path is

$$\lim_{N\to\infty}\Pr\left[NW_N - \ln N \leq t\right] = 1 - e^{-t}e^{e^{-t}}\int_{e^{-t}}^{\infty}\frac{e^{-u}}{u}du \tag{4}$$

Janson was the first one to compute the asymptotics of $NW_N - \ln N$ in [10], where he gave a short proof that $NW_N - \ln N$ converges in distribution to the convolution of the Gumbel distribution with the logistic distribution, which is the difference of two independent Gumbel random variables, related by the reflection formula for the Gamma function

$$\frac{\pi x}{\sin \pi x} = \Gamma\left(1+x\right)\Gamma\left(1-x\right),$$

since $\Gamma\left(1+x\right)$ is the pgf of a Gumbel random variable. Hence, Janson showed that

$$NW_N - \ln N \xrightarrow{d} V_1 + V_2 - V_3, \tag{5}$$

where V_1, V_2 and V_3 are independent Gumbel distributed random variables.

2.2 The shortest path tree versus the minimal spanning tree

For large N, the asymptotic average weight of the complete shortest path tree is

$$E\left[W_{\text{SPT}}\right] = \zeta\left(2\right) + O\left(\frac{1}{N}\right) \tag{6}$$

and the corresponding result for the variance is

$$\text{Var}\left[W_{\text{SPT}}\right] = \frac{4\zeta\left(3\right)}{N} + o\left(\frac{1}{N}\right) \tag{7}$$

while the scaled weight of the SPT tends to a Gaussian. In particular,

$$\sqrt{N}\left(W_{\text{SPT}} - \zeta\left(2\right)\right) \xrightarrow{d} N\left(0, \sigma^2_{\text{SPT}}\right)$$

where $\sigma^2_{\text{SPT}} = 4\zeta\left(3\right) \simeq 4.80823$. These results are established in [20].

Earlier Frieze [6] has computed the average weight of the *minimum* spanning tree W_{MST} in the complete graph with exponential with mean 1 link weights for large N as

$$E\left[W_{\text{MST}}\right] \to \zeta\left(3\right)$$

Janson [9], and later Wästlund [24] and Janson and Wästlund [11], completed Frieze's result by proving that the scaled weight of the MST tends to a Gaussian,

$$\sqrt{N}\left(W_{\text{MST}} - \zeta\left(3\right)\right) \xrightarrow{d} N\left(0, \sigma^2_{\text{MST}}\right),$$

where $\sigma^2_{\mathrm{MST}} = 6\zeta(4) - 4\zeta(3) \simeq 1.6857$.

The shortest path tree rooted at an arbitrary node is an instance of a spanning tree, but whose weight W_{SPT} is generally larger than that of the minimum spanning tree W_{MST}. The shortest path tree is often used as a heuristic to approximate the Steiner tree, for example, in multicasting. As discussed in [21, p. 399], the ratio $\frac{\zeta(2)}{\zeta(3)} \simeq 1.367$ indicates that the use of the SPT never performs, on average, more that 37% worse than the optimal Steiner tree.

2.3 The random assignment problem

The random assignment problem, as explained in [2], is the stochastic variant of the following task: choose an assignment of N jobs to N machines with the objective to minimize the total cost of performing the N jobs, given the $N \times N$ matrix C where the element c_{ij} equals the cost of performing job i on machine j. The assignment problem thus consists of determining the permutation π that minimizes the sum $\sum_{j=1}^{N} c_{j,\pi(j)}$. Probability enters in the most simple setting when the elements c_{ij} are i.i.d. exponentially random variables with mean 1. The corresponding random assignment problem (RAP) investigates the properties of the random variable $R_N = \min_\pi \sum_{j=1}^{N} c_{j,\pi(j)}$.

The RAP has a long history of which parts are overviewed in [2] and [25]. Here, we only illustrate the remarkable similarity with the shortest path tree problem. A basic result and analog of (6) is

$$E[R_N] = \sum_{k=1}^{N} \frac{1}{k^2} \to \zeta(2) \tag{8}$$

which was asymptotically proved by Aldous [2], and for finite N, independently, by Linusson and Wästlund [12] and by Nair, Prabhakar and Sharma [13]. Earlier, Parisi [14] had conjectured (8) based on simulations and Coppersmith and Sorkin [5] have extended the conjecture to partial assignments.

In [2], Aldous also shows that $c(1, \pi(1))$ converges, for large N, to the pdf given in the right-hand side of (23) below, the (scaled) density of the weight of an arbitrary link in the URT. In addition, Aldous shows that

$$\lim_{N\to\infty} \Pr[c(1,\pi(1)) \text{ is } k\text{-th smallest of the entries } c_{11}, c_{12}, \ldots, c_{1N}] = 2^{-k}$$

which is the asymptotic analogon of the probability that the degree of a node in the URT is k (see e.g. [21, p. 369]).

All these asymptotic equivalences suggest that the underlying tree structure in both problems, the Poisson-weighted infinite tree (PWIT) in the RAP in [2] and the URT in the shortest path tree problem, have asymptotically many same properties, although both trees are not precisely equal. The overwhelming similarity between these two different problems is striking and may be worth exploring in more depth.

3 Distribution of the weight of a link in the shortest path tree

A URT U consisting of $N+1$ nodes and with the root labeled by zero can be represented as

$$U = (0 \longleftarrow 1)(n_2 \longleftarrow 2)\ldots(n_N \longleftarrow N) \tag{9}$$

where $(n_j \longleftarrow j)$ means that the j-th discovered node is attached to node $n_j \in [0, j-1]$. Hence, n_j is the predecessor of j and this relation is indicated by $\longleftarrow$. The total number of URTs with $N+1$ nodes equals $N!$. The weight W_{SPT} of a shortest path tree from the root 0 to all N other nodes, which is a URT U, is with (1) and $v_0 = 0$ and $n_1 = 0$,

$$\begin{aligned} W_{\text{SPT}} &= (v_1 - v_0) + (v_2 - v_{n_2}) + \ldots + (v_N - v_{n_N}) \\ &= \sum_{j=1}^{N} \left(v_j - v_{n_j}\right) = \sum_{j=1}^{N} \sum_{n=n_j+1}^{j} \tau_n \end{aligned}$$

In the URT, the integer n_j is uniformly distributed over the interval $[0, j-1]$. It is more convenient to use a discrete uniform random variable on $[1, j]$ which we define as $A_j = n_j + 1$. We rewrite

$$W_{\text{SPT}} = \sum_{j=1}^{N} \sum_{n=A_j}^{j} \tau_n = \sum_{j=1}^{N} \sum_{n=1}^{j} 1_{\{A_j \leq n\}} \tau_n = \sum_{n=1}^{N} \tau_n \left(\sum_{j=n}^{N} 1_{\{A_j \leq n\}} \right)$$

The set $\{A_j\}_{1 \leq j \leq N}$ are independent random variables with $\Pr[A_j = k] = \frac{1}{j}$ for $k \in [1, j]$. In addition, we define for $n \in [1, N]$ the random variables

$$B_n = \sum_{j=n}^{N} 1_{\{A_j \leq n\}} \tag{10}$$

to obtain

$$W_{\text{SPT}} = \sum_{n=1}^{N} B_n \tau_n \tag{11}$$

The n random variables $B_1, B_2, \ldots, B_n$ are dependent. The mean of the random variable $B_n \leq \sum_{j=n}^{N} 1 = N + 1 - n$ follows from (10) as

$$E[B_n] = \sum_{j=n}^{N} E\left[1_{\{A_j \leq n\}}\right] = \sum_{j=n}^{N} \Pr[A_j \leq n] = \sum_{j=n}^{N} \frac{n}{j} \tag{12}$$

Let w^* denote the weight of a random link $n_j \longleftarrow j$ in the URT consisting of $N+1$ nodes, then we have

$$w_j^* = v_{n_j} - v_j = \sum_{n=n_j+1}^{j} \tau_n = \sum_{n=A_j}^{j} \tau_n = \sum_{n=1}^{j} 1_{\{A_j \leq n\}} \tau_n \tag{13}$$

We can thus write the weight w^* of a random link as the random variable,

$$\begin{aligned} w^* &= \sum_{j=1}^{N} w_j^* 1_{\{n_j \text{ is end node}\}} \\ &= \sum_{j=1}^{N} 1_{\{n_j \text{ is end node}\}} \sum_{n=1}^{j} 1_{\{A_j \leq n\}} \tau_n \end{aligned} \tag{14}$$

We now compute the weight of a random link in the shortest path tree (or complete URT) from the root to all N other nodes in the graph.

Theorem 1 *The weight w^* of a random link in a URT consisting of $N+1$ nodes possesses the pgf*

$$E\left[e^{-zw^*}\right] = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\prod_{n=k}^{j}\frac{n\left(N+1-n\right)}{z+n(N+1-n)} \tag{15}$$

Proof: A random link is uniformly chosen out of the N links in the URT, which means that any node j has equal probability to be the end node of that link. Then, with the definition (13) applied to a specific URT U specified in (9),

$$E\left[e^{-zw^*}\middle| U\right] = \frac{1}{N}\sum_{j=1}^{N}E\left[\exp\left(-z\sum_{n=n_j+1}^{j}\tau_n\right)\right]$$

Since any URT is equally probable, unconditioning yields

$$E\left[e^{-zw^*}\right] = \frac{1}{N!}\sum_{n_1=0}^{0}\sum_{n_2=0}^{1}\cdots\sum_{n_N=0}^{N-1}\frac{1}{N}\sum_{j=1}^{N}E\left[\exp\left(-z\sum_{n=n_j+1}^{j}\tau_n\right)\right]$$

Since all τ_n are independent, we have

$$E\left[\exp\left(-z\sum_{n=n_j+1}^{j}\tau_n\right)\right] = \prod_{n=n_j+1}^{j}E\left[e^{-z\tau_n}\right] = \prod_{n=n_j+1}^{j}\frac{n\left(N+1-n\right)}{z+n(N+1-n)}$$

and

$$\begin{aligned}
E\left[e^{-zw^*}\right] &= \frac{1}{N!}\sum_{n_1=0}^{0}\sum_{n_2=0}^{1}\cdots\sum_{n_N=0}^{N-1}\frac{1}{N}\sum_{j=1}^{N}\prod_{n=n_j+1}^{j}\frac{n\left(N+1-n\right)}{z+n(N+1-n)} \\
&= \frac{1}{N!N}\sum_{n_1=0}^{0}\sum_{n_2=0}^{1}\cdots\sum_{n_N=0}^{N-1}\left(\prod_{n=n_1+1}^{1}\frac{n\left(N+1-n\right)}{z+n(N+1-n)}+\prod_{n=n_2+1}^{2}\frac{n\left(N+1-n\right)}{z+n(N+1-n)}\right. \\
&\quad \left.+\ldots+\prod_{n=n_N+1}^{N}\frac{n\left(N+1-n\right)}{z+n(N+1-n)}\right)
\end{aligned}$$

The j-th term $V_j(z)$ in the above expression for $E\left[e^{-zw^*}\right]$ is

$$\begin{aligned}
V_j(z) &= \frac{1}{N!N}\sum_{n_1=0}^{0}\sum_{n_2=0}^{1}\cdots\sum_{n_N=0}^{N-1}\prod_{n=n_j+1}^{j}\frac{n\left(N+1-n\right)}{z+n(N+1-n)} \\
&= \frac{1}{N!N}\sum_{n_j=0}^{j-1}\prod_{n=n_j+1}^{j}\frac{n\left(N+1-n\right)}{z+n(N+1-n)}\underbrace{\sum_{n_1=0}^{0}\sum_{n_2=0}^{1}\cdots\sum_{n_N=0}^{N-1}}_{\text{no sum over } n_j}1 \\
&= \frac{1}{jN}\sum_{n_j=1}^{j}\prod_{n=n_j}^{j}\frac{n\left(N+1-n\right)}{z+n(N+1-n)}
\end{aligned}$$

Hence, we arrive at (15). $\square$

3.1 The average weight of a random link in the URT

We first compute the average weight of a link in the shortest path tree as the negative of the derivative of (15) with respect to z and evaluated at $z=0$. This results in

$$\begin{aligned} E\left[w^*\right] &= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\sum_{n=k}^{j}\frac{1}{n(N+1-n)} \\ &= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\frac{1}{n(N+1-n)}\sum_{k=1}^{n}1 \\ &= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\frac{1}{(N+1-n)} \end{aligned} \tag{16}$$

Using the identity $\sum_{j=1}^{N}\frac{1}{j}\sum_{k=N+1-j}^{N}\frac{1}{k}=\sum_{j=1}^{N}\frac{1}{j^2}$ proved in [20], we arrive at

$$E\left[w^*\right]=\frac{1}{N}\sum_{n=1}^{N}\frac{1}{n^2}$$

This results can also obtained directly from $W_{\text{SPT}}=\sum_{j=1}^{N}w_j^*$ after taking the expectation since $E\left[W_{\text{SPT}}\right]=\sum_{n=1}^{N}\frac{1}{n^2}$ as first proved in [19], and refined in [20].

A third method starts from (14). Taking the expectation of both sides gives the average weight of a link in the SPT as

$$\begin{aligned} E\left[w^*\right] &= \sum_{j=1}^{N}E\left[1_{\{n_j \text{ is end node}\}}\sum_{n=1}^{j}1_{\{A_j\le n\}}\tau_n\right] \\ &= \frac{1}{N}\sum_{j=1}^{N}\sum_{n=1}^{j}E\left[1_{\{A_j\le n\}}\tau_n\right]=\frac{1}{N}\sum_{j=1}^{N}\sum_{n=1}^{j}E\left[1_{\{A_j\le n\}}\right]E\left[\tau_n\right] \\ &= \frac{1}{N}\sum_{j=1}^{N}\sum_{n=1}^{j}\frac{n}{j}\frac{1}{n\left(N+1-n\right)}=\frac{1}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\frac{1}{\left(N+1-n\right)} \end{aligned}$$

which is again (16).

3.2 The variance of the weight of a random link

The second moment is the second derivative of (15) with respect to z, evaluated at $z=0$,

$$\begin{aligned} E\left[\left(w^*\right)^2\right] &= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\left(\left(\sum_{n=k}^{j}\frac{1}{n(N+1-n)}\right)^2+\sum_{n=k}^{j}\frac{1}{(n(N+1-n))^2}\right) \\ &= \frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\sum_{n=k}^{j}\frac{1}{(n(N+1-n))^2}+\frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\sum_{n=k+1}^{j}\sum_{m=k}^{n-1}\frac{1}{nm(N+1-n)(N+1-m)} \end{aligned}$$

The first term is

$$\begin{aligned} T_1 &= \frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\sum_{n=k}^{j}\frac{1}{(n(N+1-n))^2}=\frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\sum_{k=1}^{n}\frac{1}{(n(N+1-n))^2} \\ &= \frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\frac{1}{n(N+1-n)^2} \end{aligned}$$

Using the identity, proved in [20],

$$2\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\frac{1}{n\left(N+1-n\right)^{2}}=\frac{1}{(N+1)^{2}}\left(\sum_{n=1}^{N}\frac{1}{n}\right)^{2}+\frac{3}{(N+1)^{2}}\sum_{n=1}^{N}\frac{1}{n^{2}}+\frac{2}{(N+1)}\left(2\sum_{k=1}^{N}\frac{1}{k^{3}}-\sum_{k=1}^{N}\frac{1}{k^{2}}\sum_{j=1}^{k}\frac{1}{j}\right)$$

we obtain

$$T_{1}=\frac{1}{N(N+1)^{2}}\left(\sum_{n=1}^{N}\frac{1}{n}\right)^{2}+\frac{3}{N(N+1)^{2}}\sum_{n=1}^{N}\frac{1}{n^{2}}+\frac{2}{N(N+1)}\left(2\sum_{k=1}^{N}\frac{1}{k^{3}}-\sum_{k=1}^{N}\frac{1}{k^{2}}\sum_{j=1}^{k}\frac{1}{j}\right)$$

The second term is

$$T_{2}=\frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{k=1}^{j}\sum_{n=k+1}^{j}\sum_{m=k}^{n-1}\frac{1}{nm(N+1-n)(N+1-m)}$$

Reversing the k- and n- sum, followed by a reversal of the k- and m-sum, yields

$$\begin{aligned}T_{2}&=\frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=2}^{j}\sum_{k=1}^{n-1}\sum_{m=k}^{n-1}\frac{1}{nm(N+1-n)(N+1-m)}\\&=\frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=2}^{j}\sum_{m=1}^{n-1}\frac{1}{nm(N+1-n)(N+1-m)}\sum_{k=1}^{m}1\\&=\frac{2}{N}\sum_{j=1}^{N}\frac{1}{j}\sum_{n=2}^{j}\frac{1}{n(N+1-n)}\sum_{m=1}^{n-1}\frac{1}{N+1-m}\end{aligned}$$

Using the identity, proved in [20],

$$2\sum_{j=1}^{N}\frac{1}{j}\sum_{n=1}^{j}\frac{1}{n\left(N+1-n\right)}\sum_{m=1}^{n-1}\frac{1}{N+1-m}=\frac{2}{N+1}\sum_{k=1}^{N}\frac{1}{k^{2}}\sum_{j=1}^{k}\frac{1}{j}-\frac{3}{\left(N+1\right)^{2}}\sum_{k=1}^{N}\frac{1}{k^{2}}-\frac{1}{\left(N+1\right)^{2}}\left(\sum_{n=1}^{N}\frac{1}{n}\right)^{2}$$

we find

$$T_{2}=\frac{2}{N(N+1)}\sum_{k=1}^{N}\frac{1}{k^{2}}\sum_{j=1}^{k}\frac{1}{j}-\frac{3}{N\left(N+1\right)^{2}}\sum_{k=1}^{N}\frac{1}{k^{2}}-\frac{1}{N\left(N+1\right)^{2}}\left(\sum_{n=1}^{N}\frac{1}{n}\right)^{2}$$

Hence,

$$E\left[\left(w^{*}\right)^{2}\right]=T_{1}+T_{2}=\frac{4}{N(N+1)}\sum_{k=1}^{N}\frac{1}{k^{3}}$$

from which the variance $\mathrm{Var}[w^{*}]=E\left[\left(w^{*}\right)^{2}\right]-\left(E\left[w^{*}\right]\right)^{2}$ follows as

$$\mathrm{Var}[w^{*}]=\frac{4}{N(N+1)}\sum_{k=1}^{N}\frac{1}{k^{3}}-\frac{1}{N^{2}}\left(\sum_{n=1}^{N}\frac{1}{n^{2}}\right)^{2}\qquad(17)$$

In summary, for large N, the average link weight scales as $E\left[w^{*}\right]=\frac{1}{N}\frac{\pi^{2}}{6}\simeq\frac{1.64493}{N}$ with standard deviation $\sigma_{w*}=\sqrt{\mathrm{Var}[w^{*}]}$ equal to $\sigma_{w*}=\frac{\sqrt{4\zeta(3)-\zeta^{2}(2)}}{N}+O\left(N^{-2}\right)$ and $\sqrt{4\zeta\left(3\right)-\zeta^{2}\left(2\right)}\simeq1.44997$. Thus, the standard deviation is about the same as the mean itself.

3.3 Asymptotic distribution of w^*

Theorem 1 gives the exact pgf, from which we here derive the asymptotic distribution. A generalization of the Riemann Zeta function $\zeta(s)$ is the Hurwitz Zeta function, defined for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(a) \geq 0$ as

$$\zeta(s,a) = \sum_{k=1}^{\infty} \frac{1}{(a+k)^s} \tag{18}$$

which shows that $\zeta(s,0) = \zeta(s)$.

Theorem 2 *The weight w^* of a random link in an infinitely large URT possesses, for $|x| < 1$, the asymptotic pgf*

$$\lim_{N\to\infty} E\left[e^{-Nxw^*}\right] = \frac{1}{(x+1)^2} + x\sum_{j=1}^{\infty} \frac{1}{(x+1+j)^2 (x+j)} \tag{19}$$

that is written in terms of the Hurwitz Zeta function as

$$\lim_{N\to\infty} E\left[e^{-Nxw^*}\right] = 1 - x\zeta(2,x) \tag{20}$$

Proof: Following a similar procedure as in [22], we write

$$z + n(N+1-n) = \left(\sqrt{\left(\frac{N+1}{2}\right)^2 + z} + \frac{N+1}{2} - n\right)\left(\sqrt{\left(\frac{N+1}{2}\right)^2 + z} - (\frac{N+1}{2} - n)\right)$$

and define $y = \sqrt{(\frac{N+1}{2})^2 + z}$. Then,

$$\begin{aligned}
\prod_{n=k}^{j} \frac{n(N+1-n)}{z+n(N+1-n)} &= \frac{j!\,(N+1-k)!}{(k-1)!\,(N-j)!} \prod_{n=k}^{j} \frac{1}{(y+\frac{N+1}{2}-n)} \prod_{n=k}^{j} \frac{1}{(y-\frac{N+1}{2}+n)} \\
&= \frac{j!\,(N+1-k)!}{(k-1)!\,(N-j)!} \frac{\Gamma\left(y+\frac{N+1}{2}-j\right)}{\Gamma\left(y+\frac{N+1}{2}-k+1\right)} \frac{\Gamma\left(y-\frac{N+1}{2}+k\right)}{\Gamma\left(y-\frac{N+1}{2}+j+1\right)}
\end{aligned}$$

and, substituted in (15), yields

$$E\left[e^{-zw^*}\right] = \frac{1}{N}\sum_{j=1}^{N} \frac{(j-1)!\Gamma\left(y+\frac{N+1}{2}-j\right)}{(N-j)!\Gamma\left(y-\frac{N+1}{2}+j+1\right)} \sum_{k=1}^{j} \frac{(N+1-k)!}{(k-1)!} \frac{\Gamma\left(y-\frac{N+1}{2}+k\right)}{\Gamma\left(y+\frac{N+1}{2}-k+1\right)}$$

For large N and $|z| < N$, we have[3] that $y = \sqrt{(\frac{N+1}{2})^2 + z} \sim \frac{N+1}{2} + \frac{z}{N}$ such that

$$E\left[e^{-zw^*}\right] \sim \frac{1}{N}\sum_{j=1}^{N} \frac{(j-1)!\Gamma\left(N+\frac{z}{N}-j+1\right)}{(N-j)!\Gamma\left(\frac{z}{N}+j+1\right)} \sum_{k=1}^{j} \frac{(N+1-k)!}{(k-1)!} \frac{\Gamma\left(\frac{z}{N}+k\right)}{\Gamma\left(N+\frac{z}{N}-k+2\right)}$$

We now introduce the scaling $z = Nx$, where $|x| < 1$ since $|z| < N$,

$$E\left[e^{-Nxw^*}\right] \sim \frac{1}{N}\sum_{j=1}^{N} \frac{\Gamma(j)\,\Gamma(N+1-j+x)}{\Gamma(x+j+1)\,\Gamma(N+1-j)} \sum_{k=1}^{j} \frac{\Gamma(x+k)}{\Gamma(k)} \frac{\Gamma(N+1-k+1)}{\Gamma(N+1-k+x+1)} \tag{21}$$

[3] The notation $f(x) \sim g(x)$ for large x means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Let $m = N + 1 - k$ in the k-sum such that

$$E\left[e^{-Nxw^*}\right] \sim \frac{1}{N}\sum_{j=1}^{N}\frac{\Gamma(j)}{\Gamma(x+j+1)}\frac{\Gamma(N+1-j+x)}{\Gamma(N+1-j)}\sum_{m=N+1-j}^{N}\frac{\Gamma(m+1)}{\Gamma(m+x+1)}\frac{\Gamma(N+1-m+x)}{\Gamma(N+1-m)}$$

Introducing (46), proved in Lemma 4 in the appendix,

$$E\left[e^{-Nxw^*}\right] \sim \frac{\Gamma(N+x+2)}{N\Gamma(x)}\sum_{j=1}^{N}\frac{1}{\Gamma(x+j+1)}\frac{\Gamma(N+1-j+x)}{\Gamma(N+1-j)}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma(j+x+k+1)}{(x+k+1)}\frac{\Gamma(k+x)}{\Gamma(N+2x+k+2)}$$

$$= \frac{1}{N\Gamma(x)}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma(k+x)}{(x+k+1)}\frac{\Gamma(N+x+2)}{\Gamma(N+2x+k+2)}\sum_{j=1}^{N}\frac{\Gamma(x+j+1+k)}{\Gamma(x+j+1)}\frac{\Gamma(N+1-j+x)}{\Gamma(N+1-j)}$$

Using the alternative series (50) for the j-sum yields

$$E\left[e^{-Nxw^*}\right] \sim \frac{x}{N}\sum_{k=0}^{\infty}\Gamma(k+x)\Gamma(x+1+k)\frac{\Gamma(N+x+2)}{\Gamma(N+2x+k+2)}\sum_{j=0}^{N-1}\frac{1}{(k-j)!}\frac{\Gamma(N+x+1)}{\Gamma^2(x+2+j)\Gamma(N-j)}$$

$$= \frac{x}{N}\sum_{j=0}^{N-1}\frac{\Gamma(N+x+1)}{\Gamma^2(x+2+j)\Gamma(N-j)}\sum_{k=0}^{\infty}\frac{\Gamma(k+x)\Gamma(x+1+k)}{(k-j)!}\frac{\Gamma(N+x+2)}{\Gamma(N+2x+k+2)}$$

where the reversal is allowed by absolute convergence and convergence of the k-sum, because the terms decrease as $O\left(k^{j-2-N}\right)$ for large k. The latter is verified by applying the asymptotic expansion [1, 6.1.7] of the Gamma function, $\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1+O\left(z^{-1}\right)\right)$, for large $z = k$ and fixed a and b. We now apply this asymptotic expansion to the k-sum for large N,

$$E\left[e^{-Nxw^*}\right] \sim xN^{-x-1}\sum_{j=0}^{N-1}\frac{\Gamma(N+x+1)}{\Gamma^2(x+2+j)\Gamma(N-j)}\sum_{k=j}^{\infty}\frac{\Gamma(k+x)\Gamma(x+1+k)}{(k-j)!}N^{-k}\left(1+O\left(N^{-1}\right)\right)$$

$$= xN^{-x-1}\sum_{j=0}^{N-1}\frac{\Gamma(N+x+1)}{\Gamma^2(x+2+j)\Gamma(N-j)}\left\{\Gamma(j+x)\Gamma(x+1+j)N^{-j}+O\left(N^{-j-1}\right)\right\}$$

$$= xN^{-x-1}\sum_{j=0}^{N-1}\frac{\Gamma(N+x+1)\Gamma(j+x)\Gamma(x+1+j)}{\Gamma^2(x+2+j)\Gamma(N-j)}N^{-j}$$

$$+O\left(N^{-x-2}\sum_{j=0}^{N-1}\frac{\Gamma(N+x+1)}{\Gamma^2(x+2+j)\Gamma(N-j)}N^{-j}\right)$$

Again applying the asymptotic expansion of the Gamma function to the j-sum yields

$$E\left[e^{-Nxw^*}\right] = x\sum_{j=0}^{N-1}\frac{\Gamma(j+x)\Gamma(x+1+j)}{\Gamma^2(x+2+j)}+O\left(N^{-1}\right)$$

$$= x\sum_{j=0}^{N-1}\frac{1}{(x+1+j)^2(x+j)}+O\left(N^{-1}\right)$$

from which we obtain the asymptotic pgf (19) of Nw^*. Partial fraction decomposition gives

$$\sum_{j=0}^{N-1}\frac{1}{(x+1+j)^2(x+j)} = \sum_{j=0}^{N-1}\frac{1}{(x+j)} - \sum_{j=0}^{N-1}\frac{1}{(x+1+j)} - \sum_{j=0}^{N-1}\frac{1}{(x+1+j)^2}$$

$$= \frac{1}{x} - \frac{1}{x+N} - \sum_{j=1}^{N}\frac{1}{(x+j)^2}$$

and

$$E\left[e^{-Nxw^*}\right]=1-x\sum_{j=1}^{N}\frac{1}{(x+j)^2}+O\left(N^{-1}\right)$$

After taking the limit $N\to\infty$ and using the definition of the Hurwitz Zeta function (18), we find (20). $\square$

The Hurwitz form (20) of the pgf $E\left[e^{-Nxw^*}\right]$ is particularly well suited to compute all moments. We verify immediately from (20), by differentiation with respect to x evaluated at $x=0$, that

$$\lim_{N\to\infty}E\left[Nw^*\right]=\zeta(2)$$

More generally, since the k-th derivative of the Hurwitz Zeta function $\zeta(s,x)$ in (18) with respect to x is

$$\frac{d^k\zeta(s,x)}{dx^k}=(-1)^k\frac{\Gamma(s+k)}{\Gamma(s)}\zeta(s+k,x)$$

we obtain from (20), by Leibniz' differentiation rule,

$$\begin{aligned}\lim_{N\to\infty}E\left[(Nw^*)^k\right]&=(-1)^k\left.\frac{d^k}{dx^k}\left(1-x\zeta(2,x)\right)\right|_{x=0}\\&=(-1)^{k-1}\left.\sum_{j=0}^{k}\binom{k}{j}\frac{d^j}{dx^j}(x)\,\frac{d^{k-j}}{dx^{k-j}}\left(\zeta(2,x)\right)\right|_{x=0}\\&=(-1)^{k-1}\left.x\frac{d^k}{dx^k}\left(\zeta(2,x)\right)+k\frac{d^{k-1}}{dx^{k-1}}\left(\zeta(2,x)\right)\right|_{x=0}\end{aligned}$$

Thus, any k-th moment of the asymptotic random variable $\lim_{N\to\infty}Nw^*$ is

$$\lim_{N\to\infty}E\left[(Nw^*)^k\right]=kk!\zeta(k+1)\tag{22}$$

Corollary 1 *The asymptotic pdf of the weight w^* of a random link in an infinitely large URT is, for $t\geq 0$,*

$$\lim_{N\to\infty}f_{Nw^*}(t)=\frac{\left(t-1+e^{-t}\right)e^{-t}}{\left(1-e^{-t}\right)^2}\tag{23}$$

Proof: The inverse Laplace transform of (19) is

$$f_{Nw^*}(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{xt}dx}{(x+1)^2}+\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{xe^{xt}dx}{(x+1+j)^2(x+j)}$$

where $c>0$. When closing the contour over the negative $\mathrm{Re}(x)$-plane equals, we encounter a simple pole at $x=-j$ and a double pole at $x=-j-1$. By Cauchy residue theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{xe^{xt}dx}{(x+1+j)^2(x+j)}&=\lim_{x\to-j}\frac{xe^{xt}}{(x+1+j)^2}+\lim_{x\to-j-1}\frac{d}{dx}\frac{xe^{xt}}{(x+j)}\\&=(j+1)\,te^{-(j+1)t}-je^{-jt}\left(1-e^{-t}\right)\end{aligned}$$

Hence,

$$f_{Nw^*}(t)=\sum_{j=0}^{\infty}(j+1)\,te^{-(j+1)t}-je^{-jt}\left(1-e^{-t}\right)=\left(t-1+e^{-t}\right)\sum_{j=1}^{\infty}je^{-jt}$$

After computing the sum, we find (23). $\square$

Observe that direct application of the asymptotic expansion of the Gamma function, for large N and fixed x, into (21) is only valid for fixed j,

$$\sum_{k=1}^{j} \frac{\Gamma(x+k)}{\Gamma(k)} \frac{\Gamma(N-k+2)}{\Gamma(N-k+(x+2))} = \sum_{k=1}^{j} \frac{\Gamma(x+k)}{\Gamma(k)} N^{-x}\left(1+O\left(N^{-1}\right)\right)$$
$$= N^{-x}\frac{\Gamma(x+j+1)}{(x+1)\Gamma(j)} + O\left(N^{-x-1}\right)$$

By ignoring this restriction, we would erroneously deduce

$$E\left[e^{-Nxw^*}\right] \sim \frac{N^{-x-1}}{x+1}\sum_{j=1}^{N}\frac{\Gamma(N+x-j+1)}{\Gamma(N-j+1)} + O\left(N^{-x-2}\sum_{j=1}^{N}\frac{(j-1)!\Gamma(N+x-j+1)}{(N-j)!\Gamma(x+j+1)}\right)$$
$$\sim \frac{N^{-x-1}}{(x+1)^2}\frac{\Gamma(N+x+1)}{\Gamma(N)} + O\left(N^{-2}\sum_{j=1}^{N}\frac{\Gamma(j)}{\Gamma(j+x+1)}\right)$$

Since $\sum_{j=1}^{N}\frac{\Gamma(j)}{\Gamma(j+x+1)} = \frac{1}{x^2\Gamma(x)} - \frac{\Gamma(N+1)}{x\Gamma(N+x+1)} = O\left(N^{-x}\right)$, it is tempting to conclude that

$$E\left[e^{-Nxw^*}\right] \sim \frac{1}{(x+1)^2} + O\left(N^{-x-2}\right)$$

whose leading term is the pgf of the sum of two exponential random variables with mean 1. In this way, only the first term in the exact asymptotic pgf (19) is found, which shows that this heuristic derivation is therefore demonstrably incorrect.

4 Distribution of extreme interattachment times

Since the set of link weights in the URT, $\{(v_1 - v_0), (v_2 - v_{n_2}), \ldots, (v_N - v_{n_N})\}$, are dependent random variables, we focus in the sequel on the set of the independent interattachment times to the URT.

4.1 The minimum interattachment time to the URT

Since the set of $\{\tau_j\}_{1\le j\le N}$ are independent, exponential random variables with mean $\frac{1}{j(N+1-j)}$, the minimum [21, p. 51] is again exponentially distributed with mean $\frac{1}{\sum_{j=1}^{N} j(N+1-j)}$. From [15, pp. 1]

$$\sum_{j=1}^{N} j(N+1-j) = \frac{N(N+1)(N+2)}{3!} = \binom{N+2}{3}$$

we find that the pdf of the minimum interattachment time to the URT is

$$f_{\min_{1\le j\le N}\tau_j}(x) = \frac{N(N+1)(N+2)}{3!}\exp\left(-\frac{N(N+1)(N+2)}{3!}x\right)$$

The pdf of the minimum link weight in the whole complete graph K_{N+1}, where all link weights are i.i.d. with mean 1, is $f_{\min_{l\in K_{N+1}} w_l}(x) = \frac{N(N+1)}{2}\exp\left(-\frac{N(N+1)}{2}x\right)$, where w_l is the weight of link l. Hence, the minimum interattachment time can be significantly smaller than the minimum link weight in K_{N+1}, and thus in any URT.

4.2 The maximum interattachment time to the URT

Analogously, the maximum interattachment time to the URT satisfies

$$\Pr\left[\max_{1\leq j\leq N}\tau_j \leq x\right] = \prod_{j=1}^{N}\left(1-e^{-j(N+1-j)x}\right) \tag{24}$$

The remainder of the article is devoted to the maximum interattachment time to the URT $\tau_{\max} = \max_{1\leq j\leq N}\tau_j$. We start with the Theorem 3, from which the relation between $\tau_{\max}$ and w^* for large N will be deduced.

Theorem 3 *The asymptotic law of $\tau_{\max}$ for large N is*

$$\lim_{N\to\infty}\Pr\left[N\tau_{\max}\leq\xi\right] = \prod_{m=1}^{\infty}\left(1-e^{-\xi m}\right)^2 \tag{25}$$

We give two proofs. In Appendix B, we present an analytic proof that, beside the correct scaling $x=\frac{\xi}{N}$, actually gives a more precise result including an error term. Once the correct scaling law is known, we can verify Theorem 3 more elegantly. The short proof is given here.

Proof: We rewrite (24) as

$$\Pr\left[w_{\max}\leq x\right] = \prod_{j=1}^{a}\left(1-e^{-j(N+1-j)x}\right)\prod_{j=a+1}^{N}\left(1-e^{-j(N+1-j)x}\right)$$

Letting $k=N+1-j$ in the last product, then $1\leq k\leq N-a$

$$\prod_{j=1}^{N}\left(1-e^{-j(N+1-j)x}\right) = \prod_{j=1}^{a}\left(1-e^{-j(N+1-j)x}\right)\prod_{k=1}^{N-a}\left(1-e^{-(N+1-k)kx}\right)$$

Choosing $a=N-a$, results in $a=\frac{N}{2}$, where the integer nature of a for large N can be ignored. With the scaling $x=\frac{\xi}{N}$, we have

$$\begin{aligned}\Pr\left[w_{\max}\leq\frac{\xi}{N}\right] &= \prod_{j=1}^{\frac{N}{2}}\left(1-e^{-j\xi(1-\frac{j-1}{N})}\right)^2\\ &= \prod_{j=1}^{\alpha N^{\beta}-1}\left(1-e^{-j\xi(1-\frac{j-1}{N})}\right)^2\prod_{j=\alpha N^{\beta}}^{\frac{N}{2}}\left(1-e^{-j\xi(1-\frac{j-1}{N})}\right)^2\end{aligned} \tag{26}$$

where $\alpha>0$ and $0<\beta<1$ such that $1-\frac{j-1}{N}=1-o(N)$ in the first product. In the second product, all exponentials vanish when $N\to\infty$ such that

$$\lim_{N\to\infty}\prod_{j=\alpha N^{\beta}}^{\frac{N}{2}}\left(1-e^{-j\xi(1-\frac{j-1}{N})}\right)^2 = 1$$

Taking the limit $N\to\infty$ in (26) finally leads to (25). $\square$

5 The Dedekind Eta function

The Dedekind Eta function [15] is, for $t \in \mathbb{C}$ with $\operatorname{Im}(t) \geq 0$,

$$\eta(t) = e^{\pi i t/12} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i t m}\right) \tag{27}$$

and it obeys the modular transformation equations [15, p. 163],

$$\begin{aligned} \eta(t+b) &= e^{\pi i b/12} \eta(t) \\ \eta\left(\frac{at+b}{ct+d}\right) &= f(a,b,c,d) \sqrt{\frac{ct+d}{i}}\, \eta(t) \end{aligned}$$

where a, b, c, d are integers and $f(a,b,c,d)$ is specified in [15, p. 163] in terms the Legendre-Jacobi symbol of number theory. In particular,

$$\eta\left(-\frac{1}{t}\right) = \eta(t) \sqrt{\frac{t}{i}} \tag{28}$$

Let us denote the asymptotic random variable by $T_{\max} = \lim_{N\to\infty} N\tau_{\max}$. By (27) and (28), (25) can be expressed as

$$\Pr\left[T_{\max} \leq \xi\right] = \prod_{m=1}^{\infty} \left(1 - e^{-\xi m}\right)^2 = e^{\frac{\xi}{12}} \eta^2\left(\frac{\xi i}{2\pi}\right) \tag{29}$$

$$= \frac{2\pi}{\xi} e^{\frac{\xi}{12}} \eta^2\left(\frac{2\pi}{\xi} i\right) \tag{30}$$

Hence, a functional equation for the distribution of $T_{\max}$ follows as

$$\Pr\left[T_{\max} \leq 2\pi t\right] = \frac{e^{\frac{\pi}{6}\left(t - \frac{1}{t}\right)}}{t} \Pr\left[T_{\max} \leq \frac{2\pi}{t}\right]$$

We found the appearance of the Dedekind function in probability theory surprising since it may hint to a connection with analytic number theory. The sequel is devoted to compute the moments and pgf of the asymptotic random variable $T_{\max}$, by using amazing properties of the Dedekind Eta function.

5.1 Taylor series of $\Pr\left[T_{\max} \leq \xi\right]$

The computation of the Taylor series of $\Pr\left[T_{\max} \leq -\log q\right] = \prod_{m=1}^{\infty}(1-q^m)^2$ where $q = e^{-\xi}$ requires a brief review of some classical results. Euler discovered the series

$$\prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(3n+1)} \qquad |q| < 1 \tag{31}$$

from which the famous pentagonal number theorem is deduced [4, p. 124][15, p. 173]. Jacobi found that

$$\prod_{m=1}^{\infty} (1 - q^m)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{\frac{1}{2}n(n+1)} \qquad |q| < 1 \tag{32}$$

and the generating function of the number of partitions $p(m)$ is,

$$\prod_{m=1}^{\infty}(1-q^m)^{-1}=\sum_{n=0}^{\infty}p(n)\,q^n \qquad |q|<1 \tag{33}$$

where $p(0)=1$, $p(1)=1$, $p(2)=2$, $p(3)=3$, $p(4)=5$, $p(5)=7$, $p(6)=11$, after which the prime sequence stops since $p(7)=15$, $p(8)=22$ and so on. The "singularly happy" collaboration of Hardy and Ramanujan, as coined by Littlewood [3, Section 5.1] and narrated in [7], has led to a beautiful and remarkably accurate asymptotic formula for $p(m)$, that was later perfected by Rademacher (see [3, 15]). At last, we mention Ramanujan's tau function that features a similar functional equation as the Riemann Zeta function [7, Chapter X], defined by the generating function $\sum_{n=1}^{\infty}\tau(n)\,q^n = q\prod_{m=1}^{\infty}(1-q^m)^{24}$.

These results on integer powers of $\prod_{m=1}^{\infty}(1-q^m)$ provide several ways to compute the Taylor series of $\Pr[T_{\max}\leq -\log q]$. Multiplying (32), rewritten as a power series in q, and (33) yields

$$\begin{aligned}\prod_{m=1}^{\infty}(1-q^m)^2 &= \sum_{m=0}^{\infty}(-1)^k(2k+1)\,1_{\left\{m=\frac{k(k+1)}{2}\right\}}q^m\sum_{n=0}^{\infty}p(n)\,q^n\\ &=\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}p(m-n)(-1)^k(2k+1)\,1_{\left\{n=\frac{k(k+1)}{2}\right\}}\right\}q^m\end{aligned}$$

Hence, the Taylor expansion for $\Pr[T_{\max}\leq -\log q]$ is

$$\prod_{m=1}^{\infty}(1-q^m)^2=\sum_{m=0}^{\infty}a_m q^m \tag{34}$$

where the Taylor coefficients are all integers,

$$a_m=\sum_{k=0}^{\left[\frac{1}{2}\left(\sqrt{8m+1}-1\right)\right]}(-1)^k(2k+1)\,p\left(m-\frac{k(k+1)}{2}\right) \tag{35}$$

and $[x]$ denote the largest integer smaller of equal to x. We list the first 22 coefficients,

$$\begin{array}{lll@{\qquad}lll}
a_0 &=& 1 & a_1 &=& -2\\
a_2 &=& -1 & a_3 &=& 2\\
a_4 &=& 1 & a_5 &=& 2\\
a_6 &=& -2 & a_7 &=& 0\\
a_8 &=& -2 & a_9 &=& -2\\
a_{10} &=& 1 & a_{11} &=& 0\\
a_{12} &=& 0 & a_{13} &=& 2\\
a_{14} &=& 3 & a_{15} &=& -2\\
a_{16} &=& 2 & a_{17} &=& 0\\
a_{18} &=& 0 & a_{19} &=& -2\\
a_{20} &=& -2 & a_{21} &=& 0
\end{array}$$

These Taylor coefficients a_m of $\Pr[T_{\max}\leq -\log q]$ apparently lack a nice number theoretic interpretation and they form a less elegant series[4] than the Taylor coefficients of the first and third power of

[4] Taylor coefficients of $\prod_{m=1}^{\infty}(1-q^m)^x$ for any complex x can be computed exactly (in terms of the divisor function $\sigma(n)=\sum_{d|n}d$), which straightforwardly extends the machinery in the next subsection from the present case of interest $x=2$ to any x. Numerical evaluations of the first 100 Taylor coefficients for positive integers $x=k$ shows that none of the cases $k=2$ nor $k\in[4,100]$ exhibits a regular structure, only the cases $k=1$ and $k=3$. Order estimates of the high order Taylor coefficients at negative, integer values of k have been found by Meinardus (see e.g. [3, Chapter 6]).

$\prod_{m=1}^{\infty}(1-q^m)$.

Finally, we remark that the Taylor series (34) is, in fact, also a convergent asymptotic series for large ξ, such that

$$\begin{aligned}\Pr\left[T_{\max} \leq \xi\right] &= \prod_{m=1}^{\infty}(1-e^{-m\xi})^2 = \sum_{m=0}^{\infty} a_m e^{-m\xi} \\ &= 1 - 2e^{-\xi} - e^{-2\xi} + 2e^{-3\xi} + O\left(e^{-4\xi}\right)\end{aligned} \tag{36}$$

5.2 The moments of $T_{\max}$

We compute the Mellin transform using (25) and (34),

$$\begin{aligned}E\left[T_{\max}^{\beta}\right] &= \int_0^{\infty} \xi^{\beta} \frac{d}{d\xi}\left(\prod_{m=1}^{\infty}(1-e^{-\xi m})\right)^2 d\xi \\ &= \int_0^{\infty} \xi^{\beta} \frac{d}{d\xi}\left(\sum_{m=0}^{\infty} a_m e^{-\xi m}\right) d\xi \\ &= \sum_{m=1}^{\infty} a_m \left(-m\right) \int_0^{\infty} \xi^{\beta} e^{-\xi m} d\xi\end{aligned}$$

Thus[5],

$$E\left[T_{\max}^{\beta}\right] = -\Gamma\left(\beta+1\right)\sum_{m=1}^{\infty}\frac{a_m}{m^{\beta}} \tag{37}$$

is a Dirichlet series that converges for $\beta \geq 0$ as seen from (34) because $\sum_{m=1}^{\infty} a_m = -1$. Moreover, for large β, the Dirichlet series (37) reveals that

$$\frac{E\left[T_{\max}^{\beta}\right]}{\Gamma\left(\beta+1\right)} = 2 + 2^{-\beta} - \sum_{m=3}^{\infty}\frac{a_m}{m^{\beta}} = 2\left(1+2^{-\beta-1}\right) + O\left(3^{-\beta}\right) \tag{38}$$

A well-known fact of Dirichlet series is the slow convergence, especially for small values of β: For $m \in [1000, 2000]$, we found $E\left[\omega_{\max}\right] = 1.70 \pm 0.01$ and in $m \in [2000, 3000]$, $E\left[\omega_{\max}\right] = 1.703 + 0.005$ and, similarly, $\mathrm{Var}[\omega_{\max}] \approx 1.042$. One of the spectacular properties of the Dedekind transform (28) – essentially a modular transform that also is characteristic to Jacobi's theta-functions –, is that they enable amazingly fast convergence. Indeed, after partial integration,

$$E\left[T_{\max}^{\beta}\right] = \beta\int_0^{\infty} \xi^{\beta-1}\left(1 - \prod_{m=1}^{\infty}(1-e^{-\xi m})^2\right) dt$$

we write the integrand in terms of the Dedekind function (29),

$$E\left[T_{\max}^{\beta}\right] = \beta\left(2\pi\right)^{\beta}\int_0^{\infty} u^{\beta-1}\left(1 - e^{\frac{\pi u}{6}}\eta^2\left(iu\right)\right) du \tag{39}$$

[5] A power series $\sum_{m=0}^{\infty} b_m \left(e^{-x}\right)^m$, convergent for $x > 0$, may be multiplied by x^{s-1} where $s > 0$, and integrated term-by-term, provided only that the resulting series is convergent [16, Sec. 1.79, p. 47].

We split the integration interval as,

$$\int_0^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du = \int_0^c u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du + \int_c^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du$$
$$= \frac{c^\beta}{\beta} - \int_0^c u^{\beta-1}e^{\frac{\pi u}{6}}\eta^2(iu)\,du + \int_c^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du$$

and apply the transform (28) to the argument in the first integral

$$\int_0^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du = \frac{c^\beta}{\beta} - \int_0^c u^{\beta-2}e^{\frac{\pi u}{6}}\eta^2\left(\frac{i}{u}\right)du + \int_c^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du$$
$$= \frac{c^\beta}{\beta} - \int_{\frac{1}{c}}^\infty u^{-\beta}e^{\frac{\pi}{6u}}\eta^2(iu)\,du + \int_c^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du$$

The procedure followed here actually mimics Riemann's famous second derivation of the functional equation of the Riemann Zeta function [17, Chapter 2] using Jacobi's (third) theta-function.

Partial integration of the last integral results in

$$\int_c^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du = -\frac{c^\beta}{\beta}\left(1-e^{\frac{\pi c}{6}}\eta^2(ic)\right) + \frac{1}{\beta}\int_c^\infty u^\beta \frac{d}{du}\left(e^{\frac{\pi u}{6}}\eta^2(iu)\right)du$$

Hence,

$$\int_0^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du = \frac{c^\beta}{\beta}e^{\frac{\pi c}{6}}\eta^2(ic) - \int_{\frac{1}{c}}^\infty u^{-\beta}e^{\frac{\pi}{6u}}\eta^2(iu)\,du + \frac{1}{\beta}\int_c^\infty u^\beta \frac{d}{du}\left(e^{\frac{\pi u}{6}}\eta^2(iu)\right)du$$

Introducing (34) yields

$$\int_0^\infty u^{\beta-1}\left(1-e^{\frac{\pi u}{6}}\eta^2(iu)\right)du = \frac{c^\beta}{\beta}\sum_{m=0}^\infty a_m e^{-2\pi mc} - \sum_{m=0}^\infty a_m \int_{\frac{1}{c}}^\infty u^{-\beta}e^{\frac{\pi}{6u}}e^{-\left(\frac{\pi}{6}+2\pi m\right)u}du$$
$$-\frac{2\pi}{\beta}\sum_{m=1}^\infty m a_m \int_c^\infty u^\beta e^{-2\pi um}du$$

We proceed further for the moments, where $\beta = k > 0$ are positive integers, because then the last integral is

$$\int_c^\infty u^k e^{-2\pi um}du = \frac{1}{(2\pi m)^{k+1}}\int_{2\pi mc}^\infty x^k e^{-x}dx = \frac{\Gamma(k+1, 2\pi mc)}{(2\pi m)^{k+1}}$$
$$= \frac{k!e^{-2\pi mc}}{(2\pi m)^{k+1}}\sum_{j=0}^k \frac{(2\pi mc)^j}{j!}$$

where we have used the expansion of the incomplete Gamma function [1, 6.5]. Thus,

$$E\left[T_{\max}^k\right] = (2\pi c)^k\left\{1 - k!\sum_{m=1}^\infty \frac{a_m e^{-2\pi mc}}{(2\pi mc)^k}\sum_{j=0}^{k-1}\frac{(2\pi mc)^j}{j!}\right\} - k(2\pi)^k\sum_{m=0}^\infty a_m\int_{\frac{1}{c}}^\infty u^{-k}e^{\frac{\pi}{6u}}e^{-\left(\frac{\pi}{6}+2\pi m\right)u}du$$

Unfortunately, the remaining integral cannot be evaluated in closed form[6], but, it is bounded by

$$\int_{\frac{1}{c}}^\infty u^{-\beta}e^{\frac{\pi}{6u}}e^{-\left(\frac{\pi}{6}+2\pi m\right)u}du < e^{\frac{\pi c}{6}}c^\beta e^{-\left(\frac{\pi}{6}+2\pi m\right)\frac{1}{c}} < e^{\frac{\pi}{6}\left(c-\frac{1}{c}\right)}c^\beta$$

[6] There are results in terms of Bessel functions, but they are far less attractive than the integral itself. By partial integration, a recursion similarly to that of the incomplete Gamma function can be derived from which an asymptotic series is obtained.

Hence, by choosing c small enough for $\beta > 0$, the integral can be made sufficiently small to neglect. However, choosing c very small, enlarges the factor $\frac{e^{-2\pi mc}}{(2\pi mc)^k}$ and the need to evaluate more terms in m-sum to achieve a prescribed level of accuracy. A minimization of terms in the m-sum is achieved by making both integrals about equally large, thus, for $c = 1$. This choice – Riemann's original choice – expresses the average of all (integer powers) of $T_{\max}$ in terms of a fast converging series

$$E\left[T_{\max}^k\right] = (2\pi)^k \left\{1 - k! \sum_{m=1}^{\infty} \frac{a_m e^{-2\pi m}}{(2\pi m)^k} \sum_{j=0}^{k-1} \frac{(2\pi m)^j}{j!}\right\} - k\,(2\pi)^k \sum_{m=0}^{\infty} a_m \int_1^{\infty} u^{-k} e^{\frac{\pi}{6u}} e^{-\left(\frac{\pi}{6}+2\pi m\right)u} du$$

Compared with the Dirichlet series, only 5 terms in the m-sum provide an accuracy of 10 digits. Thus, we find, with 15 digits accuracy, $E\left[T_{\max}\right] = 1.702955978947701$ and $\mathrm{Var}[T_{\max}] = 1.040903835036823$. We list a table of the first 10 moments,

$$\begin{array}{lclllcl}
E\left[T_{\max}\right] & = & 1.702955979 & & E\left[T_{\max}^2\right] & = & 3.940962901 \\
E\left[T_{\max}^3\right] & = & 12.19707256 & & E\left[T_{\max}^4\right] & = & 48.78787933 \\
E\left[T_{\max}^5\right] & = & 242.6083708 & & E\left[T_{\max}^6\right] & = & 1449.044606 \\
E\left[T_{\max}^7\right] & = & 10114.37171 & & E\left[T_{\max}^8\right] & = & 80784.44155 \\
E\left[T_{\max}^9\right] & = & 726430.2004 & & E\left[T_{\max}^{10}\right] & = & 7261016.766
\end{array}$$

In summary, in addition to the relation $E\left[\tau_{\min}\right] = \sqrt{\mathrm{Var}\left[\tau_{\min}\right]} \simeq \frac{6}{N^3}$ for large N, the following first order estimates

$$\begin{array}{cc}
E\left[\tau_{\max}\right] \simeq \frac{1.703}{N} & \mathrm{Var}\left[\tau_{\max}\right] \simeq \frac{1.041}{N} \\
E\left[w^*\right] \simeq \frac{1.645}{N} & \mathrm{Var}\left[w^*\right] \simeq \frac{1.45}{N}
\end{array}$$

illustrate the curious point that, for large N, the mean of a random weight in the URT and of the maximum of the interattachment times to the URT are close to each other!

5.3 Asymptotic pgf of $T_{\max}$

The probability generating function of the random variable $T_{\max} = \lim_{N\to\infty} N\tau_{\max}$ is

$$\varphi_{T_{\max}}(z) = E\left[e^{-zT_{\max}}\right] = \int_0^{\infty} e^{-z\xi} \frac{d}{d\xi}\left(\prod_{m=1}^{\infty}\left(1 - e^{-\xi m}\right)\right)^2 d\xi \tag{40}$$

which converges for $\mathrm{Re}\,(z) > -1$ as follows from (36). Introducing the power series (34)

$$\varphi_{T_{\max}}(z) = \int_0^{\infty} e^{-z\xi} \frac{d}{d\xi}\left(\sum_{m=0}^{\infty} a_m e^{-\xi m}\right) d\xi$$

and reversing the sum and integral, yields

$$\varphi_{T_{\max}}(z) = -\sum_{m=1}^{\infty} \frac{m a_m}{m+z} = 1 + z\sum_{m=1}^{\infty} \frac{a_m}{m+z} \tag{41}$$

The series (41) is a partial fraction expansion that converges everywhere, except at the poles $z = -m$ with $m \in \mathbb{N}$. Indeed, we have that

$$\left|\sum_{m=1}^{\infty} \frac{a_m}{m+z}\right| \leq \sum_{m=1}^{\infty} \frac{|a_m|}{|m+z|} \leq \sum_{m=1}^{\infty} \frac{|a_m|}{m-|z|} = \sum_{m=1}^{[|z|]} \frac{|a_m|}{m-|z|} + \sum_{m=1+|z|}^{\infty} \frac{|a_m|}{m\left(1-\frac{|z|}{m}\right)}$$

and

$$\sum_{m=1+|z|}^{\infty} \frac{|a_m|}{m\left(1-\frac{|z|}{m}\right)} = \sum_{m=1+|z|}^{\infty} \frac{|a_m|}{m} \sum_{k=0}^{\infty} \frac{|z|^k}{m^k} = \sum_{m=1+|z|}^{\infty} \frac{|a_m|}{m} \left(1+\frac{|z|}{m}+O\left(m^{-2}\right)\right)$$
$$= \sum_{m=1+|z|}^{\infty} \frac{|a_m|}{m} + |z|\, O\left(\sum_{m=1+|z|}^{\infty} \frac{|a_m|}{m^2}\right)$$

Since the Dirichlet series $\sum_{m=1}^{\infty} \frac{a_m}{m^\beta}$ converges for $\beta \geq 0$, it converges absolutely for $\beta \geq 1$ [16, p. 292]. This proves convergence of (41) for any complex z, except at the poles $z = -m$. Due to the pole at $z = -1$, the Taylor series of $\varphi_{T_{\max}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k E\left[T_{\max}^k\right]}{k!} z^k$ around $z = 0$ has a radius of convergence equal to 1, a fact that also follows from (38).

The slow converge of the partial fraction expansion (41) again suggests us to invoke the Dedekind transform (28) to obtain fast converging series. After partial integration of (40),

$$\varphi_{T_{\max}}(z) = z \int_0^{\infty} e^{-z\xi} \left(\prod_{m=1}^{\infty} \left(1-e^{-\xi m}\right)\right)^2 d\xi \tag{42}$$

we introduce (29) into (42)

$$\varphi_{T_{\max}}(z) = z \int_0^{\infty} e^{-z\xi} e^{\frac{\xi}{12}} \eta^2 \left(\frac{\xi i}{2\pi}\right) d\xi$$

Applying the Dedekind transform (30) combined with (34) gives

$$\eta^2 \left(\frac{2\pi}{\xi} i\right) = e^{-\frac{(2\pi)^2}{12\xi}} \sum_{m=0}^{\infty} a_m e^{-\frac{(2\pi)^2}{\xi} m}$$

such that

$$\varphi_{T_{\max}}(z) = 2\pi z \sum_{m=0}^{\infty} a_m \int_0^{\infty} \frac{d\xi}{\xi} e^{-\left(z-\frac{1}{12}\right)\xi} e^{-\frac{(2\pi)^2}{\xi}\left(m+\frac{1}{12}\right)}$$

For $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(b) > 0$ and any $s \in \mathbb{C}$, the integral is written in terms of the modified Bessel function [26, Section 6.22]

$$\int_0^{\infty} x^{s-1} e^{-ax-b/x} dx = 2\left(\frac{b}{a}\right)^{s/2} K_s\left(2\sqrt{ab}\right) \tag{43}$$

and we obtain, only for $\operatorname{Re}(z) > \frac{1}{12}$,

$$\varphi_{T_{\max}}(z) = 4\pi z \sum_{m=0}^{\infty} a_m K_0 \left(4\pi \sqrt{\left(z-\frac{1}{12}\right)\left(m+\frac{1}{12}\right)}\right) \tag{44}$$

This series (44) converges amazingly faster than (41): for $z = \frac{1}{2}$, only 9 terms in the m-sum achieve a 10 digits accuracy, whereas (41) provides only 2 digits with 3000 terms; for $z = 1$, four terms in (44) provide 10 digits accuracy, while (41) remains at 2 digits with 3000 terms.

Following a similar approach, we present a related series for $\varphi_{T_{\max}}(z)$. We use Euler's pentagonal number series (31),

$$\prod_{m=1}^{\infty} \left(1-e^{-\xi m}\right)^2 = e^{\frac{\xi}{24}} \eta\left(\frac{\xi i}{2\pi}\right) \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{1}{2}n(3n+1)\xi}$$

The Dedekind transform gives

$$\eta\left(\frac{\xi i}{2\pi}\right)=\sqrt{\frac{2\pi}{\xi}}\eta\left(\frac{2\pi}{\xi}i\right)$$

Since $\eta\left(\frac{\xi}{2\pi}i\right)=e^{-\frac{\xi}{24}}\sum_{n=-\infty}^{\infty}(-1)^{n}e^{-\frac{1}{2}n(3n+1)\xi}$, it follows that

$$\eta\left(\frac{2\pi}{\xi}i\right)=e^{-\frac{(2\pi)^2}{24\xi}}\sum_{k=-\infty}^{\infty}(-1)^{k}e^{-\frac{1}{2}k(3k+1)\frac{(2\pi)^2}{\xi}}$$

such that

$$\varphi_{T_{\max}}(z)=z\sqrt{2\pi}\sum_{k=-\infty}^{\infty}(-1)^{k}\sum_{n=-\infty}^{\infty}(-1)^{n}\int_{0}^{\infty}\xi^{\frac{1}{2}-1}e^{-\xi\left(z-\frac{1}{24}+\frac{1}{2}n(3n+1)\right)}e^{-\frac{1}{\xi}\left(\frac{(2\pi)^2}{24}+\frac{1}{2}k(3k+1)(2\pi)^2\right)}d\xi$$

Invoking the integral (43) for the modified Bessel function, we have

$$\begin{aligned}I&=\int_{0}^{\infty}\xi^{\frac{1}{2}-1}e^{-\xi\left(z-\frac{1}{24}+\frac{1}{2}n(3n+1)\right)}e^{-\frac{1}{\xi}\left(\frac{(2\pi)^2}{24}+\frac{1}{2}k(3k+1)(2\pi)^2\right)}\\&=2\sqrt{2\pi}\left(\frac{\frac{1}{24}+\frac{k(3k+1)}{2}}{z-\frac{1}{24}+\frac{n(3n+1)}{2}}\right)^{\frac{1}{4}}K_{\frac{1}{2}}\left(4\pi\sqrt{\left(z-\frac{1}{24}+\frac{n(3n+1)}{2}\right)\left(\frac{1}{24}+\frac{k(3k+1)}{2}\right)}\right)\end{aligned}$$

An exact expression for $K_{n+\frac{1}{2}}(z)$ exists [1, 10.2.17], in particular, $K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}$, which, when applied, leads to

$$\varphi_{T_{\max}}(z)=4\pi z\sum_{k=-\infty}^{\infty}(-1)^{k}\sum_{n=-\infty}^{\infty}(-1)^{n}\frac{e^{-4\pi\sqrt{\left(z-\frac{1}{24}+\frac{1}{2}n(3n+1)\right)\left(\frac{1}{24}+\frac{1}{2}k(3k+1)\right)}}}{\sqrt{24z-1+12n(3n+1)}}\tag{45}$$

We still see that the term $n=0$ limits the validity to $\operatorname{Re}(z)>\frac{1}{24}$. Only a few terms in the k- and n-series already provide a good accuracy.

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A Sums of Gamma functions

Lemma 4 *For any integer N and $\operatorname{Re}(x) > 0$, we have that*

$$\sum_{m=N+1-j}^{N} \frac{\Gamma(m+1)}{\Gamma(m+x+1)} \frac{\Gamma(N+1-m+x)}{\Gamma(N+1-m)} = \frac{\Gamma(N+x+2)}{\Gamma(x)\Gamma(j)} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(j+x+k+1)}{(x+k+1)} \frac{\Gamma(k+x)}{\Gamma(N+2x+k+2)} \tag{46}$$

Proof: We introduce the Beta function integral [1, 6.2.1],

$$B(w,z) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

which is valid for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$, in

$$T = \sum_{m=N+1-j}^{N} \frac{\Gamma(m+1)}{\Gamma(m+x+1)} \frac{\Gamma(N+1-m+x)}{\Gamma(N+1-m)}$$

such that, for $\mathrm{Re}(x) > 0$,

$$\begin{aligned} T &= \frac{1}{\Gamma(x)} \int_0^1 dt\, (1-t)^{x-1} \sum_{m=N+1-j}^{N} \frac{\Gamma(N+1-m+x)}{\Gamma(N+1-m)} t^m \\ &= \frac{1}{\Gamma(x)} \int_0^1 dt\, (1-t)^{x-1} \sum_{k=1}^{j} \frac{\Gamma(k+x)}{\Gamma(k)} t^{N+1-k} \\ &= \frac{1}{\Gamma(x)} \int_0^1 dt\, (1-t)^{x-1}\, t^N \sum_{k=0}^{j-1} \frac{\Gamma(k+1+x)}{k!} \left(\frac{1}{t}\right)^k \end{aligned}$$

Introducing Euler's integral for the Gamma function into the k-sum yields

$$\sum_{k=0}^{j-1} \frac{\Gamma(k+1+x)}{k!} \left(\frac{1}{t}\right)^k = \sum_{k=0}^{j-1} \frac{1}{k!} \left(\frac{1}{t}\right)^k \int_0^\infty u^{k+x} e^{-u} du = \int_0^\infty u^x e^{-u} \sum_{k=0}^{j-1} \frac{1}{k!} \left(\frac{u}{t}\right)^k du$$

We now employ

$$\sum_{k=0}^{j-1} \frac{\left(\frac{u}{t}\right)^k}{k!} = \frac{e^{\frac{u}{t}} \Gamma\left(j, \frac{u}{t}\right)}{(j-1)!} = \frac{e^{\frac{u}{t}}}{(j-1)!} \int_{\frac{u}{t}}^\infty y^{j-1} e^{-y} dy$$

where the incomplete Gamma function is defined [1, 6.5.3] as $\Gamma(a,x) = \int_x^\infty y^{a-1} e^{-y} dy$ such that

$$\begin{aligned} \sum_{k=0}^{j-1} \frac{\Gamma(k+1+x)}{k!} \left(\frac{1}{t}\right)^k &= \frac{1}{\Gamma(j)} \int_0^\infty du\, u^x e^{-u} e^{\frac{u}{t}} \int_{\frac{u}{t}}^\infty y^{j-1} e^{-y} dy \\ &= \frac{t^{x+1}}{\Gamma(j)} \int_0^\infty dw\, w^x e^{-wt} e^w \int_w^\infty y^{j-1} e^{-y} dy \end{aligned}$$

Partial integration yields

$$\sum_{k=0}^{j-1} \frac{\Gamma(k+1+x)}{k!} \left(\frac{1}{t}\right)^k = \frac{t^{x+1}}{\Gamma(j)} \int_0^\infty dw\, w^{j-1} e^{-w} \int_0^w u^x e^{u(1-t)} du$$

We compute the u-integral as

$$\int_0^w u^x e^{u(1-t)} du = \sum_{k=0}^\infty \frac{(1-t)^k}{k!} \int_0^w u^{x+k} du = \sum_{k=0}^\infty \frac{(1-t)^k}{k!} \frac{w^{x+k+1}}{(x+k+1)}$$

Hence,

$$\begin{aligned} \sum_{k=0}^{j-1} \frac{\Gamma(k+1+x)}{k!} \left(\frac{1}{t}\right)^k &= \frac{t^{x+1}}{\Gamma(j)} \sum_{k=0}^\infty \frac{(1-t)^k}{k!} \frac{1}{(x+k+1)} \int_0^\infty dw\, w^{j+x+k+1-1} e^{-w} \\ &= \frac{t^{x+1}}{\Gamma(j)} \sum_{k=0}^\infty \frac{(1-t)^k}{k!} \frac{\Gamma(j+x+k+1)}{(x+k+1)} \end{aligned}$$

Substituting this series, results in

$$\begin{aligned} T &= \frac{1}{\Gamma(x)\Gamma(j)} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(j+x+k+1)}{(x+k+1)} \int_0^1 dt\, (1-t)^{x+k-1}\, t^{N+x+1} \\ &= \frac{1}{\Gamma(x)\Gamma(j)} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(j+x+k+1)}{(x+k+1)} \frac{\Gamma(N+x+2)\Gamma(k+x)}{\Gamma(N+2x+k+2)} \end{aligned}$$

which establishes (46). □

Although proven under the restriction that $\mathrm{Re}(x) > 0$, Lemma 4 seems also valid for negative real x, but clearly not for $x = 0$.

Lemma 5 *The sum*

$$H_p(y) = \frac{1}{\Gamma(y+p)} \sum_{j=p+1}^{N} \frac{\Gamma(y+j+k-p)}{\Gamma(y+j)} \frac{\Gamma(N+y+p-j)}{\Gamma(N+1-j)} \tag{47}$$

obeys the recursion

$$H_p(y) = \frac{\Gamma(y+1+k)}{\Gamma^2(y+p+1)} \frac{\Gamma(N+y)}{\Gamma(N-p)} + (k-p)\, H_{p+1}(y) \tag{48}$$

and equals

$$H_p(y) = \sum_{j=0}^{N-p-1} \frac{(k-p)!}{(k-p-j)!} \frac{\Gamma(y+1+k)}{\Gamma^2(y+p+1+j)} \frac{\Gamma(N+y)}{\Gamma(N-p-j)} \tag{49}$$

Proof: Using the binomial recursion $\binom{a-j}{b-j} = \binom{a-j+1}{b-j} - \binom{a-j}{b-j-1}$ yields

$$\begin{aligned} H_p(y) &= \sum_{j=p+1}^{N} \frac{\Gamma(y+j+k-p)}{\Gamma(y+j)} \binom{N+y+p-1-j+1}{N-j} - \sum_{j=p+1}^{N-1} \frac{\Gamma(y+j+k-p)}{\Gamma(y+j)} \binom{N+y+p-1-j}{N-j-1} \\ &= \sum_{j=p+1}^{N} \frac{\Gamma(y+j+k-p)}{\Gamma(y+j)} \binom{N+y+p-1-j+1}{N-j} - \sum_{j=p+2}^{N} \frac{\Gamma(y+j+k-p-1)}{\Gamma(y+j-1)} \binom{N+y+p-j}{N-j} \\ &= \frac{\Gamma(y+1+k)}{\Gamma(y+p+1)} \binom{N+y-1}{N-p-1} + \sum_{j=p+2}^{N} \left\{ \frac{\Gamma(y+j+k-p)}{\Gamma(y+j)} - \frac{\Gamma(y+j+k-p-1)}{\Gamma(y+j-1)} \right\} \binom{N+y+p-j}{N-j} \end{aligned}$$

With

$$\frac{\Gamma(y+j+k-p)}{\Gamma(y+j)} - \frac{\Gamma(y+j+k-p-1)}{\Gamma(y+j-1)} = (k-p) \frac{\Gamma(y+j+k-p-1)}{\Gamma(y+j)}$$

we obtain

$$H_p(y) = \frac{\Gamma(y+1+k)}{\Gamma^2(y+p+1)} \frac{\Gamma(N+y)}{\Gamma(N-p)} + (k-p) \sum_{j=(p+1)+1}^{N} \frac{\Gamma(y+j+k-(p+1))}{\Gamma(y+j)} \binom{N+y+(p+1)-1-j}{N-j}$$

which equals the recursion (48). Iterating the recursion (48) q-times yields

$$H_p(y) = \sum_{j=0}^{q-1} \frac{(k-p)!}{(k-p-j)!} \frac{\Gamma(y+1+k)}{\Gamma^2(y+p+1+j)} \frac{\Gamma(N+y)}{\Gamma(N-p-j)} + \frac{(k-p)!}{(k-p-q)!} H_{p+q}(y)$$

Since $H_N(y) = 0$, we arrive with $q = N - p$ at (49). □

From the definition (47)

$$G=\sum_{j=1}^{N}\frac{\Gamma\left(x+j+1+k\right)}{\Gamma\left(x+j+1\right)}\frac{\Gamma\left(N+1-j+x\right)}{\Gamma\left(N+1-j\right)}=\Gamma\left(x+1\right)H_0\left(x+1\right)$$

we obtain with (49)

$$G=\Gamma\left(x+1\right)\Gamma\left(x+2+k\right)\Gamma\left(N+x+1\right)\sum_{j=0}^{N-1}\frac{k!}{(k-j)!}\frac{1}{\Gamma^2\left(x+2+j\right)\Gamma\left(N-j\right)} \tag{50}$$

B Proof of Theorem 3

We start by considering the logarithm of (24),

$$\begin{aligned}-\log\Pr\left[\tau_{\max}\leq x\right]&=-\sum_{j=1}^{N}\log\left(1-e^{-j(N+1-j)x}\right)=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{N}e^{-j(N+1-j)kx}\\&=\sum_{k=1}^{\infty}\frac{e^{-\frac{(N+1)^2}{4}kx}}{k}\sum_{j=1}^{N}e^{kx\left(j-\frac{N+1}{2}\right)^2}\end{aligned} \tag{51}$$

Lemma 6 formally shows, provided the q-sum converges, that

$$\sum_{j=1}^{N}e^{kx\left(j-\frac{N+1}{2}\right)^2}=2-e^{kx(N+1)^2/4}+\frac{N+1}{2}\sum_{q=0}^{\infty}\frac{B_{2q}}{(2q)!}\left(4kx\right)^q\frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}M\left(q+\frac{1}{2},\frac{3}{2},kx(N+1)^2/4\right)$$

We now embark on the limiting process for large N. For large z and $\mathrm{Re}\left(z\right)>0$, the asymptotic expansion of the Kummer function [1, 13.1.4] is $\frac{\Gamma(a)}{\Gamma(b)}M\left(a,b,z\right)=e^z z^{a-b}\left(1+O\left(|z|^{-1}\right)\right)$. Thus,

$$\sum_{j=1}^{N}e^{kx\left(j-\frac{N+1}{2}\right)^2}=2-e^{kx(N+1)^2/4}+\frac{2e^{kx(N+1)^2/4}}{kx(N+1)}\sum_{q=0}^{\infty}\frac{B_{2q}}{(2q)!}\left(kx(N+1)\right)^{2q}\left(1+O\left(\frac{1}{kx(N+1)^2}\right)\right)$$

The generating function of the Bernoulli numbers,

$$\frac{t}{e^t-1}=-\frac{t}{2}+\sum_{n=0}^{\infty}B_{2n}\frac{t^{2n}}{(2n)!}\qquad\text{for}\quad |t|<2\pi$$

shows that the q-sum converges, provided $kx(N+1)<2\pi$,

$$\sum_{q=0}^{\infty}\frac{B_{2q}}{(2q)!}\left(kx(N+1)\right)^{2q}=\frac{kx(N+1)}{e^{kx(N+1)}-1}+\frac{kx(N+1)}{2}$$

such that, for $kx(N+1)<2\pi$,

$$\sum_{j=1}^{N}e^{kx\left(j-\frac{N+1}{2}\right)^2}=2+\frac{2e^{kx(N+1)^2/4}}{e^{kx(N+1)}-1}+2e^{kx(N+1)^2/4}\left(\frac{1}{e^{kx(N+1)}-1}+\frac{1}{2}\right)O\left(\frac{1}{kx(N+1)^2}\right)$$

Substituting this series into (51) yields

$$\begin{aligned}-\log\Pr\left[\tau_{\max}\leq x\right]&=\sum_{k=1}^{\infty}\frac{e^{-\frac{(N+1)^2}{4}kx}}{k}\sum_{j=1}^{N}e^{kx\left(j-\frac{N+1}{2}\right)^2}\\&=2\left(1+O\left(\frac{1}{x(N+1)^2}\right)\right)\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{kx(N+1)}-1}\end{aligned}$$

Now,

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{e^{2\pi kt}-1}=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{m=1}^{\infty}e^{-2\pi kmt}=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\left(e^{-2\pi mt}\right)^k}{k}$$
$$=-\sum_{m=1}^{\infty}\log\left(1-e^{-2\pi mt}\right)=-\log\left(\prod_{m=1}^{\infty}(1-e^{-2\pi mt})\right)$$

such that, for large N,

$$\log\Pr\left[\tau_{\max}\leq x\right]=2\log\left(\prod_{m=1}^{\infty}(1-e^{-x(N+1)m})\right)+O\left(x^{-1}N^{-2}\right)$$

In order to have a finite limit as $N\to\infty$, we need to scale x as $x\,(N+1)=\xi$ and this proves (25).
$\square$

Lemma 6 *Provided the q-sum converges, there holds that*

$$\sum_{j=1}^{N}e^{y\left(j-\frac{N+1}{2}\right)^2}=2-e^{y(N+1)^2/4}+\frac{N+1}{2}\sum_{q=0}^{\infty}\frac{B_{2q}}{(2q)!}\,(4y)^q\,\frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}M\left(q+\frac{1}{2},\frac{3}{2},y(N+1)^2/4\right)\quad(52)$$

where B_k are the Bernoulli numbers and where the Kummer function [1, 13.1.2] is

$$M\left(a,b,z\right)=\frac{\Gamma\left(b\right)}{\Gamma\left(a\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(a+k\right)}{\Gamma\left(b+k\right)}\frac{z^k}{k!}\quad(53)$$

Proof: We rewrite the j-sum as

$$\sum_{j=1}^{N}e^{y\left(j-\frac{N+1}{2}\right)^2}=\sum_{j=-(N-1)/2}^{(N-1)/2}e^{yj^2}=1+2\sum_{j=1}^{(N-1)/2}e^{yj^2}=1+2\sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{j=1}^{(N-1)/2}j^{2m}$$

Applying the relation [1, 23.1.4] involving Bernoulli numbers B_k,

$$\sum_{k=1}^{j-1}k^m=\frac{j^{m+1}}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_kj^{-k}$$

yields

$$\sum_{j=1}^{N}e^{y\left(j-\frac{N+1}{2}\right)^2}=1+(N+1)\sum_{m=0}^{\infty}\frac{\left(y(N+1)^2/4\right)^m}{m!\,(2m+1)}\sum_{q=0}^{2m}\binom{2m+1}{q}B_q\left(\frac{2}{N+1}\right)^q$$

We reverse the m- and k-series,

$$\sum_{j=1}^{N}e^{y\left(j-\frac{N+1}{2}\right)^2}=1+(N+1)\sum_{q=0}^{\infty}\frac{B_q}{q!}\left(\frac{2}{N+1}\right)^q\sum_{m=\left[\frac{q+1}{2}\right]}^{\infty}\frac{(2m)!\left(y(N+1)^2/4\right)^m}{m!(2m+1-q)!}$$
$$=1+(N+1)\sum_{q=0}^{\infty}\frac{B_q}{q!}\left(\frac{2}{N+1}\right)^{q-2\left[\frac{q+1}{2}\right]}y^{\left[\frac{q+1}{2}\right]}\sum_{m=0}^{\infty}\frac{(2m+2\left[\frac{q+1}{2}\right])!\left(y(N+1)^2/4\right)^m}{\left(m+\left[\frac{q+1}{2}\right]\right)!(2m+2\left[\frac{q+1}{2}\right]+1-q)!}$$

which simplifies, since $B_{2q+1} = 0$ for $q > 0$, to

$$\sum_{j=1}^{N} e^{y\left(j-\frac{N+1}{2}\right)^2} = 1 + (N+1)\sum_{m=0}^{\infty} \frac{\left(y(N+1)^2/4\right)^m}{(m)!(2m+1)}$$
$$- \left(\frac{(N+1)}{2}\right)^2 y \sum_{m=0}^{\infty} \frac{\left(y(N+1)^2/4\right)^m}{(m+1)!}$$
$$+ (N+1)\sum_{q=1}^{\infty} \frac{B_{2q}}{(2q)!} y^q \sum_{m=0}^{\infty} \frac{(2(m+q))!\left(y(N+1)^2/4\right)^m}{(m+q)!(2m+1)!}$$

The reversal of m- and k-sum is only allowed provided the q-sum converges. The second sum is

$$\sum_{m=0}^{\infty} \frac{\left(y(N+1)^2/4\right)^m}{(m+1)!} = \frac{1}{y(N+1)^2/4}\left(e^{y(N+1)^2/4} - 1\right)$$

The remaining m-sums can be rewritten, after using the duplication formula $\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ of the Gamma function [1, 6.1.18], in terms of the Kummer function (53). Indeed, with $w = y(N+1)^2/4$, we have that

$$S_q = \sum_{m=0}^{\infty} \frac{(2(m+q))!w^m}{(m+q)!(2m+1)!} = \frac{2^{2q}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{\left(m+q-\frac{1}{2}\right)!\,(4w)^m}{(2m+1)!}$$
$$= 2^{2q-1} \sum_{m=0}^{\infty} \frac{\left(m+q-\frac{1}{2}\right)!}{\Gamma\left(m+\frac{3}{2}\right)} \frac{w^m}{m!}$$
$$= 2^{2q-1} \frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} M\left(q+\frac{1}{2}, \frac{3}{2}, w\right)$$

and, the first m-sum is $\sum_{m=0}^{\infty} \frac{\left(y(N+1)^2/4\right)^m}{(m)!(2m+1)} = S_0$. Combining all yields (52). $\square$