# An Epidemic Perspective on the Cut Size in Networks 

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#### Abstract

An epidemic spreads over the network via infectious links between healthy and infected nodes. The rate of increase in the number of infected nodes depends on the number of infectious links, called the cut size. After reviewing results on the cut size, its average and variance are computed in a simplified case, where each node has equal infection probability $v$, independent of all the other nodes. Although the simplified case is approximate for an epidemic process, high degree variance graphs are shown to diffuse information and diseases in the fastest and most bursty way. Moreover, variations of the cut size divided by the mean cut size decrease with $v$ and are minimum for any graph at $v=\frac{1}{2}$.


## 1 Introduction

Motivated by its crucial role in the spead of epidemics on networks as reviewed below, we investigate the cut size, which equals the number of links between two partitions of a graph. The determination of and tight bounds on the cut size continue to play a central role in graph theory. Here, we first review in Section 2 the state of the art about the cut size from an epidemic perspective, where a viral item propagates from infected nodes towards healthy nodes through links of the cut-set. Additional results are given in the Appendices. In Section 3, we provide a probabilistic estimate for a simplified case, where all nodes possess a same probability to belong to the infected partition.

We consider an unweighted, undirected graph $G$ containing a set $\mathcal{N}$ of $N$ nodes (also called vertices) and a set $\mathcal{L}$ of $L$ links (also called edges). The topology of the graph is represented by a symmetric $N \times N$ adjacency matrix $A$. In a SIS epidemic process on $G$, the viral state of a node $i$ at time $t$ is specified by a Bernoulli random variable $X_{i}(t) \in\{0,1\}: X_{i}(t)=0$ for a healthy, but susceptible node and $X_{i}(t)=1$ for an infected node. A node $i$ at time $t$ can be in one of the two states: infected, with probability $v_{i}(t)=\operatorname{Pr}\left[X_{i}(t)=1\right]$ or healthy, with probability $1-v_{i}(t)$, but susceptible to the infection. We assume that the curing process per node $i$ is a Poisson process with rate $\delta$ and that the infection rate per link is a Poisson process with rate $\beta$. Obviously, only an infected node can infect its direct neighbors, that are still healthy. Both the curing and infection Poisson process are independent.

[^0]The effective infection rate, sometimes also called the spreading rate [1], is defined by $\tau=\frac{\beta}{\delta}$. This is the general continuous-time Markovian description of the simplest type of a SIS epidemic process on a network. We do not consider non-Markovian epidemics [2, 3, 4] nor self-infections [5]. The exact Markovian SIS governing equation $[6,7]$ for the infection probability of node $i$,

$$
\begin{equation*}
\frac{d E\left[X_{i}\right]}{d t}=E\left[-\delta X_{i}+\beta\left(1-X_{i}\right) \sum_{k=1}^{N} a_{k i} X_{k}\right] \tag{1}
\end{equation*}
$$

shows that the time-derivative of the infection probability $E\left[X_{i}\right]=\operatorname{Pr}\left[X_{i}=1\right]$ of a node $i$ consists of the expectation of two competing processes, expressed in the Bernoulli random variable $X_{i} \in\{0,1\}$ : (1) while node $i$ is healthy, i.e. not infected ( $1-X_{i}$ ), all infected neighbors $\sum_{k=1}^{N} a_{k i} X_{k}$ of node $i$ try to infect the node $i$ with rate $\beta$ and (2) while node $i$ is infected $X_{i}$, the node $i$ is cured at rate $\delta$.

We define the fraction of infected nodes by

$$
\begin{equation*}
S(t)=\frac{1}{N} \sum_{i=1}^{N} X_{i}(t) \tag{2}
\end{equation*}
$$

We denote the average fraction of infected nodes, also called the prevalence, by

$$
\begin{equation*}
y(t ; \tau)=E[S(t)]=\frac{1}{N} \sum_{i=1}^{N} E\left[X_{i}(t)\right]=\frac{1}{N} \sum_{i=1}^{N} v_{i}(t) \tag{3}
\end{equation*}
$$

that obeys, as demonstrated in [8] and further studied in [9, 10], the differential equation

$$
\begin{equation*}
\frac{d y\left(t^{*} ; \tau\right)}{d t^{*}}=-y\left(t^{*} ; \tau\right)+\frac{\tau}{N} E\left[w\left(t^{*} ; \tau\right)^{T} Q w\left(t^{*} ; \tau\right)\right] \tag{4}
\end{equation*}
$$

where $t^{*}=\delta t$ is the scaled time, $Q=\Delta-A$ is the Laplacian of the graph $G$ with $\Delta=\operatorname{diag}\left(d_{1}, d_{2}, \ldots, d_{N}\right)$ and $d_{i}$ is the degree of node $i$ in $G$, and the Bernoulli vector $w=\left(X_{1}, X_{2}, \ldots, X_{N}\right)$.

## 2 Cut-set of a graph

We apply the basic [11, p. 72] Laplacian property ${ }^{1} x^{T} Q x=\sum_{l \in \mathcal{L}}\left(x_{l^{+}}-x_{l^{-}}\right)^{2}$, where the link $l$ connects the node $l^{+}$and node $l^{-}$, to the Bernoulli vector $w=\left(X_{1}, X_{2}, \ldots, X_{N}\right)$,

$$
\begin{equation*}
w^{T} Q w=\sum_{l \in \mathcal{L}}\left(X_{l^{+}}-X_{l^{-}}\right)^{2} \tag{5}
\end{equation*}
$$

[^1]Since any nodal infection state $X_{i} \in\{0,1\}$, we observe that $\left(X_{l^{+}}-X_{l^{-}}\right)^{2}$ is zero, if both endnodes are in the same state, while $\left(X_{l^{+}}-X_{l^{-}}\right)^{2}=1$, if both endnodes of link $l$ are in a different state. Hence,

$$
w^{T} Q w=\sum_{l \in \mathcal{L}}\left(X_{l^{+}}-X_{l^{-}}\right)^{2} \leq L
$$

where equality in the upper bound can be attained for bipartite graphs. Only the infectious links between infected and healthy endnodes contribute to $w^{T} Q w$. Alternatively, the quadratic form $w^{T} Q w$ equals the cut size, which is the number of links in the cut-set, defined in turn as the set of links with one healthy endnode, while the other endnode is infected (illustrated in Fig. 1 in [12]). When $\mathcal{V}$ denotes the set of $|\mathcal{V}|=V=N S$ infected or viral nodes in the graph $G$ and its complement $\mathcal{V}^{c}$ the set of $\left|\mathcal{V}^{c}\right|=N(1-S)$ healthy nodes, then the cut-set is often denoted by $\partial \mathcal{V}$ with corresponding cut size equal to

$$
|\partial \mathcal{V}|=w^{T} Q w
$$

where $w_{j}=1$ if node $j \in \mathcal{V}$, else $w_{j}=0$.
The probability that there is an infectious link directed from node $i$ to $j$ is

$$
\begin{equation*}
\operatorname{Pr}\left[X_{i}=1, X_{j}=0\right]=E\left[X_{i}\left(1-X_{j}\right)\right]=E\left[X_{i}\right]-E\left[X_{i} X_{j}\right] \tag{6}
\end{equation*}
$$

which clearly vanishes if $i=j$. The net probability flux follows as

$$
\Phi_{i j}=\operatorname{Pr}\left[X_{i}=1, X_{j}=0\right]-\operatorname{Pr}\left[X_{i}=0, X_{j}=1\right]=E\left[X_{i}\right]-E\left[X_{j}\right]
$$

Physically, we may interpret the right-handside as the "probability potential" difference responsible for a net "probability" current. With the definition (6), the SIS governing equation (1) becomes

$$
\begin{equation*}
\frac{d E\left[X_{i}\right]}{d t}+\delta E\left[X_{i}\right]=\beta \sum_{k=1}^{N} a_{k i} \operatorname{Pr}\left[X_{k}=1, X_{i}=0\right] \tag{7}
\end{equation*}
$$

Summing (7) over all nodes and rewritten in terms of the prevalence (3) yields

$$
\frac{d y(t ; \tau)}{d t}+\delta y(t ; \tau)=\frac{\beta}{N} \sum_{i=1}^{N} \sum_{k=1}^{N} a_{k i} \operatorname{Pr}\left[X_{k}=1, X_{i}=0\right]
$$

Comparison with the earlier expression (4) shows that the average number of links in the cut-set between healthy and infected nodes is ${ }^{2}$

$$
\begin{equation*}
E\left[w^{T} Q w\right]=\sum_{i=1}^{N} \sum_{k=1}^{N} a_{k i} \operatorname{Pr}\left[X_{k}=1, X_{i}=0\right] \tag{8}
\end{equation*}
$$

[^2]The expected cutsize equals the joint probability over all node pairs in which neighboring nodes in the graph are in a different infectious state; also, rewritten as

$$
E[|\partial \mathcal{V}|]=\sum_{l \in \mathcal{L}} \operatorname{Pr}\left[X_{l^{+}}=1, X_{l^{-}}=0\right]+\operatorname{Pr}\left[X_{l^{-}}=1, X_{l^{+}}=0\right]
$$

These joint probabilities $\operatorname{Pr}\left[X_{k}=1, X_{i}=0\right]$ in a Markovian SIS process can be determined precisely as shown in $[6,7]$, but require the knowledge of the joint probabilities over all triples of nodal states, which in turn requires all combinations of four nodal states and so on. Eventually, the exact determination leads to $2^{N}$ linear differential equations and this exponentially large number in $N$ is a finger print of the NP-hard nature of the cutsize problem.

### 2.1 Cut size: topological perspective

If one cluster consists of a single node $j$, the cut size equals $|\partial \mathcal{V}|=d_{j}$, the degree of node $j$. Clearly, the minimum cut size for $V=1$ equals $|\partial \mathcal{V}|_{\text {min }}=d_{\text {min }}$ and the maximum cut size is $|\partial \mathcal{V}|_{\max }=d_{\max }$. If one cluster consists of two nodes $j$ and $l$, then $|\partial \mathcal{V}|=d_{j}+d_{l}-a_{j l}$ and the cut size for $V=2$ is bounded by $d_{\min }+d_{(N-1)}-1 \leq|\partial \mathcal{V}| \leq d_{\max }+d_{(2)}$, where $d_{(i)}$ denotes the $i$-th largest degree in $G$ so that $d_{\max }=d_{(1)} \geq d_{(2)} \geq \ldots \geq d_{(N)}=d_{\min }$. In general for a cluster with $V$ nodes, the cut size $|\partial \mathcal{V}|=w^{T} Q w=w^{T}(\Delta-A) w$ is

$$
\begin{equation*}
|\partial \mathcal{V}|=\sum_{i \in \mathcal{V}} d_{i}-\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} a_{i j} \tag{9}
\end{equation*}
$$

where the last sum $w^{T} A w$ equals all links in the subgraph $G \mathcal{V}$ of $G$ on $\mathcal{V}$ nodes. Since $\sum_{i=N}^{N-V+1} d_{(i)} \leq$ $\sum_{i \in \mathcal{V}} d_{i} \leq \sum_{i=1}^{V} d_{(i)}$ and $0 \leq \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} a_{i j} \leq 2\binom{V}{2}$, but $\sum_{i=N}^{N-V+1} d_{(i)}-2\binom{V}{2} \leq 0$, the cut size in any connected graph is bounded by

$$
1 \leq|\partial \mathcal{V}| \leq \sum_{i=1}^{V} d_{(i)}
$$

which illustrates the complicated dependence on the number $V$ of infected nodes. In summary, we find the "topological" bounds of the cut size in a connected graph,

$$
\begin{equation*}
1 \leq|\partial \mathcal{V}| \leq L \tag{10}
\end{equation*}
$$

### 2.2 Cut size: spectral perspective

Since $\mu_{N}=0$ and $\mu_{N-1}>0$ in a connected graph, the quadratic form (36) provides the powerful spectral representation ${ }^{3}$ for the cut size

$$
\begin{equation*}
|\partial \mathcal{V}|=\sum_{j=1}^{N-1} \zeta_{j}^{2} \mu_{j} \tag{11}
\end{equation*}
$$

which consists of $N-1$ terms, whereas the topological counter part $|\partial \mathcal{V}|=\sum_{l \in \mathcal{L}}\left(w_{l^{+}}-w_{l^{-}}\right)^{2}$ contains at least $N-1$ terms in any connected graph.

[^3]We introduce the definition (41) of the difference vector $\varpi$ into the quadratic form, invoking the characteristic property $Q u=0$ of any Laplacian,

$$
w^{T} Q w=\varpi^{T} Q \varpi
$$

After invoking the inequality [7, (5.4) on p. 99],

$$
\mu_{N-1} \leq \frac{\varpi^{T} Q \varpi}{\varpi^{T} \varpi}=\frac{\sum_{k=1}^{N-1} \mu_{k}\left(\varpi^{T} z_{k}\right)^{2}}{\sum_{k=1}^{N-1}\left(\varpi^{T} z_{k}\right)^{2}} \leq \mu_{1}
$$

we find with (43) the "spectral" bounds of the cut size in a connected graph

$$
\begin{equation*}
N S(1-S) \mu_{N-1} \leq|\partial \mathcal{V}| \leq N S(1-S) \mu_{1} \tag{12}
\end{equation*}
$$

These upper and lower bounds were derived earlier in [9, Appendix B. 2 ] by a different method.
If $S=\frac{1}{N}$, then the topological lower bound in (10) equals $d_{\min }$, while the spectral lower bound in (12) is $\left(1-\frac{1}{N}\right) \mu_{N-1}<\mu_{N-1} \leq d_{\text {min }}$, where the latter inequality is deduced in [11, p. 82]. The argument illustrates that the topological lower bound (10) is tighter for $S=\frac{1}{N}$ than the spectral one (12). The upper bound in (12), NS $(1-S) \mu_{1} \leq \frac{N}{4} \mu_{1}$, may vary between $\frac{L}{2} \leq \frac{N}{4} \mu_{1} \leq \frac{N^{2}}{4}$, while the topological upper bound $L$ can lie in between these bounds, so that a general comparison is difficult. Although producing tighter upper and lower bounds than in (12) is difficult, we recently found [13] for $1 \leq K<N$,

$$
\begin{equation*}
\sum_{k=1}^{K-1} s_{k^{\prime}}^{2}\left(\mu_{k^{\prime}}-\mu_{K^{\prime}}\right)+N S(1-S) \mu_{K^{\prime}} \leq|\partial \mathcal{V}| \leq \sum_{k=1}^{K-1} s_{k}^{2}\left(\mu_{k}-\mu_{K}\right)+N S(1-S) \mu_{K} \tag{13}
\end{equation*}
$$

where $k^{\prime}=N-k, K^{\prime}=N-K$ and $s_{k}^{2}=\max \left(\left(\sum_{i=1}^{N S}\left(z_{k}^{\downarrow}\right)_{i}\right)^{2},\left(\sum_{i=1}^{(1-S) N}\left(z_{k}^{\downarrow}\right)_{i}\right)^{2}\right)$ in which the vector $z_{k}^{\downarrow}$ has the same components as the eigenvector $z_{k}$, but ordered as $\left(z_{k}^{\downarrow}\right)_{1} \geq\left(z_{k}^{\downarrow}\right)_{2} \geq \cdots \geq\left(z_{k}^{\downarrow}\right)_{N}$. If $K=1$, then (13) reduces to (12).

### 2.3 Isoperimetric bound on the cut size $|\partial \mathcal{V}|$ in a connected graph

Chung [14] presents a comprehensive study on isoperimetric inequalities in graphs, from which we mention

Theorem 1 (Theorem 5 in [14]) Suppose that a graph $G$ with $N$ nodes has average degree $d_{a v}$. Then, for any two disjoint subsets $X$ and $Y$ of nodes in $G$, the number $e(X, Y)$ of (ordered) links with first endpoint in $X$ and the second endpoint in $Y$ satisfies

$$
\begin{equation*}
\left|e(X, Y)-\frac{d_{a v}}{N}\right| X||Y|| \leq \frac{\max _{1 \leq j \leq N-1}\left|\mu_{j}-d_{a v}\right|}{N} \sqrt{|X|(N-|X|)|Y|(N-|Y|)} \tag{14}
\end{equation*}
$$

For the specific case, where set $Y=G \backslash X$ is the complement of set $X$ so that $N-|X|=|Y|$, we provide here a slightly better bound than (14) and discuss, in particular, the right-hand side error bound.

Theorem 2 The number $|\partial \mathcal{V}|$ of infective links in a graph $G$ on $N$ nodes and with $L$ links, in which a fraction $S$ of the nodes is infected, is upper bounded by

$$
\begin{equation*}
\left||\partial \mathcal{V}|-\frac{2 L N}{N-1} S(1-S)\right| \leq \max _{1 \leq j \leq N-1}\left|\mu_{j}-\frac{N d_{a v}}{N-1}\right| N S(1-S) \tag{15}
\end{equation*}
$$

Proof: The spectral form (11), slightly rewritten for a real number $\theta$ as,

$$
|\partial \mathcal{V}|-\theta \sum_{j=1}^{N-1} \zeta_{j}^{2}=\sum_{j=1}^{N-1}\left(\mu_{j}-\theta\right) \zeta_{j}^{2}
$$

leads, invoking (45), to

$$
|\partial \mathcal{V}|-\theta N S(1-S)=\sum_{j=1}^{N-1}\left(\mu_{j}-\theta\right) \zeta_{j}^{2}
$$

from which it follows that

$$
\begin{equation*}
||\partial \mathcal{V}|-\theta N S(1-S)| \leq \sum_{j=1}^{N-1}\left|\mu_{j}-\theta\right| \zeta_{j}^{2} \tag{16}
\end{equation*}
$$

The right-hand side is upper bounded by the Cauchy-Schwarz inequality (see e.g. [7, p. 107]) as

$$
\begin{equation*}
\left(\sum_{j=1}^{N-1}\left|\mu_{j}-\theta\right| \zeta_{j}^{2}\right)^{2} \leq \sum_{j=1}^{N-1}\left(\mu_{j}-\theta\right)^{2} \sum_{j=1}^{N-1} \zeta_{j}^{4} \tag{17}
\end{equation*}
$$

We proceed by determining the value of $\theta$ that minimizes the right-hand side. The sum $\sum_{j=1}^{N-1}\left(\mu_{j}-\theta\right)^{2}$ is minimized when $\theta^{*}=\frac{1}{N-1} \sum_{j=1}^{N-1} \mu_{j}$, which either follows by straightforward calculus or directly using the variational principle [7, p. 13] of the variance of a random variable. Furthermore, since any $N \times N$ Laplacian matrix $Q$ possesses a zero eigenvalue $\mu_{N}=0$, we find that the minimizer

$$
\theta^{*}=\frac{1}{N-1} \sum_{j=1}^{N} \mu_{j}=\frac{N}{N-1} E[\mu]
$$

where, switching to the stochastic counterpart, $E[\mu]$ denotes the expectation of a randomly chosen eigenvalue of the Laplacian matrix $Q$, which also equals $E[\mu]=E[D]$, but the variance $\operatorname{Var}[\mu]=$ $\operatorname{Var}[D]+E[D]$ as shown in [11, p. 68-69]. Thus, the minimizer $\theta^{*}=E\left[\mu_{+}\right]=\frac{N}{N-1} E[\mu]$ equals the mean of the positive Laplacian eigenvalues, which is slightly larger than the average degree $d_{a v}=\frac{2 L}{N}$ in the graph. Introducing the minimizer $\theta^{*}=\frac{N}{N-1} d_{a v}$ into (16) results into

$$
\left||\partial \mathcal{V}|-\frac{N}{N-1} d_{a v} N S(1-S)\right| \leq \sum_{j=1}^{N-1}\left|\mu_{j}-\frac{N}{N-1} d_{a v}\right| \zeta_{j}^{2}
$$

We upper bound the right-hand side in a rather crude way as

$$
\sum_{j=1}^{N-1}\left|\mu_{j}-\frac{N}{N-1} d_{a v}\right| \zeta_{j}^{2} \leq \max _{1 \leq j \leq N-1}\left|\mu_{j}-\frac{N}{N-1} d_{a v}\right| \sum_{j=1}^{N-1} \zeta_{j}^{2}
$$

where equality is only attained when $\mu_{j}-\frac{N}{N-1} d_{a v}$ is the same for any $j$, and thus only for the complete graph. Invoking (45), we arrive at the isoperimetric inequality (15).

We rephrase (15) in terms of Chung's Theorem 1. The isoperimetric inequality (15) approximates the number of infective links $|\partial \mathcal{V}|=e(X, Y)$ by $\frac{d_{a v}}{N-1}(N S)(N-N S)$, the product of the number $N S=|X|$ of infected nodes and the number $|Y|=N-N S$ of susceptible nodes multiplied by a constant probability $\frac{d_{a v}}{N-1}$ that an infectious link exists between with cluster $X$ and $Y$. In contrast to Chung's Theorem 1, the isoperimetric inequality (15) is exact for the complete graph $K_{N}$, where $\mu_{j}=N$ for $1 \leq j<N$, thus

$$
\mid \partial \mathcal{V} \|_{K_{N}}=N S(N-N S)
$$

is the largest possible in any graph on $N$ nodes, given a same Bernoulli vector $w$. If one node, say $l$, is infected and $S=\frac{1}{N}$, then $w=e_{l}$ so that $|\partial \mathcal{V}|=d_{l}$ and the bound on the isoperimetric inequality (15) $\left|d_{l}-d_{a v}\right| \leq \max _{1 \leq j \leq N-1}\left|\frac{N-1}{N} \mu_{j}-d_{a v}\right|$ is sharp. We have mentioned that the upper bound in (15) is rather crude, so that the approximation $w^{T} Q w \approx \frac{2 L N}{N-1} S(1-S)$ is generally better than the upper bound indicates, which has motivated the probabilistic approach in Section 3.

We end this section by providing an alternative approach to the isoperimetric inequality. Consider

$$
\begin{aligned}
\sum_{j=1}^{N-1}\left(\mu_{j}-\theta\right)^{2} \zeta_{j}^{2} & =\sum_{j=1}^{N-1} \mu_{j}^{2} \zeta_{j}^{2}-2 \theta \sum_{j=1}^{N-1} \mu_{j} \zeta_{j}^{2}+\theta^{2} \sum_{j=1}^{N-1} \zeta_{j}^{2} \\
& =w^{T} Q^{2} w-2 \theta w^{T} Q w+\theta^{2} N S(1-S)
\end{aligned}
$$

which is minimized if $\theta$ equals

$$
\widetilde{\theta}=\frac{w^{T} Q w}{N S(1-S)}
$$

yielding

$$
\begin{equation*}
\sum_{j=1}^{N-1}\left(\mu_{j}-\widetilde{\theta}\right)^{2} \zeta_{j}^{2}=w^{T} Q^{2} w-\frac{\left(w^{T} Q w\right)^{2}}{N S(1-S)} \geq 0 \tag{18}
\end{equation*}
$$

and

$$
\sum_{j=1}^{N-1}\left(\mu_{j}-\widetilde{\theta}\right) \zeta_{j}^{2}=0
$$

Inequality (17) indicates that $\sum_{j=1}^{N-1}\left|\mu_{j}-\theta\right| \zeta_{j}^{2}$ is minimized if $\theta$ equals $\theta^{*}=E\left[\mu_{+}\right]=\frac{N}{N-1} E[\mu]=$ $\frac{N}{N-1} d_{a v}$. The difference $h=\widetilde{\theta}-\theta^{*}$ between the two minimizers,

$$
h=\frac{|\partial \mathcal{V}|}{N S(1-S)}-\frac{N}{N-1} d_{a v}
$$

reflects the precise isoperimetric equality. A sharp isoperimetric inequality thus requires to accurately lower and upper bound $|h|$.

## 3 When components of the Bernoulli vector $w$ are independent

In this section, we provide a probabilistic estimate of the cut size $|\partial \mathcal{V}|$ in the special case, where we assume that
(a) each component $w_{k}$ of the $N \times 1$ Bernoulli vector $w$ is a Bernoulli random variable with mean $E\left[w_{k}\right]=v$ and
(b) each component of $w$ is independent from any other component.

The assumption (a) and (b) mean that each node $k$ has equal probability $v$ to be infected, independent of the infection state of any other node. Clearly, this assumption is confining. For example, in any irregular graph, where not all nodes have the same degree, some nodes have higher probability of infection than others. Examples are hubs (or large degree nodes) in networks. In reality, the infection of some clusters $\mathcal{V}$ is thus more probable than that of others. However, without the assumptions (a) and (b), an analysis as presented here is intractable and we can only resort to the SIS governing equation (1) or the prevalence differential equation (4), that cannot be solved analytically so far (not even for the complete graph [7, p. 456]).

Assumption (a) that $E\left[w_{k}\right]=v$ for all $1 \leq k \leq N$ nodes leads to $E[w]=v . u$. In terms of the definition (3) of the prevalence, we observe that $y=\frac{1}{N} E\left[w^{T} u\right]=v$. Under the assumption (a) of equal expectation $E\left[w_{k}\right]=v$ for all $1 \leq k \leq N$, the expectation of the random vector $\zeta=Z^{T} w$ is

$$
E[\zeta]=Z^{T} E[w]=v Z^{T} u=\sqrt{N} v e_{N}
$$

or, component-wise,

$$
\left\{\begin{array}{l}
E\left[\zeta_{j}\right]=0 \quad \text { for } 1 \leq j<N  \tag{19}\\
E\left[\zeta_{N}\right]=v \sqrt{N}
\end{array}\right.
$$

The joint expectation follows from (39) as

$$
\begin{aligned}
E\left[\zeta_{j} \zeta_{m}\right] & =E\left[\sum_{k=1}^{N} \sum_{l=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{l} w_{k} w_{l}\right]=\sum_{k=1}^{N} \sum_{l=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{l} E\left[w_{k} w_{l}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} E\left[w_{k}\right]+\sum_{k=1}^{N} \sum_{l=1 ; l \neq k}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{l} E\left[w_{k} w_{l}\right]
\end{aligned}
$$

Introducing the assumption (b) of independence, which implies that $E\left[w_{k} w_{l}\right]=E\left[w_{k}\right] E\left[w_{l}\right]$, we obtain

$$
\begin{aligned}
E\left[\zeta_{j} \zeta_{m}\right] & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} E\left[w_{k}\right]+\sum_{k=1}^{N}\left(z_{j}\right)_{k} E\left[w_{k}\right] \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l} E\left[w_{l}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} E\left[w_{k}\right]+\sum_{k=1}^{N}\left(z_{j}\right)_{k} E\left[w_{k}\right]\left(\sum_{l=1}^{N}\left(z_{m}\right)_{l} E\left[w_{l}\right]-\left(z_{m}\right)_{k} E\left[w_{k}\right]\right)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\zeta_{j} \zeta_{m}\right] & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} E\left[w_{k}\right]\left(1-E\left[w_{k}\right]\right)+\sum_{k=1}^{N}\left(z_{j}\right)_{k} E\left[w_{k}\right] \sum_{l=1}^{N}\left(z_{m}\right)_{l} E\left[w_{l}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} E\left[w_{k}\right]\left(1-E\left[w_{k}\right]\right)+E\left[\zeta_{j}\right] E\left[\zeta_{m}\right]
\end{aligned}
$$

from which the covariance follows, with (19) and $\operatorname{Var}\left[w_{k}\right]=E\left[w_{k}\right]\left(1-E\left[w_{k}\right]\right)$, as

$$
\operatorname{Cov}\left[\zeta_{j}, \zeta_{m}\right]=E\left[\zeta_{j} \zeta_{m}\right]-E\left[\zeta_{j}\right] E\left[\zeta_{m}\right]=\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \operatorname{Var}\left[w_{k}\right]
$$

Invoking the assumption (a) that $E\left[w_{k}\right]=v$, independent of the nodal index $k$, and due to "double orthogonality" [15], $\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k}=\delta_{j m}$, and $z_{j}^{T} u=0$ for $j<N$, we find for $j<N$ that the covariance simplifies to ${ }^{4}$

$$
\operatorname{Cov}\left[\zeta_{j}, \zeta_{m}\right]=v(1-v) \delta_{j m}
$$

Finally, invoking (19), we arrive at the joint expectations

$$
\left\{\begin{array}{c}
E\left[\zeta_{j} \zeta_{m}\right]=v(1-v) \delta_{j m} \quad \text { for } 1 \leq j<N  \tag{20}\\
E\left[\zeta_{N}^{2}\right]=v+v^{2}(N-1) \quad \text { when } j=m=N
\end{array}\right.
$$

In summary, under the assumptions (a) that each component $w_{k}$ is a Bernoulli random variable with mean $E\left[w_{k}\right]=v$ and (b) independent of any other component $w_{m}$ (with $m \neq k$ ), the projection $\zeta_{j}=w^{T} z_{j}$ with index $1 \leq j<N$ is a zero mean random variable with variance $\operatorname{Var}\left[\zeta_{j}\right]=v(1-v)$, and uncorrelated to any other $\zeta_{m}$ with $1 \leq m \neq j \leq N$. However, the random variables $\zeta_{j}$ and $\zeta_{m}$ are not independent! Indeed, invoking (46) computed in Appendix B

$$
E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}^{2}+v^{2}(1-v)^{2}\left(1+2 \delta_{j m}\right)
$$

and $E\left[\zeta_{j}^{2}\right] E\left[\zeta_{m}^{2}\right]=v^{2}(1-v)^{2}$, which follows from (20), illustrates the dependence ${ }^{5}$ between the random variables $\zeta_{j}$ and $\zeta_{m}$.

The random variable $\zeta_{N}$ has mean $E\left[\zeta_{N}\right]=v \sqrt{N}$, but the same $\operatorname{Var}\left[\zeta_{N}\right]=v(1-v)$ as $\operatorname{Var}\left[\zeta_{j}\right]$ with $1 \leq j<N$. Since each $w_{k}$ is a Bernoulli random variable with equal expectation $E\left[w_{k}\right]=v$, the variance $\operatorname{Var}\left[w_{k}\right]=v(1-v)$ is thus the same as the variance of each dependent random variable $\zeta_{m}$, which agrees with the geometric interpretation of the orthogonal matrix $Z$ as a rotation operator [16, Sec. 8.2]. Indeed, the subspace spanned by all possible Bernoulli vectors $w$ is rotated by $Z$ into the subspace of all $\zeta$ vectors without changing the distances (due to (40), both vectors have equal Euclidean norm $\|w\|=\|\zeta\|)$. Hence, the variation in distance of all those vectors $w$ around the mean vector $E[w]=v u$ is thus equal to the variation of all $\zeta$ vectors around their mean vector $E[\zeta]=\sqrt{N} v e_{N}$.

$$
\begin{aligned}
& { }^{4} \text { The dual case, where } w_{j}=\sum_{k=1}^{N}\left(z_{k}\right)_{j} \zeta_{k} \text { follows from (38), similarly yields with (20), } \\
& \qquad \operatorname{Cov}\left[w_{j}, w_{m}\right]=E\left[w_{j} w_{m}\right]-E\left[w_{j}\right] E\left[w_{m}\right]=\sum_{k=1}^{N}\left(z_{k}\right)_{j}\left(z_{k}\right)_{m} \operatorname{Var}\left[\zeta_{k}\right]
\end{aligned}
$$

Invoking $\operatorname{Var}\left[\zeta_{k}\right]=v(1-v)$ for all $1 \leq k \leq N$ and double orthogonality $\sum_{k=1}^{N}\left(z_{k}\right)_{j}\left(z_{k}\right)_{m}=\delta_{j m}$ then results in

$$
\operatorname{Cov}\left[w_{j}, w_{m}\right]=v(1-v) \delta_{j m}
$$

[^4]
### 3.1 Probability generating function $\varphi_{\zeta_{j}}(t)$ of $\zeta_{j}$

Invoking the independence assumption (b) among components $w_{k}$ of the Bernoulli vector $w$, the probability generating function of random variable $\zeta_{j}$, defined in (39), equals

$$
\begin{aligned}
\varphi_{\zeta_{j}}(t) & =E\left[e^{-t \zeta_{j}}\right]=E\left[e^{-t \sum_{k=1}^{N}\left(z_{j}\right)_{k} w_{k}}\right]=\prod_{k=1}^{N} E\left[e^{-t\left(z_{j}\right)_{k} w_{k}}\right] \\
& =\prod_{k=1}^{N}\left(\operatorname{Pr}\left[w_{k}=0\right]+e^{-t\left(z_{j}\right)_{k}} \operatorname{Pr}\left[w_{k}=1\right]\right)
\end{aligned}
$$

Let us denote $\operatorname{Pr}\left[w_{k}=1\right]=v_{k}$, where $v_{k}$ is the probability that node $k$ belongs to the infected cluster, then

$$
\varphi_{\zeta_{j}}(t)=\prod_{k=1}^{N}\left(1-v_{k}+e^{-t\left(z_{j}\right)_{k}} v_{k}\right)
$$

After taking the logarithm, we obtain

$$
\log \varphi_{\zeta_{j}}(t)=\sum_{k=1}^{N} \log \left(1+v_{k}\left(e^{-t\left(z_{j}\right)_{k}}-1\right)\right)
$$

We expand the logarithmic terms in a Taylor series,

$$
\log \left(1+v_{k}\left(e^{-t\left(z_{j}\right)_{k}}-1\right)\right)=\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} v_{k}^{r}\left(e^{-t\left(z_{j}\right)_{k}}-1\right)^{r}
$$

Invoking the generating function of the Stirling numbers $\mathcal{S}_{m}^{(k)}$ of the Second Kind [17],

$$
\left(e^{x}-1\right)^{k}=k!\sum_{m=k}^{\infty} \mathcal{S}_{m}^{(k)} \frac{x^{m}}{m!}
$$

yields

$$
\log \left(1+v_{k}\left(e^{-t\left(z_{j}\right)_{k}}-1\right)\right)=\sum_{r=1}^{\infty}(-1)^{r-1} v_{k}^{r}(r-1)!\sum_{m=r}^{\infty} \mathcal{S}_{m}^{(r)} \frac{(-1)^{m}\left(z_{j}\right)_{k}^{m}}{m!} t^{m}
$$

Reversing the $r$-and $m$-sum results in

$$
\log \left(1+v_{k}\left(e^{-t\left(z_{j}\right)_{k}}-1\right)\right)=\sum_{m=1}^{\infty}\left(\left(z_{j}\right)_{k}^{m} \sum_{r=1}^{m}(-1)^{r-1}(r-1)!\mathcal{S}_{m}^{(r)} v_{k}^{r}\right) \frac{(-1)^{m} t^{m}}{m!}
$$

and

$$
\log \varphi_{\zeta_{j}}(t)=\sum_{m=1}^{\infty}\left(\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m} p_{m}\left(v_{k}\right)\right) \frac{(-1)^{m} t^{m}}{m!}
$$

where $p_{m}(v)$ is a polynomial of order $m$ in $v$,

$$
\begin{equation*}
p_{m}(v)=\sum_{r=1}^{m}(-1)^{r-1}(r-1)!\mathcal{S}_{m}^{(r)} v^{r} \tag{21}
\end{equation*}
$$

for example,

$$
\begin{aligned}
& p_{1}(v)=v \\
& p_{2}(v)=v(1-v) \\
& p_{3}(v)=v(1-v)(1-2 v) \\
& p_{4}(v)=v(1-v)\left(1-6 v+6 v^{2}\right)
\end{aligned}
$$

Using $\mathcal{S}_{m}^{(m)}=1$ and $\mathcal{S}_{m}^{(1)}=1$, the first term for $m=1$ equals $\sum_{k=1}^{N}\left(z_{j}\right)_{k} v_{k}$ and the second term equals $\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(v_{k}-v_{k}^{2}\right)$, which are difficult to evaluate, unless all probabilities $v_{k}=v$ are the same. So far, the derivation is general and holds for any linear transformation of the Benoulli vector $w$ with independent components.

Confining to the case $v_{k}=v$ (assumption (a)), we can exploit the orthogonality properties $\sum_{k=1}^{N}\left(z_{j}\right)_{k}=\sqrt{N} \delta_{j N}$ and $\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}=1$ for any Laplacian matrix $Q$, and we finally arrive for $j<N$ at

$$
\begin{equation*}
\log \varphi_{\zeta_{j}}(t)=v(1-v) \frac{t^{2}}{2}+\sum_{m=3}^{\infty}\left(\frac{(-1)^{m} p_{m}(v)}{m!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m}\right) t^{m} \tag{22}
\end{equation*}
$$

Expression (22) illustrates that each $\zeta_{j}$ for $1 \leq j<N$ possesses a same first zero moment and a same second moment (i.e. $v(1-v)$ ) and that only higher moments are different, because, generally, $\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m} \neq \sum_{k=1}^{N}\left(z_{i}\right)_{k}^{m}$ for $m>2$.

### 3.1.1 The polynomial $p_{m}(v)$

Since $\left|\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m}\right| \leq \sum_{k=1}^{N}\left|\left(z_{j}\right)_{k}\right|^{m}$ decreases with $m$, because $\left|\left(z_{j}\right)_{k}\right|<1$ by (35) and thus $\left|\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m}\right| \leq$ $\sum_{k=1}^{N}\left|\left(z_{j}\right)_{k}\right|^{2}=1$, the series (22) converges faster than the generating function of the polynomials $p_{m}(v)$,

$$
\log \left(1+v\left(e^{-t}-1\right)\right)=\sum_{m=1}^{\infty} p_{m}(v) \frac{(-1)^{m} t^{m}}{m!}
$$

After differentiation with respect to $t$, we obtain

$$
\frac{v e^{-t}}{1+v\left(e^{-t}-1\right)}=\sum_{m=1}^{\infty} p_{m}(v) \frac{(-1)^{m-1} t^{m-1}}{(m-1)!}=\sum_{m=0}^{\infty} p_{m+1}(v) \frac{(-1)^{m} t^{m}}{m!}
$$

Since $\frac{v e^{-t}}{1+v\left(e^{-t}-1\right)}=\frac{1}{\frac{1-v}{v} e^{t}+1}=\frac{1}{1+e^{t+\log \frac{1-v}{v}}}$ and $\log \frac{1-v}{v}=-\log \frac{v}{1-v}$, we observe that the transform $v \rightarrow 1-v$ results in the symmetry equation

$$
p_{m}(v)=(-1)^{m} p_{m}(1-v) \quad \text { for } m>1
$$

The invariance of the cut size $|\partial \mathcal{V}|$ in (11) under the transform $v \rightarrow 1-v$ is natural, because $v \rightarrow 1-v$ transforms the set $\mathcal{V}$ of infected nodes to its complement $\mathcal{V}^{c}$, the healthy nodes and vice versa, but does not change the number $|\partial \mathcal{V}|$ of infectious links.

Perhaps more important, the generating function of the polynomials $p_{m}(v)$ is a member of the Fermi-Dirac integrals, because

$$
\frac{1}{1+e^{t+\log \frac{1-v}{v}}}=F_{-1}\left(\log \frac{v}{1-v}-t\right)
$$

where the Fermi-Dirac integral of order $p$, for $\operatorname{Re}(p)>-1$, is defined as

$$
\begin{equation*}
F_{p}(y)=\frac{1}{\Gamma(p+1)} \int_{0}^{\infty} \frac{x^{p}}{1+e^{x-y}} d x \tag{23}
\end{equation*}
$$

from which the complex integral for the Fermi-Dirac integral, for any $p \in \mathbb{C}$,

$$
\begin{equation*}
F_{p}(y)=-\frac{\Gamma(-p)}{2 \pi i} \int_{C} \frac{(-z)^{p} d z}{1+e^{z-y}} \tag{24}
\end{equation*}
$$

with contour $C$ starting at $+\infty$ above the positive real axis, encircling the origin and returning to $+\infty$ below the positive real axis, be deduced [18, Chapter 13]. Since $F_{p}(0)=\eta(p+1)$, where the Eta function $\eta(s)=\left(1-2^{1-s}\right) \zeta(s)$, the zero argument Fermi-Dirac integral is connected to the Riemann Zeta function $\zeta(s)$ investigated in [19]. By differentiating (24) with respect to $y$, we find a functional equation, valid for all complex $y$ and $p$,

$$
\begin{equation*}
\frac{d F_{p}(y)}{d y}=F_{p-1}(y) \tag{25}
\end{equation*}
$$

It follows from the generating function and (25) that

$$
p_{m+1}(v)=\left.(-1)^{m} \frac{d^{m}}{d t^{m}} F_{-1}\left(\log \frac{v}{1-v}-t\right)\right|_{t=0}=F_{-1-m}\left(\log \frac{v}{1-v}\right)
$$

Thus for $m>0$, the polynomial $p_{m}(v)$ equals the negative, integer order Fermi-Dirac integral

$$
\begin{equation*}
p_{m}(v)=F_{-m}\left(\log \frac{v}{1-v}\right) \tag{26}
\end{equation*}
$$

There exists a wealth of properties of the Fermi-Dirac integrals, for which we refer to [18, Chapter 13].
We add to the theory of the Fermi-Dirac integrals by demonstrating that all zeros of the polynomial $p_{m}(v)$ are real and lie in the interval $[0,1]$. Applying the functional equation (25) to $p_{m}(v)=$ $F_{-m}\left(\log \frac{v}{1-v}\right)$ yields

$$
\begin{equation*}
p_{m}(v)=(1-v) v p_{m-1}^{\prime}(v) \tag{27}
\end{equation*}
$$

Suppose that $p_{m}(v)$ has only real zeros in $[0,1]$, then also $p_{m}^{\prime}(v)$ has all zeros in $[0,1]$ by Gauss's theorem [11, p. 296] or Rolle's theorem. Relation (27) then demonstrates that also $p_{m+1}(v)$ has all zeros in $[0,1]$. The induction is initiated for $m=3$, which then proves the truth for all $m$.

### 3.1.2 The moments $E\left[\zeta_{j}^{m}\right]$

The probability density function (50) of $f_{\zeta_{j}}(x)=\frac{d}{d x} \operatorname{Pr}\left[\zeta_{j} \leq x\right]$ is computed in Appendix C as

$$
f_{\zeta_{j}}(x)=\frac{e^{-\frac{x^{2}}{2 v(1-v)}}}{\sqrt{2 \pi(v(1-v))}}\left(1+h_{j}(x)\right)
$$

where $h_{j}(x)$ specifies the deviation from the Gaussian distribution. For our purpose, the investigation of the cut size, we further confine to the moments $E\left[\zeta_{j}^{m}\right]$.

Our characteristic coefficients $s[k, m]$ of a complex function $f(z)=\sum_{k=0}^{\infty} f_{k} z^{k}$, defined as

$$
\begin{equation*}
s[k, m]=\sum_{\sum_{i=1}^{k} j_{i}=m ; j_{i}>0} \prod_{i=1}^{k} f_{j_{i}} \tag{28}
\end{equation*}
$$

and introduced in [20, Sec. 2 and Appendix], can be shown to provide the Taylor series for any $x$,

$$
\begin{equation*}
e^{x f(z)}=e^{x f_{0}}\left(1+\sum_{m=1}^{\infty}\left[\sum_{k=1}^{m} \frac{x^{k}}{k!} s[k, m]\right] z^{m}\right) \tag{29}
\end{equation*}
$$

Applying (29) to the function $\log \varphi_{\zeta_{j}}(t) \equiv \sum_{k=0}^{\infty} f_{k}(j) z^{k}$ in (22) for $1 \leq j<N$, with Taylor coefficients $f_{0}(j)=f_{1}(j)=0, f_{2}(j)=\frac{v(1-v)}{2}$ and $f_{m}(j)=\frac{(-1)^{m} p_{m}(v)}{m!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m}$ for $m \geq 3$, we find

$$
\varphi_{\zeta_{j}}(t)=1+\sum_{m=2}^{\infty}\left[\sum_{k=1}^{m} \frac{1}{k!} s_{\log \varphi_{\zeta_{j}}(t)}[k, m]\right] t^{m}
$$

Since $\varphi_{\zeta_{j}}(t)=E\left[e^{-t \zeta_{j}}\right]=\sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} t^{m} E\left[\zeta_{j}^{m}\right]$, equating corresponding powers in $t$ yields $E\left[\zeta_{j}\right]=0$ and, for $m>1$,

$$
\begin{equation*}
E\left[\zeta_{j}^{m}\right]=(-1)^{m} m!\sum_{k=1}^{m} \frac{1}{k!} s_{\log \varphi_{\zeta_{j}}(t)}[k, m] \tag{30}
\end{equation*}
$$

The characteristic coefficients $s\left[k, m\right.$ ] satisfy a recursion [20, eq. (3)], from which, denoting $Z_{j}(m)=$ $\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m}$, the first few moments $E\left[\zeta_{j}^{m}\right]$ are computed as ${ }^{6}$

$$
\begin{aligned}
E\left[\zeta_{j}\right] & =v \sqrt{N} \delta_{j N} \\
E\left[\zeta_{j}^{2}\right] & =v(1-v) \\
E\left[\zeta_{j}^{2}\right] & =\left(v-3 v^{2}+2 v^{3}\right) Z_{j}(3) \\
E\left[\zeta_{j}^{4}\right] & =\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) Z_{j}(4)+3(1-v)^{2} v^{2} \\
E\left[\zeta_{j}^{5}\right] & =10(1-v) v\left(v-3 v^{2}+2 v^{3}\right) Z_{j}(3)+\left(v-15 v^{2}+50 v^{3}-60 v^{4}+24 v^{5}\right) Z_{j}(5) \\
E\left[\zeta_{j}^{6}\right] & =10\left(2 v^{3}-3 v^{2}+v\right)^{2} Z_{j}^{2}(3)+15(1-v) v\left(-6 v^{4}+12 v^{3}-7 v^{2}+v\right) Z_{j}(4) \\
& \left(-120 v^{6}+360 v^{5}-390 v^{4}+180 v^{3}-31 v^{2}+v\right) Z_{j}(6)+15(1-v)^{3} v^{3}
\end{aligned}
$$

### 3.2 The number of links in the cut-set

The spectral decomposition (11) of the cut size $|\partial \mathcal{V}|=\sum_{j=1}^{N-1} \zeta_{j}^{2} \mu_{j}$, the number of links in the cut-set of a connected graph specified by the Bernoulli vector $w$, is a weighted linear combination of dependent random variables $\zeta_{j}^{2}$, with mean $E\left[\zeta_{j}^{2}\right]=v(1-v)$ and variance $\operatorname{Var}\left[\zeta_{j}^{2}\right]=E\left[\zeta_{j}^{4}\right]-\left(E\left[\zeta_{j}^{2}\right]\right)^{2}=$ $\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) Z_{j}(4)+2(1-v)^{2} v^{2}$. Unfortunately, the dependence among the random variables $\zeta_{j}^{2}$ prevents the application of the powerful method of probability generating functions [7],

$$
\varphi_{|\partial \mathcal{V}|}(t)=E\left[e^{-t|\partial \mathcal{V}|}\right]=E\left[e^{-t \sum_{j=1}^{N-1} \zeta_{j}^{2} \mu_{j}}\right] \neq \prod_{j=1}^{N-1} E\left[e^{-\zeta_{j}^{2} t \mu_{j}}\right]
$$

so that a simple relation between $\varphi_{\zeta_{j}^{2}}(t)=E\left[e^{-t \zeta_{j}^{2}}\right]$, which is generally computable ${ }^{7}$ as shown in Section 3.1 and 3.1.2, and $\varphi_{|\partial \mathcal{V}|}(t)=E\left[e^{-t|\mathcal{V}|}\right]$ is unlikely to exist. One may question as well

[^5]whether the spectral approach (11) of the cut size $|\partial \mathcal{V}|$ has actually advantages over the topologicial one in (5)? Nevertheless, the first few moments $E\left[|\partial \mathcal{V}|^{k}\right]$ can be computed,

Theorem 3 In a graph $G$ with $N$ nodes and L links, there are two partitions, the set of all infected nodes and the set of all healthy nodes. The $N \times 1$ vector $w$ is a Bernoulli random vector that specifies whether a node $k$ in $G$ belongs to the infected partition if $w_{k}=1$, else the node $k$ belongs to the other partition $\left(w_{k}=0\right)$. Each node $k$ is assumed to have equal probability, independent of all the other nodes, to be infected with mean $E\left[w_{k}\right]=\operatorname{Pr}\left[w_{k}=1\right]=v$ for all $1 \leq k \leq N$ nodes. Under these assumptions, the average cut size, i.e. the number of links in the cut-set, equals

$$
\begin{equation*}
E[|\partial \mathcal{V}|]=2 L v(1-v) \tag{31}
\end{equation*}
$$

and the corresponding variance of the cut size is

$$
\begin{equation*}
\operatorname{Var}[|\partial \mathcal{V}|]=\left(v-5 v^{2}+8 v^{3}-4 v^{4}\right) \sum_{k=1}^{N} d_{k}^{2}+4 L v^{2}(1-v)^{2} \tag{32}
\end{equation*}
$$

where $d_{k}$ is the degree of node $k$.
Proof $^{8}$ : Under the above assumptions (a) and (b), the average number (31) of links in the cut-set follows from (11), (20) and $\sum_{j=1}^{N-1} \mu_{j}=2 L$ (see e.g. [11, p. 68]) as

$$
E[|\partial \mathcal{V}|]=\sum_{j=1}^{N-1} E\left[\zeta_{j}^{2}\right] \mu_{j}=v(1-v) \sum_{j=1}^{N-1} \mu_{j}=2 L v(1-v)
$$

Next, we compute

$$
E\left[|\partial \mathcal{V}|^{2}\right]=E\left[\sum_{j=1}^{N-1} \sum_{m=1}^{N-1} \mu_{j} \mu_{m} \zeta_{j}^{2} \zeta_{m}^{2}\right]=\sum_{j=1}^{N-1} \sum_{m=1}^{N-1} \mu_{j} \mu_{m} E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]
$$

Invoking (46) from Appendix B

$$
E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}^{2}+v^{2}(1-v)^{2}\left(1+2 \delta_{j m}\right)
$$

we obtain

$$
\begin{aligned}
E\left[|\partial \mathcal{V}|^{2}\right] & =\sum_{j=1}^{N-1} \sum_{m=1}^{N-1} \mu_{j} \mu_{m} E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right] \\
& =v^{2}\left(1-v^{2}\right) \sum_{j=1}^{N-1} \sum_{m=1}^{N-1} \mu_{j} \mu_{m}\left(1+2 \delta_{j m}\right)+\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) \sum_{k=1}^{N}\left(\sum_{j=1}^{N-1} \mu_{j}\left(z_{j}\right)_{k}^{2}\right)^{2}
\end{aligned}
$$

Since $\sum_{j=1}^{N-1} \mu_{j}\left(z_{j}\right)_{k}^{2}=d_{k}$ (see e.g. [15]), we have

$$
E\left[|\partial \mathcal{V}|^{2}\right]=\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) \sum_{k=1}^{N} d_{k}^{2}+(2 L v(1-v))^{2}+2 v^{2}\left(1-v^{2}\right) \sum_{j=1}^{N-1} \mu_{j}^{2}
$$

[^6]Finally, with $\sum_{j=1}^{N-1} \mu_{j}^{2}=2 L+\sum_{k=1}^{N} d_{k}^{2}$ (see [11, art. 70, p. 68]), we arrive at ${ }^{9}$

$$
E\left[|\partial \mathcal{V}|^{2}\right]=\left(v-5 v^{2}+8 v^{3}-4 v^{4}\right) \sum_{k=1}^{N} d_{k}^{2}+(2 L v(1-v))^{2}+4 L v^{2}\left(1-v^{2}\right)
$$

The variance $\operatorname{Var}[|\partial \mathcal{V}|]=E\left[|\partial \mathcal{V}|^{2}\right]-(E[|\partial \mathcal{V}|])^{2}$ in (32) then follows with (31).

### 3.3 Consequences of Theorem 3

Rewriting (31) as

$$
E[|\partial \mathcal{V}|]=\frac{d_{a v}}{N} N v(N-N v)
$$

shows agreement with Chung's Theorem 1, but differs in the prefactor $\frac{d_{a v}}{N}$ instead of $\frac{d_{a v}}{N-1}$ from our Theorem 2 and Omic's Theorem 4 in Appendix D. The difference between (31) and the latter two Theorems lies in the consideration of a partition of $m$ infected nodes, while here only the average $N v$ of the infected nodes is defined. The difference is comparable to the Erdős-Rényi random graph with link existence probability $p$ versus the Erdős-Rényi random graph with precisely $L$ links.

Since $v-5 v^{2}+8 v^{3}-4 v^{4}=v(1-v)(2 v-1)^{2}$ and using $\frac{1}{N} \sum_{k=1}^{N} d_{k}^{2}=\operatorname{Var}[D]+(E[D])^{2}$ and $E[D]=d_{a v}=\frac{2 L}{N}$, the variance is rewritten as

$$
\frac{\operatorname{Var}[|\partial \mathcal{V}|]}{2 N v(1-v)}=2\left(v-\frac{1}{2}\right)^{2} E\left[D^{2}\right]+\left(\frac{1}{4}-\left(v-\frac{1}{2}\right)^{2}\right) E[D]
$$

and illustrates that $\operatorname{Var}[|\partial \mathcal{V}|] \sim v(1-v)(2 v-1)^{2} N E\left[D^{2}\right]$ when $N \rightarrow \infty$.
The Chebyshev inequality [7, p. 104],

$$
\begin{equation*}
\operatorname{Pr}[|X-E[X]| \geq t] \leq \frac{\sigma^{2}}{t^{2}} \tag{33}
\end{equation*}
$$

quantifies the spread of a random variable $X$ around its mean $E[X]$. The smaller $\sigma=\sqrt{\operatorname{Var}[X]}$, the more concentrated $X$ is around the mean. Applying the Chebyshev inequality (33) to the cut size yields

$$
\operatorname{Pr}[||\partial \mathcal{V}|-2 L v(1-v)| \geq x] \leq \frac{v(1-v)(2 v-1)^{2} \sum_{k=1}^{N} d_{k}^{2}+4 v^{2}(1-v)^{2} L}{x^{2}}
$$

Let $t=\frac{x}{2 L v(1-v)}$, then we arrive at

$$
\begin{equation*}
\operatorname{Pr}\left[\left|\frac{|\partial \mathcal{V}|}{2 L v(1-v)}-1\right| \geq t\right] \leq \frac{1}{L t^{2}}\left(1+\frac{2\left(v-\frac{1}{2}\right)^{2}}{v(1-v)}\left(\frac{\operatorname{Var}[D]}{E[D]}+E[D]\right)\right) \tag{34}
\end{equation*}
$$

The topological upper bound (10), $w^{T} Q w \leq L$, recast in probabilistic setting as

$$
\operatorname{Pr}\left[\left|\frac{|\partial \mathcal{V}|}{2 L v(1-v)}-1\right|>\frac{1-2 v(1-v)}{2 v(1-v)}\right]=0
$$

[^7]demonstrates the weakness of the Chebyshev inequality, because, for $t>\frac{1-2 v(1-v)}{2 v(1-v)}$, the right-hand side in (34) can always be replaced by zero. Nevertheless, the Chebyshev inequality (34) provides interesting insights.

The right-hand side of Chebyshev inequality (34) is minimal for $v=\frac{1}{2}$ at which the graph is separated into two clusters of equal size on average connected by $\frac{L}{2}$ links on average. For $v=\frac{1}{2}$, the Chebyshev inequality (34) shows that the cut size $|\partial \mathcal{V}|$ tends to $\frac{L}{2}$ in any graph, when the number $L$ of links grows large. Simulations seem to indicate that the cut size $|\partial \mathcal{V}|$ for $v=\frac{1}{2}$ is close to a Gaussian distribution.

Furthermore, among all graphs with fixed number $N$ of nodes and fixed number $L$ of links, the graphs that maximize the variance $\operatorname{Var}[D]$ of the degree incur the largest variations of the number of links in the cut-set. The maximum variance graphs ${ }^{10}$ contain $s=\left[\frac{L}{N-1}\right]$ full star nodes with degree $N-1$ and one partially filled node with degree $L-s$, while all other nodes have degree $s$ or $s+1$. Generally, but subject to the assumption in Theorem 3, power-law graphs with large degree variance lead to large fluctuations in the cut size, which may cause a fast and bursty epidemic spread. Under the less realistic assumption (a) of equal nodal infection probability, Theorem 3 supports the frequently repeated claim [22] of a vanishing epidemic threshold in power law graphs, although a rigorous proof (such as a demonstration that an upper bound of the epidemic threshold vanishes for power law graphs when $N \rightarrow \infty$ ) is missing.

The Chebyshev inequality (34) shows that, among all graphs with $N$ nodes and $L$ links, the deviation from the mean number of links in the cut-set is smallest for a regular graph with degree $D=r$ and $L=\frac{1}{2} r N$, because

$$
\operatorname{Pr}\left[\left|\frac{|\partial \mathcal{V}|_{\text {reg. graph }}}{r N v(1-v)}-1\right| \geq t\right] \leq \frac{2}{r N t^{2}}\left(1+\frac{2\left(v-\frac{1}{2}\right)^{2}}{v(1-v)} r\right)
$$

which illustrates that, if the degree $r$ and stringency ${ }^{11} t$ are fixed and not a function of $N$, then $\frac{\mid \partial \mathcal{V}_{\text {reg. graph }}}{r N v(1-v)} \rightarrow 1$ almost surely for large $N$ and $0<v<1$. Moreover, the assumption (a) that each node has equal infection probability, $E\left[w_{k}\right]=v$, holds in SIS epidemics for a regular graph after sufficiently long time. The independence assumption (b) is typically made in mean-field approximations [23, 24]. The SIS prevalence differential equation (4) reduces for a regular graph with $y=v$ and (31) to $\frac{d v}{d t^{*}}=-v+r \tau v(1-v)$, with steady-state $v_{\infty}=1-\frac{1}{\tau r}$ equal to that of the N-Intertwined Mean-Field Approximation (NIMFA) [7, p. 465].

Since the Chebyshev inequality (33) can be sharpened considerably if more of the distribution is known, we infer that the isoperimetric inequalities in Section 2.3 can be improved very likely. Indeed, the isoperimetric inequality (14) reads

$$
\left|\frac{|\partial \mathcal{V}|}{2 L v(1-v)}-1\right| \leq \frac{\max \left|\mu_{1}-d_{a v}, d_{a v}-\mu_{N-1}\right|}{d_{a v}}
$$

and for large $N$ the right-hand side grows with $N$, where the Chebyshev inequality (34) rather shows an increased accuracy with $N$ (unless $\frac{\operatorname{Var}[D]}{E[D]}$ grows with $N$ ).

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[^8]
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## A Background

This section extends the analysis in $[9$, Appendix B].

## A. 1 Orthogonal eigenvector matrix $Z$ of the Laplacian $Q$

As in [15], the $N \times 1$ real vector $z_{k}$ denotes the $k$-th eigenvector of the symmetric $N \times N$ Laplacian matrix $Q$ belonging to the eigenvalue $\mu_{k}$ and the eigenvalues of the Laplacian $Q$ are ordered as $\mu_{1} \geq \mu_{2} \geq \ldots \geq \mu_{N-1} \geq \mu_{N}=0$. The vector component $\left(z_{k}\right)_{i}$ represents the eigenvector component at eigenvalue or eigenfrequency $\mu_{k}$ for node $i$. The eigenvectors are normalized and obey the orthogonality requirement $z_{k}^{T} z_{m}=\delta_{k m}$, where $\delta_{k m}$ is the Kronecker delta, which is equal to one, if $k=m$, and otherwise $\delta_{k m}=0$. The eigenvector belonging to the zero eigenvalue $\mu_{N}=0$ equals $z_{N}=\frac{u}{\sqrt{N}}$, where the all one vector $u=(1,1, \ldots, 1)$. A major advantage in the spectral theory of the Laplacian $Q$ over the adjacency matrix $A$ is the knowledge of the eigenvector $z_{N}$. Indeed, the orthogonality of the eigenvectors shows that $z_{k}^{T} u=0$, implying that the sum over all components of any eigenvector $z_{k}$ for $1 \leq k<N$ equals zero, $\sum_{j=1}^{N}\left(z_{k}\right)_{j}=0$, while $\sum_{j=1}^{N}\left(z_{k}\right)_{j}^{2}=1$. Due to $z_{k}^{T} u=0$ and the fact that an eigenvector is never equal to the zero vector, there must be at least two non-zero components $\left(z_{k}\right)_{i}$ and $\left(z_{k}\right)_{j}$. Suppose that there are precisely two such non-zero components of $z_{k}$, then the normalization $z_{k}^{T} z_{k}=1$ demonstrates that $\left(z_{k}\right)_{i}=-\left(z_{k}\right)_{l}=\frac{\sqrt{2}}{2}$. Hence, the maximum possible value of any normalized eigenvector component $z_{k}$ for $1 \leq k \leq N$ equals

$$
\begin{equation*}
\left|\left(z_{k}\right)_{i}\right| \leq \frac{\sqrt{2}}{2} \tag{35}
\end{equation*}
$$

Since the eigenvectors of $Q$ constitute an orthogonal basis, any $N \times 1$ real vector $x$ can be expressed as a linear combination of the eigenvectors $z_{1}, z_{2}, \ldots, z_{N}$ of $Q$,

$$
x=\sum_{k=1}^{N} \alpha_{k} z_{k}
$$

where the scalar product $\alpha_{k}=x^{T} z_{k}$. In terms of the $N \times N$ orthogonal matrix $Z$ with eigenvectors in its columns [15], which satisfies the orthogonality conditions $Z Z^{T}=Z^{T} Z=I$ so that $Z^{-1}=Z^{T}$, we have

$$
\alpha=Z^{T} x \text { and } x=Z \alpha
$$

illustrating the one-to-one relation between the coordinates of $x$ expressed in the standard basis $e_{1}, e_{2}, \ldots, e_{N}$, where $\left(e_{k}\right)_{j}=\delta_{k j}$, and its coordinates $\alpha$ expressed in the basis of eigenvectors $z_{1}, z_{2}, \ldots, z_{N}$ of $Q$. Moreover, an orthogonal transformation $Z$ preserves the scalar product and, thus, also the Euclidean norm, $\alpha^{T} \alpha=x^{T} x$. Just as any real symmetric matrix, the Laplacian $Q$ has the eigendecomposition $Q=Z M Z^{T}$, where $M=\operatorname{diag}\left(\mu_{i}\right)$, or, in vector form, $Q=\sum_{k=1}^{N} \mu_{k} z_{k} z_{k}^{T}$. The quadratic form $x^{T} Q x=x^{T} Z M Z^{T} x$ or $x^{T} Q x=\sum_{k=1}^{N} \sum_{m=1}^{N} \alpha_{k} \alpha_{m} z_{k}^{T} Q z_{m}$ equals

$$
\begin{equation*}
x^{T} Q x=\sum_{k=1}^{N} \alpha_{k}^{2} \mu_{k} \tag{36}
\end{equation*}
$$

## A. 2 A spectral view on the Bernoulli vector $w$

The Bernoulli vector $w$ is a so-called binary vector, because each component $w_{k}$ is either zero or one, so that $w_{k}^{r}=w_{k}$ for any real $r>0$. For any Bernoulli vector $w$, we observe from (2) that

$$
\begin{equation*}
w^{T} w=\sum_{k=1}^{N} w_{k}^{2}=\sum_{k=1}^{N} w_{k}=u^{T} w=N S \tag{37}
\end{equation*}
$$

which means that the Euclidean norm of $w$ equals the sum of its vector components $u^{T} w$, equal to the number $N S$ of infected nodes in an epidemic setting. Let us consider the transformation between the vector $\zeta=Z^{T} w$ and the Bernoulli vector $w=Z \zeta$, which is the decomposition of the Bernoulli vector $w$ in terms of the eigenvectors $z_{1}, z_{2}, \ldots, z_{N}$ of the Laplacian $Q$,

$$
\begin{equation*}
w=\sum_{k=1}^{N} \zeta_{k} z_{k} \tag{38}
\end{equation*}
$$

where

$$
\begin{equation*}
\zeta_{k}=w^{T} z_{k}=\sum_{j=1}^{N} w_{j}\left(z_{k}\right)_{j} \tag{39}
\end{equation*}
$$

is the $k$-th coordinate of the Bernoulli vector $w$ along the $k$-th eigenvector $z_{k}$ in the eigenspace of $Q$. The total number $u^{T} w$ of infected nodes in the graph equals, by the Bernoulli vector property (37) and the fact that $Z$ is an orthogonal matrix

$$
\begin{equation*}
\zeta^{T} \zeta=w^{T} w=N S \tag{40}
\end{equation*}
$$

Since the normalized vector $z_{N}=\frac{u}{\sqrt{N}}$ is the eigenvector of any Laplacian matrix $Q$ belonging to the zero eigenvalue $\mu_{N}=0$, we deduce that $\zeta_{N}=w^{T} z_{N}=\frac{w^{T} u}{\sqrt{N}}=\sqrt{N} S$, and (38) becomes

$$
w=S u+\sum_{k=1}^{N-1} \zeta_{k} z_{k}
$$

which suggests us, as in [25], to define the difference vector

$$
\begin{equation*}
\varpi=S u-w \tag{41}
\end{equation*}
$$

where $S=\frac{u^{T} w}{N}$ equals the fraction of infected nodes, defined in (2). It is immediate that the difference vector equals ${ }^{12}$

$$
\begin{equation*}
\varpi=\sum_{k=1}^{N-1}\left(\varpi^{T} z_{k}\right) z_{k} \tag{42}
\end{equation*}
$$

a linear combination of all eigenvectors of the Laplacian $Q$ belonging to positive eigenvalues (for a connected graph). The definition (41) and (42) have two direct consequences. First,

$$
\varpi^{T} u=0
$$

implying that the difference vector $\varpi$ has mean zero, is orthogonal to the eigenvector $z_{N}=\frac{u}{\sqrt{N}}$ belonging to the zero Laplacian eigenvalue $\mu_{N}=0$. Next, the norm $\|\varpi\|=\sqrt{\varpi^{T} \varpi}$ follows, using $w^{T} w=N S$, from

$$
\begin{equation*}
\varpi^{T} \varpi=(w-S u)^{T}(w-S u)=N S(1-S) \tag{43}
\end{equation*}
$$

which also equals ${ }^{13}$, invoking (42) and orthogonality of the eigenvectors,

$$
\begin{equation*}
\varpi^{T} \varpi=\sum_{k=1}^{N-1}\left(\varpi^{T} z_{k}\right)^{2} \tag{44}
\end{equation*}
$$

The difference vector $\varpi$ determines for each node $j$ in the graph the variation of its infection state $w_{j}=X_{j}$ from the mean $S$ in the graph. Since $\zeta_{k}=\varpi^{T} x_{k}=w^{T} x_{k}$ for all $1 \leq k<N$, combining (43) and (44) gives us

$$
\begin{equation*}
\sum_{j=1}^{N-1} \zeta_{j}^{2}=N S(1-S) \tag{45}
\end{equation*}
$$

Finally, a Bernoulli vector $w$, which is different from the all-one vector $u$, can never be an eigenvector of the Laplacian $Q$ of a connected graph as shown in [27, Theorem 2], which implies that the vector $\zeta=Z^{T} w$ is never proportional to a basic vector $e_{k}$ for $1 \leq k \leq N$. In other words, the probability mass in the vector $\zeta$ is never concentrated in one eigenfrequency and at least two components of vector $\zeta$ are non-zero.

[^9]
## B Computation of the joint expectation $E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]$

We compute the joint expectation by invoking definition (39) of $\zeta_{j}$ for $j<N$ and $m<N$,

$$
\begin{aligned}
E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right] & =E\left[\left(\sum_{k=1}^{N}\left(z_{j}\right)_{k} w_{k}\right)^{2}\left(\sum_{l=1}^{N}\left(z_{m}\right)_{l} w_{l}\right)^{2}\right] \\
& =E\left[\sum_{k=1}^{N} \sum_{r=1}^{N} \sum_{l=1}^{N} \sum_{q=1}^{N}\left(z_{j}\right)_{k}\left(z_{j}\right)_{r}\left(z_{m}\right)_{l}\left(z_{m}\right)_{q} w_{l} w_{r} w_{k} w_{q}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1}^{N}\left(z_{j}\right)_{r} \sum_{l=1}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{r} w_{k} w_{q}\right]
\end{aligned}
$$

We write the quadruple sum as
$E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left\{\left(z_{j}\right)_{k} \sum_{l=1}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{k}^{2} w_{q}\right]+\sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{r} w_{k} w_{q}\right]\right\}$
Exploiting the Bernoulli property $w_{k}^{2}=w_{k}$ in the product $w_{l} w_{r} w_{k} w_{q}$ of Bernoulli random variables, yields
$E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{k} w_{q}\right]+\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{l} w_{r} w_{q}\right]$
Next, the individual sums in $E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=S_{1}+S_{2}$ are rewritten as

$$
\begin{aligned}
S_{1} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left\{\left(z_{m}\right)_{k} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{q}\right]+\sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{k} w_{q}\right]\right\} \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{q}\right]+\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{k} w_{q}\right]
\end{aligned}
$$

The first sum further simplifies, with $E\left[w_{j}\right]=v$, to

$$
\begin{aligned}
R_{1} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}\left\{\left(z_{m}\right)_{k} E\left[w_{k}\right]+v \sum_{q=1 ; q \neq k}^{N}\left(z_{m}\right)_{q} E\left[w_{q}\right]\right\} \\
& =v \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}^{2}+v^{2} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}\left\{\sum_{q=1}^{N}\left(z_{m}\right)_{q}-\left(z_{m}\right)_{k}\right\}
\end{aligned}
$$

Invoking orthogonality, $\sum_{q=1}^{N}\left(z_{m}\right)_{q}=0$ for $m \neq N$, and the definition

$$
T_{0}=\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}^{2}
$$

we obtain,

$$
R_{1}=\left(v-v^{2}\right) T_{0}
$$

The second sum is

$$
\begin{aligned}
R_{2} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{k} w_{q}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l}\left\{\left(z_{m}\right)_{k} E\left[w_{l} w_{k}\right]+\left(z_{m}\right)_{l} E\left[w_{l} w_{k}\right]+\sum_{q=1 ; q \neq k, l}^{N}\left(z_{m}\right)_{q} E\left[w_{l} w_{k} w_{q}\right]\right\}
\end{aligned}
$$

Taking into account independence $E\left[w_{l} w_{k}\right]=E\left[w_{l}\right] E\left[w_{k}\right]=v^{2}$ if $l \neq k$,

$$
\begin{aligned}
R_{2} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l}\left\{v^{2}\left(z_{m}\right)_{k}+v^{2}\left(z_{m}\right)_{l}+v^{3} \sum_{q=1 ; q \neq k, l}^{N}\left(z_{m}\right)_{q}\right\} \\
& =\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l}\left\{\left(z_{m}\right)_{k}+\left(z_{m}\right)_{l}\right\} \\
& =\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l}+\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l}^{2}
\end{aligned}
$$

With

$$
\begin{aligned}
\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l} & =-\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}^{2}=-T_{0} \\
\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2} \sum_{l=1 ; l \neq k}^{N}\left(z_{m}\right)_{l}^{2} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(\sum_{l=1}^{N}\left(z_{m}\right)_{l}^{2}-\left(z_{m}\right)_{k}^{2}\right)=1-T_{0}
\end{aligned}
$$

we obtain

$$
R_{2}=-2\left(v^{2}-v^{3}\right) T_{0}+\left(v^{2}-v^{3}\right)
$$

and $S_{1}=R_{1}+R_{2}$ is

$$
S_{1}=\left(v^{2}-v^{3}\right)+\left(v-3 v^{2}+2 v^{3}\right) T_{0}
$$

Similarly, the second sum

$$
S_{2}=\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{l} w_{r} w_{q}\right]
$$

is rewritten as

$$
\begin{aligned}
S_{2} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left\{\left(z_{m}\right)_{k} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right]+\left(z_{m}\right)_{r} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right]\right. \\
& \left.+\sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{l} w_{r} w_{q}\right]\right\} \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right]+\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right] \\
& +\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{l} w_{r} w_{q}\right]
\end{aligned}
$$

Next, we must split up the $q$-sums each of the three sums in $S_{2}=V_{1}+V_{2}+V_{3}$. First,

$$
\begin{aligned}
V_{1} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left\{\left(z_{m}\right)_{k} E\left[w_{k} w_{r}\right]+\left(z_{m}\right)_{r} E\left[w_{k} w_{r}\right]+\sum_{q=1 ; q \neq k, r}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right]\right\}
\end{aligned}
$$

Invoking independence,

$$
\begin{aligned}
V_{1} & =v^{2} \sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left\{\left(z_{m}\right)_{k}+\left(z_{m}\right)_{r}+v \sum_{q=1 ; q \neq k, r}^{N}\left(z_{m}\right)_{q}\right\} \\
& =\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left\{\left(z_{m}\right)_{k}+\left(z_{m}\right)_{r}\right\} \\
& =\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k}^{2} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}+\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}
\end{aligned}
$$

With $\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k}^{2} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}=-T_{0}$ and

$$
\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}=\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}-T_{0}=\delta_{j m}-T_{0}
$$

we have

$$
V_{1}=\left(v^{2}-v^{3}\right)\left(\delta_{j m}-2 T_{0}\right)
$$

where "double" orthogonality [15], $\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k}=\delta_{j m}$, has been invoked. Similarly as for $V_{1}$, we proceed as

$$
\begin{aligned}
V_{2} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{r} w_{q}\right] \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}\left\{\left(z_{m}\right)_{k} E\left[w_{k} w_{r}\right]+\left(z_{m}\right)_{r} E\left[w_{k} w_{r}\right]+E\left[w_{k} w_{r}\right] \sum_{q=1 ; q \neq k, r}^{N}\left(z_{m}\right)_{q} E\left[w_{q}\right]\right\} \\
& =\left(v^{2}-v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}\left\{\left(z_{m}\right)_{k}+\left(z_{m}\right)_{r}\right\}=V_{1}
\end{aligned}
$$

The third sum

$$
V_{3}=\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l} \sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{l} w_{r} w_{q}\right]
$$

is treated analogously. First we decompose the $q$-sum as

$$
\begin{aligned}
\widetilde{Q} & =\sum_{q=1}^{N}\left(z_{m}\right)_{q} E\left[w_{k} w_{l} w_{r} w_{q}\right] \\
& =\left(z_{m}\right)_{k} E\left[w_{k} w_{l} w_{r}\right]+\left(z_{m}\right)_{r} E\left[w_{k} w_{l} w_{r}\right]+\left(z_{m}\right)_{l} E\left[w_{k} w_{l} w_{r}\right]+E\left[w_{k} w_{l} w_{r}\right] \sum_{q=1 ; q \neq k, l, r}^{N}\left(z_{m}\right)_{q} E\left[w_{q}\right]
\end{aligned}
$$

Invoking independence and orthogonality yields

$$
\begin{aligned}
V_{3} & =\left(v^{3}-v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l}\left\{\left(z_{m}\right)_{k}+\left(z_{m}\right)_{r}+\left(z_{m}\right)_{l}\right\} \\
& =\left(v^{3}-v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l} \\
& +\left(v^{3}-v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l} \\
& +\left(v^{3}-v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l}^{2}
\end{aligned}
$$

With

$$
\begin{aligned}
T_{1} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l}=-\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(\left(z_{m}\right)_{k}+\left(z_{m}\right)_{r}\right) \\
& =-\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k}^{2} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}-\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r} \\
& =2 T_{0}-\delta_{j m}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{2} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l}=-\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}\left(\left(z_{m}\right)_{k}+\left(z_{m}\right)_{r}\right) \\
& =-\sum_{k=1}^{N}\left(z_{m}\right)_{k}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}-\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}^{2} \\
& =2 T_{0}-\delta_{j m}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{3} & =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r} \sum_{l=1 ; l \neq k, r}^{N}\left(z_{m}\right)_{l}^{2}=\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(1-\left(z_{m}\right)_{k}^{2}-\left(z_{m}\right)_{r}^{2}\right) \\
& =\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}-\sum_{k=1}^{N}\left(z_{j}\right)_{k}\left(z_{m}\right)_{k}^{2} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}-\sum_{k=1}^{N}\left(z_{j}\right)_{k} \sum_{r=1 ; r \neq k}^{N}\left(z_{j}\right)_{r}\left(z_{m}\right)_{r}^{2} \\
& =-1+2 T_{0}
\end{aligned}
$$

we have

$$
V_{3}=\left(v^{3}-v^{4}\right)\left\{6 T_{0}-2 \delta_{j m}-1\right\}
$$

and

$$
\begin{aligned}
S_{2} & =V_{1}+V_{2}+V_{3} \\
& =2\left(v^{2}-v^{3}\right)\left(\delta_{j m}-2 T_{0}\right)+\left(v^{3}-v^{4}\right)\left\{6 T_{0}-2 \delta_{j m}-1\right\} \\
& =-\left(v^{3}-v^{4}\right)+2 \delta_{j m}\left(v^{2}-2 v^{3}+v^{4}\right)+T_{0}\left\{-4 v^{2}+10 v^{3}-6 v^{4}\right\}
\end{aligned}
$$

Finally, we arrive with $E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=S_{1}+S_{2}$ at $E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=\left(v^{2}-v^{3}\right)+\left(v-3 v^{2}+2 v^{3}\right) T_{0}-\left(v^{3}-v^{4}\right)+2 \delta_{j m}\left(v^{2}-2 v^{3}+v^{4}\right)+T_{0}\left\{-4 v^{2}+10 v^{3}-6 v^{4}\right\}$ and

$$
\begin{equation*}
E\left[\zeta_{j}^{2} \zeta_{m}^{2}\right]=\left(v-7 v^{2}+12 v^{3}-6 v^{4}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{2}\left(z_{m}\right)_{k}^{2}+v^{2}(1-v)^{2}\left(1+2 \delta_{j m}\right) \tag{46}
\end{equation*}
$$

The tedious computation is readily generalized to

$$
\begin{aligned}
E\left[\left(w^{T} Q w\right)^{k}\right] & =E\left[\sum_{j_{1}=1}^{N-1} \sum_{j_{2}=1}^{N-1} \cdots \sum_{j_{k}=1}^{N-1} \mu_{j_{1}} \mu_{j_{2}} \ldots \mu_{j_{k}} \zeta_{j_{1}}^{2} \zeta_{j_{2}}^{2} \ldots \zeta_{j_{k}}^{2}\right] \\
& =\sum_{j_{1}=1}^{N-1} \sum_{j_{2}=1}^{N-1} \cdots \sum_{j_{k}=1}^{N-1} \prod_{i=1}^{k} \mu_{j_{i}} E\left[\prod_{i=1}^{k} \zeta_{j_{i}}^{2}\right]
\end{aligned}
$$

Further, by invoking the definition (39) of $\zeta_{j}$ for $j<N$ and $m<N$, we have

$$
\begin{aligned}
E\left[\prod_{i=1}^{k} \zeta_{j_{i}}^{2}\right] & =E\left[\prod_{i=1}^{k}\left(\sum_{k=1}^{N}\left(z_{j_{i}}\right)_{k} w_{k}\right)^{2}\right] \\
& =E\left[\prod_{i=1}^{k} \sum_{m_{i}=1}^{N}\left(z_{j_{i}}\right)_{m_{i}} \sum_{l_{i}=1}^{N}\left(z_{j_{i}}\right)_{l_{i}} w_{m_{i}} w_{l_{i}}\right] \\
& =E\left[\sum_{m_{1}=1}^{N}\left(z_{j_{1}}\right)_{m_{1}} \sum_{l_{1}=1}^{N}\left(z_{j_{1}}\right)_{l_{1}} w_{m_{1}} w_{l_{1}} \cdots \sum_{m_{k}=1}^{N}\left(z_{j_{k}}\right)_{m_{k}} \sum_{l_{k}=1}^{N}\left(z_{j_{k}}\right)_{l_{k}} w_{m_{k}} w_{l_{k}}\right] \\
& =\sum_{m_{1}=1}^{N}\left(z_{j_{1}}\right)_{m_{1}} \sum_{l_{1}=1}^{N}\left(z_{j_{1}}\right)_{l_{1}} \cdots \sum_{m_{k}=1}^{N}\left(z_{j_{k}}\right)_{m_{k}} \sum_{l_{k}=1}^{N}\left(z_{j_{k}}\right)_{l_{k}} E\left[\prod_{i=1}^{k} w_{m_{i}} w_{l_{i}}\right]
\end{aligned}
$$

Since all $w_{k}$ are independent Bernoulli random variables (or indicators that are either zero or one),

$$
E\left[\prod_{i=1}^{k} w_{m_{i}} w_{l_{i}}\right]=v^{q}
$$

where $q$ equals the number of indices in the set $\left\{m_{1}, \ldots, m_{k}, l_{1}, \ldots, l_{k}\right\}$ that are different. If all indices are different, then $q=2 k$. If all but 1 pair of indices are different, then $q=2 k-1$ and, so on. Thus, if all indices are the same, then $q=1$. Further, we observe that $E\left[\prod_{i=1}^{k} w_{m_{i}} w_{l_{i}}\right]$ is independent of the $\left\{j_{1}, \ldots, j_{k}\right\}$, so that the general result $E\left[\prod_{i=1}^{k} \zeta_{j_{i}}^{2}\right]$ can be checked by the explicit expression (30) of the moments $E\left[\zeta_{j}^{m}\right]$.

Direct evaluation of the quadratic form $|\partial \mathcal{V}|=w^{T} Q w=\sum_{m=1}^{N} \sum_{l=1}^{N} q_{m l} w_{m} w_{l}$ leads to

$$
|\partial \mathcal{V}|^{k}=\sum_{m_{1}=1}^{N} \sum_{l_{1}=1}^{N} \ldots \sum_{m_{k}=1}^{N} \sum_{l_{k}=1}^{N} \prod_{i=1}^{k} q_{m_{i} l_{i}} \prod_{i=1}^{k} w_{m_{r}} w_{l_{r}}
$$

so that

$$
E\left[|\partial \mathcal{V}|^{k}\right]=\sum_{m_{1}=1}^{N} \sum_{l_{1}=1}^{N} \cdots \sum_{m_{k}=1}^{N} \sum_{l_{k}=1}^{N} \prod_{i=1}^{k} q_{m_{i} l_{i}} E\left[\prod_{i=1}^{k} w_{m_{r}} w_{l_{r}}\right]
$$

which again requires to split the multiple sum into sub-sums in which $q$ indices in the set $\left\{m_{1}, \ldots, m_{k}, l_{1}, \ldots, l_{k}\right\}$ are different. It would be convenient to find a recursion that expresses $E\left[|\partial \mathcal{V}|^{k}\right]$ into $E\left[|\partial \mathcal{V}|^{n}\right]$ with integer power $n<k$.

We have not computed higher moments $E\left[|\partial \mathcal{V}|^{k}\right]$ for $k>2$ explicitly and suggest its computation as an open problem.

## C Probability density function $f_{\zeta_{j}}(x)$ of $\zeta_{j}$

In order to derive a fast converging probability density function $f_{\zeta_{j}}(x)$, application of (29) to the function $g_{j}(z)$ with Taylor coefficients $g_{m}(j)=\frac{(-1)^{m} p_{m}(v)}{m!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{m}$ for $m \geq 3$ derived from (22), we obtain

$$
\begin{equation*}
\varphi_{\zeta_{j}}(t)=e^{t^{2} v(1-v)}\left(1+\sum_{n=3}^{\infty} c_{n}(j) t^{n}\right) \tag{47}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{n}(j)=\sum_{k=1}^{n} \frac{1}{k!} s_{g_{j}(z)}[k, n] \tag{48}
\end{equation*}
$$

Due to the recursion formula of the characteristic coefficients $s[k, m]$, we can evaluate the coefficients $c_{n}(j)$ to any desired order. For example,

$$
\begin{aligned}
c_{3}(j) & =g_{3}(j)=-\frac{v-3 v^{2}+2 v^{3}}{3!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{3} \\
c_{4}(j) & =g_{4}(j)=\frac{v-7 v^{2}+12 v^{3}-6 v^{4}}{4!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{4} \\
c_{5}(j) & =g_{5}(j)=-\frac{v-15 v^{2}+50 v^{3}-60 v^{4}+24 v^{5}}{5!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{5} \\
c_{6}(j) & =\frac{1}{2}\left(g_{3}(j)\right)^{2}+g_{6}(j) \\
& =\frac{1}{72}\left(\left(v-3 v^{2}+2 v^{3}\right) \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{3}\right)^{2}+\frac{v-31 v^{2}+180 v^{3}-390 v^{4}+360 v^{5}-120 v^{6}}{6!} \sum_{k=1}^{N}\left(z_{j}\right)_{k}^{6}
\end{aligned}
$$

We now compute the inverse Laplace transform $f_{\zeta_{j}}(x)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \varphi_{\zeta_{j}}(t) e^{t x} d t$ results. With the Laplace transform of the Gaussian [7, p. 43], we have

$$
\begin{aligned}
\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{\frac{t^{2}}{2} v(1-v)} t^{m} e^{t x} d t & =\frac{d^{m}}{d x^{m}}\left(\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{\frac{t^{2}}{2} v(1-v)} e^{t x} d t\right) \\
& =\frac{d^{m}}{d x^{m}}\left(\frac{e^{-\frac{x^{2}}{2 v(1-v)}}}{\sqrt{2 \pi(v(1-v))}}\right)
\end{aligned}
$$

Using the Hermite polynomials [28], defined by the generating function for all $z$,

$$
\begin{equation*}
e^{2 x z-z^{2}}=\sum_{m=0}^{\infty} \frac{H_{m}(x)}{m!} z^{m} \tag{49}
\end{equation*}
$$

or the Rodrigues' formula [17, Sec. 22.11 on p. 785]

$$
H_{m}(x)=(-1)^{m} e^{x^{2}} \frac{d^{m}}{d x^{m}} e^{-x^{2}}
$$

yields

$$
\frac{d^{m}}{d x^{m}}\left(\frac{e^{-\left(\frac{x}{\sqrt{2 v(1-v)}}\right)}}{\sqrt{2 \pi(v(1-v))}}\right)=\frac{e^{-\frac{x^{2}}{2 v(1-v)}}}{\sqrt{2 \pi(v(1-v))}} \frac{(-1)^{m}}{(\sqrt{2 v(1-v)})^{m}} H_{m}\left(\frac{x}{\sqrt{2 v(1-v)}}\right)
$$

and

$$
\begin{equation*}
f_{\zeta_{j}}(x)=\frac{e^{-\frac{x^{2}}{2 v(1-v)}}}{\sqrt{2 \pi(v(1-v))}}\left(1+\sum_{m=3}^{\infty} \frac{(-1)^{m} c_{m}(j)}{(\sqrt{2 v(1-v)})^{m}} H_{m}\left(\frac{x}{\sqrt{2 v(1-v)}}\right)\right) \tag{50}
\end{equation*}
$$

The expansion (50) allows us to accurately assess the deviations of the probability density function $f_{\zeta_{j}}(x)$ from a Gaussian pdf.

## D A combinatorial computation of the average cut size (updated from [29])

Let us start from the governing equation (4) of the prevalence, which we rewrite with $N y(t)=E[|\mathcal{V}(t)|]$ and the cut size $|\partial \mathcal{V}|=e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)=w^{T} Q w$ as

$$
\frac{d E[|\mathcal{V}(t)|]}{d t}=E\left[\beta e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)-\delta|\mathcal{V}(t)|\right]
$$

Formally summing both sides over all possible clusters of size $V$ yields

$$
\frac{d}{d t} E\left[\sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V}|\mathcal{V}(t)|\right]=E\left[\beta \sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V} e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)-\delta \sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V}|\mathcal{V}(t)|\right]
$$

where

$$
\sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V}|\mathcal{V}(t)|=V \sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V} 1=V\binom{N}{V}
$$

Assuming that all clusters are equally probable, the average increment $\frac{d \bar{y}(t)}{d t}$ in the number of infected nodes for a given cluster size of $V$ nodes at time $t$ is a sum over all the clusters divided by the total number of combinations $\binom{N}{V}$,

$$
N \frac{d \bar{y}(t)}{d t}=\frac{\beta}{\binom{N}{V(t)}} E\left[\sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V(t)} e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)\right]-\delta V(t)
$$

For $\frac{d \bar{y}(t)}{d t}=0$ (which fixes a time $t=\theta$ and $V(\theta)=V$ ), we find the effective infection rate $\tau=\frac{\beta}{\delta}$ as

$$
\tau=\frac{V\binom{N}{V}}{E\left[\sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V} e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)\right]}
$$

Finally, $E\left[\sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V} e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)\right]=\sum_{\forall \mathcal{V}: \mathcal{V} \subseteq \mathcal{N},|\mathcal{V}|=V} E\left[e\left(\mathcal{V}(t), \mathcal{V}^{c}(t)\right)\right]=\sum_{\forall C: C \subseteq \mathcal{N},|C|=V} e\left(C, C^{c}\right)$, where we have replaced, in the summation over all sizes of the infected set $\mathcal{V}$ of nodes, the random variable $\mathcal{V}$ by a cluster $C$ or subgraph of $G$, so that

$$
\begin{equation*}
\tau=\frac{1}{\frac{1}{\left(\frac{1}{(N)}\right.} \sum_{\forall C: C \subseteq \mathcal{N},|C|=V} \frac{e\left(C, C^{c}\right)}{V}} \tag{51}
\end{equation*}
$$

In summary, if all clusters are equally probable, the effective infection rate at an extremal time $\theta$ (obeying $\frac{d \bar{y}(t)}{d t}=0$ ) can be expressed as

$$
\tau=\frac{1}{\bar{\eta}(G, V)}
$$

where the average interconnection constant of a graph $G$ is defined as

$$
\begin{equation*}
\bar{\eta}(G, V)=\frac{1}{\binom{N}{V}} \sum_{\forall C: C \subseteq \mathcal{N},|C|=V} \frac{e\left(C, C^{c}\right)}{|C|} \tag{52}
\end{equation*}
$$

Simply stated, we have averaged the ratio $\frac{e\left(C, C^{c}\right)}{|C|}$ over all clusters $C$ with $V$ nodes, taking any of them as equiprobable, divided by the total number of $V$-sized clusters in a network with $N$ nodes.

Theorem 4 (J. Omic [29, p. 49-50]) The average interconnection constant of a graph $G$ obeys

$$
\begin{equation*}
\bar{\eta}(G, V)=\frac{d_{a v}}{N-1}(N-V) \tag{53}
\end{equation*}
$$

Proof: Assume that node $i$ with degree $d_{i}$ has $k$ of its neighbors in the cluster $C$ and, thus, $d_{i}-k$ neighbors in $C^{c}$, implying that node $i$ contributes $d_{i}-k$ links to the cut-set $e\left(C, C^{c}\right)$. With these $k+1$ nodes fixed, we need to determine the number of ways to form $C$ with the remaining $V-k-1$ nodes, which equals $\binom{N-d_{i}-1}{V-k-1}$. For each node $i$ with $d_{i}$ neighbors, there are $\binom{d_{i}}{k}$ possible ways to choose a set of $k$ neighbors in $C$. Hence, in total, there are $\binom{N-d_{i}-1}{V-k-1}\binom{d_{i}}{k}$ clusters $C$ of size $V$, that contain node $i$ with $k$ neighbors and that contribute $d_{i}-k$ links to the cut-set $e\left(C, C^{c}\right)$. The number $V$ of nodes in cluster $C$ is larger than or equal to $k+1$, while, in the complement $C^{c}$, the node $i$ connects to $d_{i}-k \leq N-V$ neighbors. Thus, the minimum number of neighbors $k$ that can be in a cluster $C$ is $k=\max \left(0, V+d_{i}-N\right)$ and the maximum number is $k=\min \left(d_{i}, V-1\right)$.

$$
\text { For each node } i \text { with } d_{i} \text { neighbors, there }{ }^{14} \text { are } \sum_{k=\max \left(0, V+d_{i}-N\right)}^{\min \left(d_{i}, V-1\right)}\binom{N-d_{i}-1}{V-k-1}\binom{d_{i}}{k}=\binom{N-1}{V-1} \text { of clusters }
$$ $C$ with $V$ nodes that contain node $i$ and that contribute to all those corresponding cut-sets a total number of links equal to

$$
\sum_{k=\max \left(0, V+d_{i}-N\right)}^{\min \left(d_{i}, V-1\right)}\left(d_{i}-k\right)\binom{N-d_{i}-1}{V-k-1}\binom{d_{i}}{k}
$$

After summing over all possible nodes and using $\left(d_{i}-k\right)\binom{d_{i}}{k}=d_{i}\binom{d_{i}-1}{k}$, we obtain

$$
\sum_{\forall C: C \subseteq \mathcal{N},|C|=V} e\left(C, C^{c}\right)=\sum_{i=1}^{N} d_{i} \sum_{k=\max \left(0, V+d_{i}-N\right)}^{\min \left(d_{i}, V-1\right)}\binom{N-d_{i}-1}{V-k-1}\binom{d_{i}-1}{k}
$$

[^10]Recalling that $\binom{V}{j}=0$ if $j \notin\{0,1, \ldots, V\}$ and with the convention that $\sum_{k=a}^{b} f(k)=0$ if $b<a$, the bounds on the double sum can be simplified,

$$
\sum_{k=\max \left(0, V+d_{i}-N\right)}^{\min \left(d_{i}, V-1\right)}\binom{N-d_{i}-1}{V-k-1}\binom{d_{i}-1}{k}=\sum_{k=0}^{V-1}\binom{N-d_{i}-1}{V-1-k}\binom{d_{i}-1}{k}
$$

The latter sum is an instance of Vandermonde's binomial identity (54) for any complex number $\alpha_{1}$ and $\alpha_{2}$,

$$
\begin{equation*}
\binom{\alpha_{1}+\alpha_{2}}{m}=\sum_{j=0}^{m}\binom{\alpha_{1}}{j}\binom{\alpha_{2}}{m-j} \tag{54}
\end{equation*}
$$

so that

$$
\sum_{k=\max \left(0, V+d_{i}-N\right)}^{\min \left(d_{i}, V-1\right)}\binom{N-d_{i}-1}{V-k-1}\binom{d_{i}-1}{k}=\binom{N-2}{V-1}
$$

Invoking the basic law of the degree $\sum_{i=1}^{N} d_{i}=2 L$, the total number of links in all possible cut-sets with $V$ nodes equals

$$
\sum_{\forall C: C \subseteq \mathcal{N},|C|=V} e\left(C, C^{c}\right)=\binom{N-2}{V-1} \sum_{i=1}^{N} d_{i}=2 L\binom{N-2}{V-1}
$$

Finally, with the definition (52) of the averaged interconnection constant and of the average degree, $d_{a v}=\frac{2 L}{N}$, we arrive at (53).

The isoperimetric inequality (15) suggests that the number of infective links $w^{T} Q w=e\left(C, C^{c}\right)$ approximately equals $\frac{d_{a v}}{N-1}(N S)(N-N S)$ plus minus a possibly large bound. The definition (52) of the averaged interconnection constant shows that

$$
\frac{1}{\binom{N}{V}} \sum_{\forall C: C \subseteq \mathcal{N},|C|=V} e\left(C, C^{c}\right)=V \bar{\eta}(G, V)
$$

which represents the average number of infective links, given that $N E[S]=V$, thus

$$
E\left[w^{T} Q w \mid N S=V\right]=V \bar{\eta}(G, V)
$$

Introducing (53) in Theorem 4 results in

$$
E\left[w^{T} Q w \mid N S=V\right]=\frac{d_{a v}}{N-1} V(N-V)
$$

which precisely agrees with $E\left[w^{T} Q w\right]=\frac{d_{a v}}{N-1} E[(N S)(N-N S)]$ from (15), conditioned to $|C|=$ $N S=V$. Furthermore ,

$$
\begin{aligned}
E\left[w^{T} Q w\right] & =\sum_{V=1}^{N-1} E\left[w^{T} Q w \mid N S=V\right] \operatorname{Pr}[N S=V] \\
& =\frac{L}{2^{N}\binom{N}{2}} \sum_{V=1}^{N-1}\binom{N}{V} V(N-V)
\end{aligned}
$$

With $\binom{N}{V} V(N-V)=N(N-1)\binom{N-2}{V-1}$ and $\sum_{V=1}^{N-1}\binom{N}{V} V(N-V)=N(N-1) \sum_{V=0}^{N-2}\binom{N-2}{V}=N(N-1) 2^{N-2}$, we arrive at

$$
E\left[w^{T} Q w\right]=\frac{2 L}{2^{N}(N(N-1))} N(N-1) 2^{N-2}=\frac{L}{2}
$$

Under the assumption that all cut-sets have equal probability to occur, the average number of links in a random cut-set equals $\frac{L}{2}$, which corresponds to the case of $v=\frac{1}{2}$ in (31) because the number $\binom{N}{V}$ of cut-sets is maximal for size $V=\frac{N}{2}$. Theorem 4 can be reformulated as

Theorem 5 The average cut size $|\partial \mathcal{V}|$ for subsets $\mathcal{V} \subset \mathcal{N}$ of size $|\mathcal{V}|=V$ is equal to

$$
\begin{equation*}
\frac{\sum_{\forall \mathcal{V},|\mathcal{V}|=V}|\partial \mathcal{V}|}{\binom{N}{V}}=\frac{\frac{2 L}{N}}{N-1} V(N-V) \tag{55}
\end{equation*}
$$

Proof: The sum of all cut sizes with $V$ nodes can be written using (9) as

$$
\begin{equation*}
\sum_{\forall \mathcal{V},|\mathcal{V}|=V}|\partial \mathcal{V}|=\sum_{\forall \mathcal{V},|\mathcal{V}|=V} \sum_{i \in \mathcal{V}} d_{i}-\sum_{\forall \mathcal{V},|\mathcal{V}|=V} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} a_{i j} \tag{56}
\end{equation*}
$$

Since each node appears $\binom{N-1}{V-1}$ times in all cut sets of size $V$, the first term in the right-hand side of (56) equals

$$
\sum_{\forall \mathcal{V},|\mathcal{V}|=V} \sum_{i \in \mathcal{V}} d_{i}=\sum_{k=1}^{N}\binom{N-1}{V-1} d_{k}=\binom{N-1}{V-1} 2 L
$$

Similarly, since each pair of nodes appears $\binom{N-2}{V-2}$ times in all cut sets of size $V$, the second term in the right-hand side of (56) becomes

$$
\sum_{\forall \mathcal{V},|\mathcal{V}|=V} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} a_{i j}=\sum_{k=1}^{N} \sum_{m=1}^{N}\binom{N-2}{V-2} a_{k m}=\binom{N-2}{V-2} 2 L
$$

so that the sum of all cut sizes with $V$ nodes is

$$
\sum_{\forall \mathcal{V},|\mathcal{V}|=V}|\partial \mathcal{V}|=2 L\left(\frac{(N-1)!}{(V-1)!(N-V)!}-\frac{(N-2)!}{(V-2)!(N-V)!}\right)
$$

Division by $\binom{N}{V}=\frac{N!}{V!(N-V)!}$ then leads to the average in (55).
Clearly, the proof of Theorem 5 is an alternative and shorter proof of Theorem 4.


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[^1]:    ${ }^{1}$ By using the definition $Q=\Delta-A$, where an element $a_{i j} \in\{0,1\}$ of the adjacency matrix $A$ specifies the existence of a link between node $i$ and $j$ in the graph, the quadratic form equals

    $$
    \begin{aligned}
    x^{T} Q x & =\sum_{l \in \mathcal{L}}\left(x_{l^{+}}-x_{l^{-}}\right)^{2}=\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i j}\left(x_{i}-x_{j}\right)^{2} \\
    & =\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i j} x_{i}^{2}+\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i j} x_{j}^{2}-\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i j} x_{i} x_{j} \\
    & =\sum_{i=1}^{N} d_{i} x_{i}^{2}-\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i j} x_{i} x_{j}=x^{T} \operatorname{diag}\left(d_{j}\right) x-x^{T} A x
    \end{aligned}
    $$

[^2]:    ${ }^{2}$ Another way (see Theorem 17.3.2 in [7, p. 458]) to establish (8) starts by applying the basic [11, p. 72] Laplacian property (5). For Bernoulli indicators, it holds that $\left(X_{l^{+}}-X_{l^{-}}\right)^{2}=X_{l^{+}}+X_{l^{-}}-2 X_{l^{+}} X_{l^{-}}=X_{l^{+}}\left(1-X_{l^{-}}\right)+X_{l^{-}}\left(1-X_{l^{+}}\right)$ and

    $$
    w^{T} Q w=\sum_{l \in \mathcal{L}}\left\{X_{l^{+}}\left(1-X_{l^{-}}\right)+X_{l^{-}}\left(1-X_{l^{+}}\right)\right\}=\sum_{i=1}^{N} \sum_{k=1}^{N} a_{k i} X_{k}\left(1-X_{i}\right)
    $$

    Taking the expectation of both sides results in (8).

[^3]:    ${ }^{3}$ Several scalings are possible. For example, define the $(N-1) \times 1$ vector $p=\left(\frac{\zeta_{1}^{2}}{N S(1-S)}, \frac{\zeta_{2}^{2}}{N S(1-S)}, \ldots, \frac{\zeta_{N-1}^{2}}{N S(1-S)}\right)$ so that $p^{T} u=1$ and the $(N-1) \times 1$ vector $\mu_{+}=\left(\frac{\mu_{1}}{d_{a v}}, \frac{\mu_{2}}{d_{a v}}, \ldots, \frac{\mu_{N-1}}{d_{a v}}\right)$ obeying $u^{T} \mu_{+}=N$, then (11) becomes $\frac{w^{T} Q w}{d_{a v} N S(1-S)}=p^{T} \mu_{+}$.

[^4]:    ${ }^{5}$ Only if the random variables $X$ and $Y$ are independent, then it holds that $\left.E[f(X) g(Y))\right]=E[f(X)] E[g(Y)]$ for any pair of functions $f$ and $g$. In the special case where $X \in\{0,1\}$ and $Y \in\{0,1\}$ are Bernoulli random variables for which $f(X)=f(0)(1-X)+f(1) X$ and $E[f(X)]=a+b E[X]$ (and similarly for $Y$ ), we observe that

    $$
    E[f(X) g(Y))]=E[(a+b X)(c+d Y)]
    $$

    Consequently, uncorrelation (i.e. $E[X Y]=E[X] E[Y]$ ) for Bernoulli random variables $X$ and $Y$ implies their independence.

[^5]:    ${ }^{6}$ We can verify $E\left[\zeta_{j}^{4}\right]$ from (46).
    ${ }^{7}$ The Taylor expansion

    $$
    E\left[e^{-z \zeta_{j}^{2}}\right]=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} E\left[\zeta_{j}^{2 k}\right] z^{k}=1-v(1-v) z+\frac{1}{2} E\left[\zeta_{j}^{4}\right] z^{2}+O\left(z^{3}\right)
    $$

    in $z$ around $z=0$ has coefficients $\frac{(-1)^{k}}{k!} E\left[\zeta_{j}^{2 k}\right]$ determined in (30).

[^6]:    ${ }^{8}$ Alternatively, we can compute (32) directly from (5), or from the quadratic form as demonstrated at the end of Appendix B.

[^7]:    ${ }^{9}$ The polynomial $r_{4}(v)=v-5 v^{2}+8 v^{3}-4 v^{4}$ obeys the invariance $v \rightarrow 1-v$ or $r_{4}(v)=r_{4}(1-v)$, as physically required.

[^8]:    ${ }^{10}$ We may deduce this result by invoking the principle of majorization [21].
    ${ }^{11}$ The same conclusion follows by letting $t=\frac{1}{N^{0.5-\varepsilon}}$ and $\varepsilon>0$ (so that $N t^{2}=\frac{N}{N^{1-2 \varepsilon}}=N^{2 \varepsilon}$ ).

[^9]:    ${ }^{12}$ Moreover, with the definition (38), it holds that $\varpi^{T} x_{k}=w^{T} x_{k}=\zeta_{k}$ for all $1 \leq k<N$ and, $\varpi^{T} x_{N}=0$, while $w^{T} x_{N}=\zeta_{N}=\sqrt{N} S$. In fact, $\zeta_{k}=\left(w^{T}+c u\right) z_{k}$, for any number $c$.
    ${ }^{13}$ Relation (44), written as $\sum_{j=1}^{N} \varpi_{j}^{2}=\sum_{k=1}^{N}\left(\varpi^{T} z_{k}\right)^{2}$ is a discrete version of Parseval's or Plancherel's theorem in the theory of Fourier transforms [26]. Geometrically, (44) arises from the fact that the Euclidean norm of a vector is maintained after an orthogonal transformation.

[^10]:    ${ }^{14}$ The number of clusters of size $V$, in which a node $i$ appears, is equal to $\binom{N-1}{V-1}$, because the number of clusters of size $V$ in which node $i$ does not appear is $\binom{N-1}{V}$ and $\binom{N}{V}-\binom{N-1}{V}=\binom{N-1}{V-1}$. This observation agrees with the direct computation of the sum by invoking Vandermonde's binomial identity.

