# Directed graphs and mysterious complex eigenvalues

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**Abstract**

After the addition of one directed link with weight $\xi > 0$ to an undirected and unweighted graph, we compute at which critical weight $\xi_c$ a complex conjugate eigenvalue in the adjacency matrix is born. Simulations illustrate that the distribution $\Pr[\xi_c \leq x]$ converges remarkably fast in the number ($N \leq 40$) of nodes in Erdős-Rényi random graphs to a limit distribution, close to a Gamma distribution. Furthermore, the critical weight $\xi_c$ nor the associated complex eigenvalue pair seem to correlate with degree, betweenness nor effective resistance.

# 1 Introduction

## 1.1 Directed graphs

A directed graph is often represented by an asymmetric adjacency matrix. Here, we claim, however, that a directed graph, represented by an asymmetric adjacency matrix, is a misleadingly simple concept and considerably differs from its undirected companion [4]. For example, the Laplacian of a an asymmetric adjacency matrix is not uniquely defined; either the row or column sums can be zero, but not both. Our argument below is based on a network science point of view: each network consists of two essential parts, (a) a *graph*, also called topology or structure which is often associated with "hardware" and (b) a *process*, also called function or service, associated to "software". In most networks, items are transported from a source node to a destination node, determined by a process that steers the tranmission over a certain set of nodes and links of the graph.

A simple graph consists of a set $\mathcal{N}$ of $N$ nodes and a set $\mathcal{L}$ of $L$ links and each link $l \in \mathcal{L}$ connects two different nodes. The basic property of a simple graph on $N$ nodes is *link existence*, which can be specified by a symmetric adjacency matrix. The *direction of a link* provides additional information to the *usage* of that link, which is related to the function or process on the graph. In flow or fluid networks (water, electricity, etc.), the underlying graph consists of pipes that allow the flow in both directions and the process (e.g. potential difference over the link), dictated by the laws of nature, determines the direction [14]. Hence, the graph is undirected, specified by a symmetric adjacency or Laplacian matrix, while the process determines the direction of a link (that may change over time with the process). If the underlying graph is made directed, it means by convention that the flow is only

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allowed in one direction, irrespective of the process. In such directed graph, we need a "non-linear element" such as a shutter in the water tube, a diode in an electrical network, a single lane with traffic bords in road networks, etc. to prevent transport in both directions. The process equations in a directed graph become essentially non-linear and the beauty of linearity or linear processes on graphs vanishes. Thus, in physical flow networks, the directions of links are naturally associated to the process.

Human-created networks lead, more often than physical flow networks, to a directed graph. The webgraph, consisting of webpages as nodes where hyperlinks point to other webpages, is usually directed, which means that, only by following the hyperlinks, a walk over webpages in one direction is constructed. Similarly in social networks as Twitter, where friendship relations are not necessarily bidirectional, the flow of tweets follows one direction. However, in these examples, it is the function or process on the graph, rather than the graph itself, that is directed. More generally, any Markov process can be represented by a directed Markov graph [11], that describes the process transitions from one state to other states. For example, an SIS epidemic process on a graph with $N$ nodes (persons) and $L$ links (contact relations between persons) can be represented by a Markov graph, in which each of the $2^N$ nodes is a possible network infection state decoding the $N$ individual infection states. The SIS Markov graph [13] is a regular, directed bipartite graph with degree $N$. In most cases, the associated probability transfer matrix of a Markov process, which is a stochastic matrix, is asymmetric; the appearence of a symmetric stochastic matrix is rather exceptional. Unfortunately, in most cases, it is difficult to express the stochastic matrix or infinitesimal generator, which is a weighted Laplacian, of the Markov process as a function of the adjacency (or another graph-related) matrix of the underlying graph. We believe that this observation holds in general for any process on a graph (e.g. in human brain networks [6]) and it would be a desirable ambition (as e.g. in [5]) to design a framework that expresses the function $F$ on the graph in terms of the adjacency matrix $A$ of the graph, thus, $F = f(A; t)$, where $t$ denotes other process parameters.

In summary, we have argued that the direction of links in a graph is determined by the process in most cases and that asymmetric matrices arise most often in a process representation, which merges or couples both structure and function of the network.

### 1.2 Complex eigenvalues

In contrast to symmetric matrices [10], an asymmetric adjacency matrix as representation of a directed graph may lead to complex eigenvalues and to a Jordan form (i.e. the asymmetric matrix is not diagonalizable). Complex conjugate eigenvalues in an otherwise completely real setting must contain a certain meaning and may refer to properties of the underlying directed graph. A fascinating question is "what is the structural or topological meaning of complex eigenvalues of the adjacency matrix?" In order to cope with directed graphs, but to avoid complex eigenvalues, a Hermitian adjacency matrix has been proposed [2]. Brualdi's [1] review on the spectra of directed graphs, with an emphasis on the spectral radius and special types of graphs, illustrates the scarceness of general strong results (such as I. Schur's famous theorem of 1909, [7, p. 310] and also [3]).

Here, we investigate a simple case of asymmetry. We start with an unweighted and undirected graph $G$ with $N$ nodes and $L$ links. The graph $G$ has a $N \times N$ symmetric adjacency matrix $A = A^T$, with spectral decomposition $A = X\Lambda X^T$, where $X$ is the orthogonal matrix with the normalized eigenvectors $x_1, x_2, \ldots, x_N$ in the columns and $\Lambda$ is the diagonal matrix with the ordered real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, in which eigenvalue $\lambda_k$ corresponds to eigenvector $x_k$. We add one directed link $l_{ij}$ from node $i$ to node $j$ in the graph $G$ and we denote the resulting, directed graph by $G' = G \cup \{l_{ij}\}$. The newly added link $l_{ij}$ has, as only link in $G'$, a link weight $w_{ij} = \xi > 0$. The adjacency matrix of the weighted, directed graph $G'$ is $A' = A + \xi e_i e_j^T$ and is asymmetric, where the real, non-negative number $\xi$ is a tuning parameter which corresponds to a link in the unweighted case if $\xi = 1$.

We address a couple of research questions. First, which critical link weight $\xi_c > 0$ of the added link $l_{ij}$ gives rise to complex eigenvalues of $A'$? Given the spectral decomposition $A = X\Lambda X^T$, can we write the eigenvalues of $A'$ in terms of $X$ and $\Lambda$? In other words, does there exist an efficient algorithm to compute the eigenvalues of $A'$, given the eigenstructure of $A$? Section 2 presents an algorithm to find the complex eigenvalues of $A' = A + \xi e_i e_j^T$ in terms of the eigenvalues and eigenvectors of the adjacency matrix $A$ of the original, undirected and unweighted graph $G$. Moreover, we also determine the critical link weight $\xi_c$ above which a complex eigenvalue is born. Simulations in Section 3 on relatively small graphs allow us to compute the probability that the asymmetric, but unweighted $\xi = 1$, matrix $A'$ has no complex eigenvalues. We found that the distribution $\Pr[\xi_c \leq x]$ in small Erdős-Rényi graphs $G_p(N)$ with $N \leq 40$ nodes converges remarkably fast to an asymptotic distribution that is likely a Gamma distribution. On the negative side, the critical link weight $\xi_c$, nor the associated complex eigenvalue were found to correlate with degree, effective resistance, betweenness, which explains "mysterious" in the title. Thus, we fail to understand the topological meaning of the critical link weight $\xi_c$ or the complex eigenstructure and add a research agenda in Section 4 to stimulate further investigations.

## 2 Resolvent approach to the matrix $A'$

The characteristic polynomial of the matrix $A'$ is[1]

$$c_{A'}(\lambda) = \det(A' - \lambda I) = \det\left(A - \lambda I + \xi e_i e_j^T\right)$$

Using the Schur-formula [10]

$$\det\left(A + C_{n\times k} D_{k\times n}^T\right) = \det A \det\left(I_k + D^T A^{-1} C\right)$$

yields[2]

$$\det\left(A - \lambda I + \xi e_i e_j^T\right) = \det(A - \lambda I)\left(1 + \xi e_j^T (A - \lambda I)^{-1} e_i\right) \tag{1}$$

[1] The expression of the coefficients $c_k(A')$ of the characteristic polynomial $c_{A'}(\lambda) = \sum_{k=0}^{N} c_k(A')\lambda^k$ in terms of the coefficients $c_k(A)$ of the original characteristic polynomial $c_A(\lambda) = \det(A - \lambda I) = \sum_{k=0}^{N} c_k(A)\lambda^k$ is difficult. Actually, the explicit formula of $c_k(A')$ as a sum over all minors [10, p. 211] indicates that all coefficients but $c_N(A')$ and $c_{N-1}(A')$ are affected by $\xi$, because the sum over all minors will always include the element $a_{ij} = \xi$.

[2] A column (row) consisting of a sum of two vectors in a determinant can be split into two determinants [7, p. 10], from which we obtain

$$\det\left(A - \lambda I + \xi e_i e_j^T\right) = \det(A - \lambda I) + \xi(-1)^{i+j} \det(A - \lambda I)_{\setminus \text{row } i \setminus \text{col } j}$$

Equating this expression and (1) yields

$$\det(A - \lambda I)\left((A - \lambda I)^{-1}_{ji}\right) = (-1)^{i+j} \det(A - \lambda I)_{\setminus \text{row } i \setminus \text{col } j}$$

The procedure can be extended to the addition of more than one link. Brualdi [1] shows that, after ordering the links as $l_1, l_2, \ldots l_L$, any adjacency matrix can be written as $A = B_{out}B_{in}^T$, where $B_{in}$ is the $N \times L$ in-incidence matrix and $(B_{in})_{ij} = 1$ if the link $l_j = (n_k, n_i)$ for some node $n_k$, otherwise $(B_{in})_{ij} = 0$, whereas the $N \times L$ out-incidence matrix $B_{out}$ has an element $(B_{out})_{ij} = 1$ if the link $l_j = (n_i, n_s)$ for some node $n_s$, otherwise $(B_{out})_{ij} = 0$. For example, for two link additions, the matrix

$$\xi_1 e_i e_j^T + \xi_2 e_k e_l^T = \begin{bmatrix} \xi_1 e_i & \xi_2 e_k \end{bmatrix} . \begin{bmatrix} e_j & e_l \end{bmatrix}^T$$

and the Schur-formula results in

$$\begin{aligned}\det\left(A - \lambda I + \xi_1 e_i e_j^T + \xi_2 e_k e_l^T\right) &= \det(A - \lambda I)\det\left(I_2 + \begin{bmatrix} e_j & e_l \end{bmatrix}^T (A-\lambda I)^{-1} \begin{bmatrix} \xi_1 e_i & \xi_2 e_k \end{bmatrix}\right) \\ &= \det(A-\lambda I)\det\begin{bmatrix} \xi_1 e_j^T (A-\lambda I)^{-1} e_i + 1 & \xi_2 e_j^T (A-\lambda I)^{-1} e_k \\ \xi_1 e_l^T (A-\lambda I)^{-1} e_i & \xi_2 e_l^T (A-\lambda I)^{-1} e_k + 1 \end{bmatrix}\end{aligned}$$

where the last determinant can be computed explicitly. The two-position change in a matrix is required in a companion study about the effect of adding a weighted directed link on the Laplacian matrix of the graph. Here, we confine ourselves to the simplest case of one link addition in the adjacency matrix of the graph.

Any eigenvalue $\eta$ of $A'$ obeys $c_{A'}(\eta) = 0$ and (1) indicates that $\eta$ is a solution of

$$-\frac{1}{\xi} = (A - \eta I)^{-1}_{ij}$$

With the resolvent [10] of an $n \times n$ symmetric matrix $C$

$$(zI - C)^{-1} = \sum_{k=1}^{n} \frac{x_k(C)\, x_k^T(C)}{z - \lambda_k(C)}$$

and $\left(x_k x_k^T\right)_{ij} = (x_k)_i (x_k)_j$, we arrive at an eigenvalue equation for $\eta$ in terms of eigenstructure of $A$

$$\sum_{k=1}^{N} \frac{(x_k)_i (x_k)_j}{\eta - \lambda_k} = \frac{1}{\xi} \tag{2}$$

provided all eigenvalues $\lambda_k$ have multiplicity $m_k = 1$ (we omit the case where the multiplicity exceeds one). We further confine ourselves to connected graphs (with irreducible adjacency matrices). The resolvent eigenvalue equation (2) also indicates that the addition of a link from $i \to j$ or in the opposite direction from $j \to i$ leads to exactly the same eigenvalues (as expected from $\det\left(A^T\right) = \det(A)$).

In summary, the eigenvalues $\eta_1, \eta_2, \ldots, \eta_N$ of the adjacency matrix $A' = A + \xi e_i e_j^T$ of the weighted, directed graph $G' = G \cup \{l_{ij}\}$ satisfy

$$f(\eta) - \frac{1}{\xi} = 0$$

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which is, indeed, an identity, because for any matrix $M$, it holds that

$$\left(M^{-1}\right)_{ji} = \frac{(\text{adj}M)_{ji}}{\det M} = \frac{(-1)^{i+j}\det M_{\backslash \text{row } i \backslash \text{col } j}}{\det M}$$

where the partial fraction expansion in (2) is denoted by the function

$$f(\eta) = \sum_{k=1}^{N} \frac{\alpha_k}{\eta - \lambda_k} \tag{3}$$

and where the coefficients[3]

$$\alpha_k = (x_k)_i (x_k)_j \tag{5}$$

are the residues of the complex function $f(\eta)$. Residues play an important role in complex function theory [8, Chapter III]. Our confinement to an adjacency matrix $A$ of the original unweighted, undirected graph with *distinct* eigenvalues $\{\lambda_k\}_{1\leq k\leq N}$ implies that the residues obey $\alpha_q = \lim_{\eta\to\lambda_q} (\eta - \lambda_q) f(\eta)$ for each $1 \leq q \leq N$. The resolvent approach is essentially a Green's function approach (see e.g. [9, Sec. III.A]).

### 2.1 The partial fraction expansion (3)

If $\alpha_q = (x_q)_i (x_q)_j = 0$, then only the value at $\eta = \lambda_q$ influences the function $f(\eta)$ and only when $\alpha_q = 0$, an eigenvalue $\eta_q = \lambda_q$ of $A'$ equals an eigenvalue $\lambda_q$ of $A$, but the term $\frac{\alpha_q}{\eta-\lambda_q}$ further plays no role in the eigenvalue equation (2) of $\eta$ when $\xi$ changes. The Perron-Frobenius Theorem [11] tells us that all components of the principal eigenvector $x_1$ of a non-negative matrix are non-negative, and positive for the adjacency matrix of a connected graph, so that the residue $\alpha_1 = (x_1)_i (x_1)_j \geq 0$, while the sign of the residues $\alpha_k$ with $k > 1$ can be both negative and positive.

Double orthogonality [12] implies that $\sum_{k=1}^{N} (x_k)_q (x_k)_r = \delta_{qr}$ for any nodal pair $(q, r)$ and since $i \neq j$, $\sum_{m=1}^{N} (x_m)_i (x_m)_j = 0$ shows that there must be both negative and positive residues. More generally, a function $g$ of a symmetric matrix $A$ is

$$g(A) = \sum_{k=1}^{N} g(\lambda_k) x_k x_k^T \tag{6}$$

from which $(A^m)_{ij} = \sum_{k=1}^{N} \lambda_k^m (x_k)_i (x_k)_j$ equals [10, p. 26, p. 33] the total number of walks with $m$-hops from node $i$ to node $j$. For $m > 1$, we observe that $(A^m)_{ij}$ does not depend upon the link $a_{ij}$ (which is absent in $G$, but added in $G'$) and, thus, $\sum_{m=1}^{N} \lambda_m (x_m)_i (x_m)_j = a_{ij} = 0$.

Introducing $\frac{1}{\eta-\lambda_k} = \frac{1}{\eta}\left(1 + \frac{\lambda_k}{\eta-\lambda_k}\right)$ into (3) yields

$$f(\eta) = \frac{1}{\eta} \sum_{k=1}^{N} (x_k)_i (x_k)_j + \frac{1}{\eta} \sum_{k=1}^{N} \frac{\lambda_k (x_k)_i (x_k)_j}{\eta - \lambda_k}$$

[3] An explicit form of the residue is proved in [12]:

**Corollary 1** *The product of the $i$-th and $j$-th component of eigenvector $x_k$ of $A$ belonging to eigenvalue $\lambda_k$ with multiplicity 1 equals*

$$(x_k)_i (x_k)_j = \frac{(-1)^{i+j+1}}{c'_A(\lambda_k)} \det\left(A_{\backslash \text{row } i \backslash \text{col } j} - \lambda_k I\right) \tag{4}$$

*where* $c'_A(\lambda) = \frac{d}{d\lambda} \det(A - \lambda I) = -\sum_{n=1}^{N} \det\left(A_{G\backslash\{n\}} - \lambda I\right)$.

Since $\sum_{k=1}^{N} (x_k)_i (x_k)_j = 0$, we arrive at

$$\eta f(\eta) = \sum_{k=1}^{N} \frac{\beta_k}{\eta - \lambda_k} \tag{7}$$

with residues

$$\beta_k = \lambda_k (x_k)_i (x_k)_j \tag{8}$$

Thus, $\eta f(\eta)$ can be treated similarly as $f(\eta)$, but with residues $\beta_k = \lambda_k \alpha_k$ instead of $\alpha_k$ and $\sum_{k=1}^{N} \beta_k = 0$ similar to $\sum_{k=1}^{N} \alpha_k = 0$ and equivalent to $\int_C f(\eta)\, d\eta = \int_C \eta f(\eta)\, d\eta = 0$ for a contour $C$ enclosing all poles at the eigenvalues of $A$. Relation (7) also indicates that $|f(\eta)|$ is large for small $|\eta|$, which is confirmed by Fig. 3. Moreover, if eigenvalues of the original adjacency matrix are non-zero, $\lambda_k \neq 0$, then we observe that $\lim_{\eta \to 0} \eta f(\eta) = 0$.

**Theorem 1** *For real $\eta > \lambda_1$, it holds that $f(\eta) > 0$, whereas $f(\eta) < 0$ for sufficiently large negative real $\eta$, certainly for $\eta < -\lambda_1 \left(1 + \frac{\lambda_1^2}{2(A^2)_{ij}}\right)$.*

**Proof:** Expanding $\frac{1}{\eta - \lambda_k} = \sum_{m=0}^{\infty} \frac{\lambda_k^m}{\eta^{m+1}}$, valid if $|\lambda_k| < |\eta|$, yields

$$f(\eta) = \sum_{m=0}^{\infty} \frac{\left(\sum_{k=1}^{N} (x_k)_i (x_k)_j \lambda_k^m\right)}{\eta^{m+1}}$$

Using $(A^m)_{ij} = \sum_{k=1}^{N} \lambda_k^m (x_k)_i (x_k)_j$ and $\left(A^0\right)_{ij} = (A)_{ij} = 0$, leads to the Laurent series

$$f(\eta) = \sum_{m=2}^{\infty} \frac{(A^m)_{ij}}{\eta^{m+1}} \qquad \text{if } |\lambda_1| < |\eta| \tag{9}$$

Clearly, the Laurent series (9) indicates that $f(\eta) > 0$ for real $\eta > \lambda_1$ (because all terms $(A^m)_{ij}$ in the Laurent series are non-negative). The second claim is demonstrated as follows. We write (9) as $f(\eta) = \frac{1}{\eta}\left(\frac{(A^2)_{ij}}{\eta^2} + \sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{\eta^m}\right)$ and if $\frac{(A^2)_{ij}}{\eta^2} + \sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{\eta^m} > 0$, then $f(\eta) < 0$ for negative $\eta = -|\eta|$. Since $(A^m)_{ij} \leq \lambda_1^m$ for any pair $(i,j)$ and any non-negative integer $m$, we have

$$\left|\sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{\eta^m}\right| < \sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{|\eta|^m} \leq \sum_{m=3}^{\infty} \left(\frac{\lambda_1}{|\eta|}\right)^m = \sum_{m=0}^{\infty} \left(\frac{\lambda_1}{|\eta|}\right)^{m+3} = \frac{\frac{\lambda_1^3}{|\eta|^2}}{|\eta| - \lambda_1}$$

The condition $\frac{(A^2)_{ij}}{\eta^2} > -\sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{\eta^m}$ is obeyed when $\frac{(A^2)_{ij}}{\eta^2} > \frac{\frac{\lambda_1^3}{|\eta|^2}}{|\eta| - \lambda_1} > \left|\sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{\eta^m}\right| \geq -\sum_{m=3}^{\infty} \frac{(A^m)_{ij}}{\eta^m}$, from which we find that $|\eta| > \lambda_1 \left(1 + \frac{\lambda_1^2}{(A^2)_{ij}}\right)$. $\square$

From Theorem 1, we deduce that the resolvent equation $f(\eta) = \frac{1}{\xi}$ has always a real solution or eigenvalue $\eta_1 > \lambda_1$, in agreement with the Perron-Frobenius theorem and the fact that the spectral radius of the adjacency matrix is larger than or equal to the largest eigenvalue of any of its subgraphs. At the other extreme of the spectrum, only a negative residue $\alpha_N < 0$ makes $f(\eta) > 0$ for $\eta < \lambda_N$ and leads to a real eigenvalue $\eta_N < \lambda_N$, since Theorem 1 states that $f(\eta) < 0$ for large $\eta$, while eventually $\lim_{\eta \to -\infty} f(\eta) = 0$. Thus, if $\alpha_N < 0$, then $\eta_N < \lambda_N$ and the eigenvalue range $\eta_1 - \eta_N > \lambda_1 - \lambda_N$. If $\alpha_N > 0$, on the other hand, then $\eta_N > \lambda_N$, but in spite of (27), we cannot conclude that $\eta_1 - \eta_N > \lambda_1 - \lambda_N$. Both extreme cases give rise to real eigenvalues of $A'$ and demonstrate that

**Corollary 2** *The real part of a possible complex eigenvalue* $\eta$ *of* $A' = A + \xi e_i e_j^T$ *is bounded by* $\lambda_N < \mathrm{Re}(\eta) < \lambda_1$.

When an eigenvalue $\eta_m$ of $A'$ approaches an eigenvalue $\lambda_m$ of $A$, then the term $\frac{\alpha_m}{\eta_m - \lambda_m}$ dominates the sum in (2), so that we can approximate

$$\eta_m \approx \lambda_m + \xi \alpha_m \tag{10}$$

which corresponds to the first-order perturbation approximation [15, 60-70] for small $\xi$. In contrast to simple interlacing [10, p. 245], where all residues are positive, the resolvent equation (2) is more complicated as stated in Theorem 2:

**Theorem 2** *Between two (real) eigenvalues of* $A$*, there can be zero or one, but at most two real eigenvalues of* $A' = A + \xi e_i e_j^T$ *as long as link weight* $\xi$ *of the added link directed from node* $i$ *to node* $j$ *is sufficiently small.*

**Proof:** The resolvent equation (2) demonstrates that only intersections of the horizontal line at $y = \frac{1}{\xi} > 0$ and the positive values of the function $y = f(\eta) > 0$ at real $\eta$ lead to real eigenvalues $\eta_m$ of the matrix $A'$. The derivative $f'(\eta) = -\sum_{k=1}^{N} \frac{\alpha_k}{(\eta - \lambda_k)^2}$ indicates that $f(\eta)$ decreases from $+\infty$ for $\eta \geq \lambda_m$ if $\alpha_m > 0$ and, vice versa, increases from $-\infty$ for $\eta \geq \lambda_m$ if $\alpha_m < 0$. There are three cases: (a) if $\alpha_m > 0$ and $\alpha_{m-1} > 0$, then $f(\eta)$ decreases from infinity at $\lambda_m$ towards some $\eta_0$ and further decreases from some $\eta_1 > \eta_0$ towards $-\infty$ at $\lambda_{m-1}$. Moreover, (10) tells us for small $\xi$ that $\lambda_m < \eta_m \leq \lambda_{m-1} < \eta_{m-1}$, implying that there is only one eigenvalue $\eta_m$ of $A'$ between $\lambda_m$ and $\lambda_{m-1}$. (b) if $\alpha_m > 0$, but $\alpha_{m-1} < 0$, then (10) shows that $\lambda_m < \eta_m \leq \eta_{m-1} < \lambda_{m-1}$ and there are two eigenvalues in the interval $(\lambda_m, \lambda_{m-1})$ for small $\xi$. (c) The other extreme, $\alpha_m < 0$, but $\alpha_{m-1} > 0$, leads to $\eta_m < \lambda_m \leq \lambda_{m-1} < \eta_{m-1}$, implying the absense of eigenvalues of $A'$ in the interval $(\lambda_m, \lambda_{m-1})$ for small $\xi$. Since $A'$ has a total of $N$ eigenvalues, for small enough $\xi$, all eigenvalues of $A'$ are real and positioned in one of the $N+1$ intervals $\{(-\infty, \lambda_N), (\lambda_N, \lambda_{N-1}), \ldots (\lambda_2, \lambda_1), (\lambda_1, \infty)\}$.

By varying $\xi$, real eigenvalues cannot cross the asymptotes at $\lambda_k$, only complex eigenvalues can turn in the complex plane around the poles at $\lambda_k$. Hence, the number of real eigenvalues for small $\xi$ cannot increase in an interval $(\lambda_m, \lambda_{m-1})$, only decrease when complex eigenvalues are born for some value of $\xi$. $\square$

As a consequence, the radius of convergence of the perturbation series in $\xi$ equals $\xi_c$ specified in (14), above which a complex zero is born. Finally, we provide bounds on the largest eigenvalue $\eta_1$ of $A'$,

$$\lambda_1 \leq \eta_1 \leq \frac{(\lambda_1 + \lambda_N) + (\lambda_1 - \lambda_N)\sqrt{1 + \frac{4\xi}{\lambda_1 - \lambda_N}}}{2} \tag{11}$$

as well as bounds for the smallest eigenvalue, when $\eta_N < \lambda_N$,

$$\frac{(\lambda_1 + \lambda_N) - (\lambda_1 - \lambda_N)\sqrt{1 + \frac{4\xi}{\lambda_1 - \lambda_N}}}{2} \leq \eta_N < \lambda_N \tag{12}$$

## 2.2 Complex eigenvalues of $A'$

If eigenvalues of the directed adjacency matrix $A' = A + \xi e_i e_j^T$ are complex, then they occur in conjugate pairs, because the characteristic polynomial $c_{A'}(\lambda) = \sum_{k=0}^{N} c_k(A')\lambda^k$ has real coefficients $c_k(A')$ and the general reflection principle [8] applies. Let $\eta_m = \sigma + i\tau$ and its conjugate $\eta_m^* = \sigma - i\tau$, then (2) is written as

$$\frac{1}{\xi} = \sum_{k=1}^{N} \frac{\alpha_k}{\sigma + i\tau - \lambda_k} = \sum_{k=1}^{N} \frac{\alpha_k(\sigma - \lambda_k)}{(\sigma - \lambda_k)^2 + \tau^2} - i\tau \sum_{k=1}^{N} \frac{\alpha_k}{(\sigma - \lambda_k)^2 + \tau^2}$$

Equating real and imaginary parts leads to equations for $\sigma$ and $\tau$ in a complex eigenvalue $\eta_m = \sigma + i\tau$ of $A'$,

$$\begin{cases} \sum_{k=1}^{N} \frac{\alpha_k \lambda_k}{(\sigma-\lambda_k)^2+\tau^2} = -\frac{1}{\xi} \\ \sum_{k=1}^{N} \frac{\alpha_k}{(\sigma-\lambda_k)^2+\tau^2} = 0 \end{cases}$$

*only valid* for $\tau \neq 0$. Indeed, the last equation with $\tau = 0$ would erroneously imply that $f'(\eta_m) = 0$ for a real eigenvalue $\eta_m = \sigma$ of $A'$.

A complex eigenvalue $\eta$ of $A'$, that satisfies the resolvent eigenvalue equation (2), also obeys

$$|f(\sigma + i\tau)| = \frac{1}{\xi} \tag{13}$$

because both $\eta = \sigma + i\tau$ and $\eta^* = \sigma - i\tau$ must satisfy (2), i.e. $f(\sigma + i\tau) = \frac{1}{\xi}$ and $f(\sigma - i\tau) = \frac{1}{\xi}$, and multiplying both equations leads to (13) for $\xi > 0$. The converse is not generally true: a solution of (13) does not always satisfy the resolvent eigenvalue equation (2), in particular, not when $f(\eta) < 0$.

When $\xi$ increases, the largest eigenvalue always increases (because $u^T A u$ increases and $\lambda_1 \geq \frac{u^T A u}{N}$) and most eigenvalues also start either increasing or decreasing. Only when an eigenvalue $\eta_m$ increases and $\eta_{m-1} \geq \eta_m$ decreases with $\xi$, they meet at a double eigenvalue of $A'$. At that moment, a slight increase of $\xi$ creates a complex eigenvalue. Differentiation of the resolvent eigenvalue equation (2) with respect to $\xi$ yields

$$\frac{d\eta_m(\xi)}{d\xi} = \frac{1}{\xi^2}\left(\sum_{k=1}^{N} \frac{\alpha_k}{(\eta_m(\xi) - \lambda_k)^2}\right)^{-1} = -\frac{1}{\xi^2}\frac{1}{f'(\eta_m)}$$

and the sign of $\frac{d\eta_m(\xi)}{d\xi}$ will be initially determined by the residue $\alpha_m$ for $\eta_m$ close to $\lambda_m$. Hence, the conditions for $\eta_m \to \eta_{m-1}$ are that $\frac{d\eta_m(\xi)}{d\xi} > 0$ and $\frac{d\eta_{m-1}(\xi)}{d\xi} < 0$, or, equivalently, $f'(\eta_m) < 0$ and $f'(\eta_{m-1}) > 0$. If a double, real zero is reached at $f'(\eta_m) = 0$ for $\xi = f(\eta_m) > 0$, then $\frac{d\eta_m(\xi)}{d\xi} \to \infty$. The double, real zero can be interpreted as an unstable bifurcation with respect to $\xi$, which jumps orthogonally off the real axis into the complex plane.

The interesting point is that complex eigenvalues, born between two real eigenvalues $\lambda_{m-1}$ and $\lambda_{m-2}$ of the original adjacency matrix $A$, may start moving in the complex plane as $\xi$ further increases. At some value of $\xi$, the real part of those complex eigenvalue can enter between other real eigenvalues $\lambda_m$ and $\lambda_{m-1}$ of $A$. Moreover, they can become real again so that, three real eigenvalues can occur between $\lambda_m$ and $\lambda_{m-1}$ (as exemplified in the examples below).

In summary, complex conjugate eigenvalues $\eta = \sigma + i\tau$ and $\eta^* = \sigma - i\tau$ of the adjacency matrix $A' = A + \xi e_i e_j^T$ of the weighted, directed graph $G' = G \cup \{l_{ij}\}$ appear when $\xi > \xi_c$, where the critical link weight strength $\xi_c$ corresponds to a double real eigenvalue and satisfies

$$\begin{cases} f'(\eta) = 0 \\ \xi_c = \frac{1}{f(\eta)} > 0 \\ f''(\eta) > 0 \end{cases} \tag{14}$$

where the latter convexity prevents a positive maximum, which does not generate a complex eigenvalue if $\xi$ is increased beyond $\xi_c$, but two real eigenvalues.

### 2.3 Algorithm to determine complex eigenvalues of $A'$

Consider the eigenvalue resolvent equation (2) when $\lambda_m < \eta < \lambda_{m-1}$. We define the function

$$\Xi(\eta) = \frac{\alpha_{m-1}}{\eta - \lambda_{m-1}} + \frac{\alpha_m}{\eta - \lambda_m} \tag{15}$$

that obeys

$$\Xi(\eta) = \frac{1}{\xi} - r_m(\eta) \tag{16}$$

as follows from the eigenvalue resolvent equation (2) where

$$r_m(\eta) = \sum_{k=1;k\neq\{m-1,m\}}^{N} \frac{\alpha_k}{\eta - \lambda_k} = f(\eta) - \Xi(\eta)$$

is a differentiable, bounded function of $\eta$ on the interval $(\lambda_m, \lambda_{m-1})$. We exclude the situation where either $\alpha_{m-1} = 0$ or $\alpha_m = 0$.

**Theorem 3** *The function $\Xi(\eta) = \frac{\alpha_{m-1}}{\eta-\lambda_{m-1}} + \frac{\alpha_m}{\eta-\lambda_m}$ possesses either a minimum or a maximum if $\alpha_m\alpha_{m-1} < 0$. In particular,*

$$\begin{array}{lll} \text{if } \alpha_m > 0 > \alpha_{m-1} & \eta_{\min} = \frac{\sqrt{\alpha_m}\lambda_{m-1}+\sqrt{-\alpha_{m-1}}\lambda_m}{\sqrt{\alpha_m}+\sqrt{-\alpha_{m-1}}} & \Xi(\eta_{\min}) = \frac{\left(\sqrt{\alpha_m}+\sqrt{-\alpha_{m-1}}\right)^2}{\lambda_{m-1}-\lambda_m} > 0 \\ \text{if } \alpha_m < 0 < \alpha_{m-1} & \eta_{\max} = \frac{\sqrt{-\alpha_m}\lambda_{m-1}+\sqrt{\alpha_{m-1}}\lambda_m}{\sqrt{-\alpha_m}+\sqrt{\alpha_{m-1}}} & \Xi(\eta_{\max}) = -\frac{\left(\sqrt{-\alpha_m}+\sqrt{\alpha_{m-1}}\right)^2}{(\lambda_{m-1}-\lambda_m)} < 0 \end{array} \tag{17}$$

*where $\eta_{\min}, \eta_{\max} \in (\lambda_m, \lambda_{m-1})$. If $\alpha_m\alpha_{m-1} > 0$, then $\Xi(\eta)$ has a zero $\eta_0 = \left(1 - \frac{\alpha_m}{\alpha_{m-1}+\alpha_m}\right)\lambda_m + \left(\frac{\alpha_m}{\alpha_{m-1}+\alpha_m}\right)\lambda_{m-1}$ lying between $\lambda_m$ and $\lambda_{m-1}$.*

Theorem 3, proved in Appendix A, leads to a good approximation of the eigenvalue $\eta_m \in (\lambda_m, \lambda_{m-1})$. In particular, if $r_m(\eta)$ is sufficiently small, then $\eta_{\min}$ is a good approximation of an eigenvalue of $A'$ with multiplicity two as follows from the resolvent eigenvalue equation (2). Theorem 3 also indicates that no complex eigenvalues can occur, irrespective the strenght of $\xi$. Such graphs do exist[4]. Since $\alpha_1 > 0$, Theorem 3 shows that a complex eigenvalue of $A'$ between $\lambda_2$ and $\lambda_1$ is very unlikely (and impossible if $r_m(\eta) = 0$), which suggests a sharpening of Corollary 2 to $\lambda_N < \mathrm{Re}(\eta) < \lambda_2$.

Given the eigenvalue decomposition $A = X\Lambda X^T$, a reasonably accurate algorithm based upon Theorem 3 to find complex eigenvalue of $A'$ is as follows. For each added directed link $l_{ij}$, compute the couple $(\eta_{\min}, \Xi(\eta_{\min}))_m$ from (17) for each eigenvalue pair $(\lambda_m, \lambda_{m-1})$ indexed from $m = 2, 3, \ldots, N$.

[4] For example, for a directed, weighted link between $i = 1$ and $j = 2$ in

We ignore $\eta_{\max}$ and $\Xi(\eta_{\max})$ in (17) of Theorem 3, because $\Xi(\eta_{\max}) < 0$ cannot be reached by varying the positive $\xi$ from zero to infinity. Let $n_\eta$ denote the number of all such couples $(\eta_{\min}, \Xi(\eta_{\min}))_m$ when $m$ runs from 2 to $N$. Next, order the $n_\eta$ couples in decreasing order based on the value of $\Xi(\eta_{\min})$. The ranked list is written as $(\eta_{\min}, \Xi(\eta_{\min}))_{(1)}, (\eta_{\min}, \Xi(\eta_{\min}))_{(2)}, \ldots, (\eta_{\min}, \Xi(\eta_{\min}))_{(n_\eta)}$ where $(\Xi(\eta_{\min}))_{(1)} \geq (\Xi(\eta_{\min}))_{(2)} \geq \cdots \geq (\Xi(\eta_{\min}))_{(n_\eta)}$. If $\xi$ increases from zero to infinity, then a first complex zero will occur approximation at $(\eta_{\min})_{(1)}$ corresponding to link weight strength $\xi \approx \frac{1}{(\Xi(\eta_{\min}))_{(1)}}$, further increasing $\xi$ generates a second complex (conjugate) zero pair at $(\eta_{\min})_{(2)}$ corresponding to link weight strength $\xi \approx \frac{1}{(\Xi(\eta_{\min}))_{(2)}}$ and so on. Hence, in total approximately $n_\eta$ complex zeros can be generated, starting from $A$ by adding link $l_{ij}$, with link weight $\xi$ varying from zero to infinity.

Of course, the above algorithm is approximate. Computations show that $r_m(\eta)$ is about the same order of magnitude as $(\Xi(\eta_{\min}))_m$, and generally, $(\eta_{\min})_m$ is a reasonably accurate approximation of the exact double zero $\widetilde{\eta}_m$ of $f(\eta)$, satisfying (14) with equality in the second inequality. The comparison between the exact $(\widetilde{\eta}, f(\widetilde{\eta}))_m$ and the corresponding approximation $(\eta_{\min}, \Xi(\eta_{\min}))_m$ is reasonable, however, the rank $m$ of $(\eta_{\min}, \Xi(\eta_{\min}))_{(m)}$ does not always agree with the rank $m$ of $(\widetilde{\eta}, f(\widetilde{\eta}))_{(m)}$, mainly due to small differences in numerical values. In contrast to the simpler function $\Xi(\eta)$ in (15), we mention that the function $f(\eta)$ in (3) can possess maxima (minima) that are positive (negative). Eigenvalues $\eta$ deduced from such extrema cannot occur in (16). Hence, the total number $n_c$ of complex eigenvalues of the resolvent eigenvalue equation (2) when $\xi$ increases from zero to infinity can be different from the approximate number $n_\eta$. We deduce from Theorem 3 that the number $n_\eta$ entirely depends on the number of sign changes in the residues $\{\alpha_m\}_{1\leq m\leq N}$ that additionally obey $\alpha_m > 0 > \alpha_{m-1}$. The actually number $n_c$ of complex eigenvalues of $A'$ depends on the residues in a more complex way and, most likely, it holds that $n_c > n_\eta$ (although we do not have a proof).

---

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

### 2.4 Examples

1. An instance of an Erdős-Rényi (ER) graph $G_{0.6}(10)$ with $N = 10$ and link density $p = 0.6$ has an adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

with eigenvalue vector

$$\lambda = (6.1143, 1.6970, 0.9551, 0.3011, 0, -0.9505, -1.2304, -1.7154, -2.5448, -2.6262)$$

Between node 9 and 10 a directed link with weight $\xi$ is placed. The correspond partial fraction function (3) is

$$f(x) = \frac{0.0982451}{x - 6.11427} + \frac{0.00220921}{x - 1.69696} - \frac{0.277967}{x - 0.955063} - \frac{0.0448324}{x - 0.301059} + \frac{0.0790953}{x + 0.950517} + \frac{0.0947427}{x + 1.2304} - \frac{0.00497767}{x + 1.71537} - \frac{0.0220569}{x + 2.54483} + \frac{0.0755413}{x + 2.62625}$$

and drawn in Fig. 1.

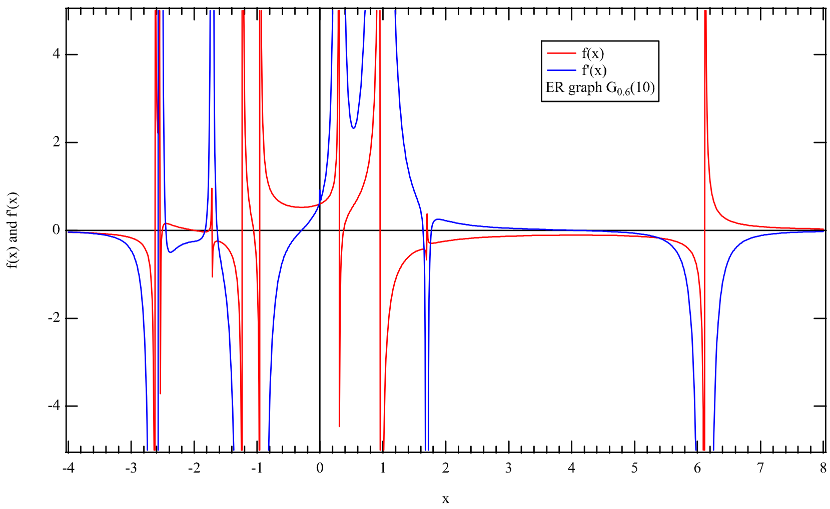


Figure 1: The partial fraction expansion $f(x)$ and its derivative $f'(x)$ versus $x$. When $f'(x) = 0$, a double zero of $f(x) - \frac{1}{\xi}$ is found, which determines the onset of a complex zero.

Our approximate algorithm finds the existence of complex eigenvalues at

$$(\eta_{\min}, \Xi(\eta_{\min}))_{(1)} = (-2.57339, 2.20153)$$
$$(\eta_{\min}, \Xi(\eta_{\min}))_{(2)} = (-0.236502, 0.194175)$$

while the exact sequence is

$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(1)} = (-2.57337, 2.1706)$$
$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(2)} = (-0.296125, 0.52218)$$
$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(3)} = (-2.4516, 0.158324)$$

The onset of the first complex eigenvalue with $\xi_c = \xi_1 = \frac{1}{2.1706} = 0.4607$ and the second with $\xi_2 = \frac{1}{0.52218} = 1.91505$ is reasonable well found by the approximate algorithm. However, the example illustrates the complication around $x = -2.45$, where $f(x)$ shows a "third order" behavior that cannot be reconstructed from $\Xi(\eta)$ so that the onset of third complex eigenvalue at $\xi_3 = \frac{1}{0.158324} = 6.31615$ is not retrieved.

2. Another instance of an Erdős-Rényi graph $G_{0.6}(10)$ has an adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with eigenvalue vector

$$\lambda = (5.6810, 1.6971, 0.8175, 0.2805, 0.2116, -0.1213, -1.4127, -1.8032, -2.1863, -3.1642)$$

Between node 1 and 5 a directed link with weight $\xi$ is placed. The correspond partial fraction function (3) is

$$\begin{aligned} f(x) = {} & \frac{0.107557}{x - 5.68101} + \frac{0.00277403}{x - 1.69715} + \frac{0.0126244}{x - 0.817474} - \frac{0.202543}{x - 0.280509} - \frac{0.172377}{x - 0.211585} + \frac{0.0492927}{x + 0.121302} \\ & + \frac{0.0927033}{x + 1.41273} - \frac{0.0133299}{x + 1.80323} - \frac{0.030405}{x + 2.1863} + \frac{0.153703}{x + 3.16417} \end{aligned} \tag{18}$$

and drawn in Fig. 2.

The approximate algorithm finds the existence of complex eigenvalues at

$$(\eta_{\min}, \Xi(\eta_{\min}))_{(1)} = (-0.0053, 1.21972)$$
$$(\eta_{\min}, \Xi(\eta_{\min}))_{(2)} = (-2.48733, 0.328092)$$

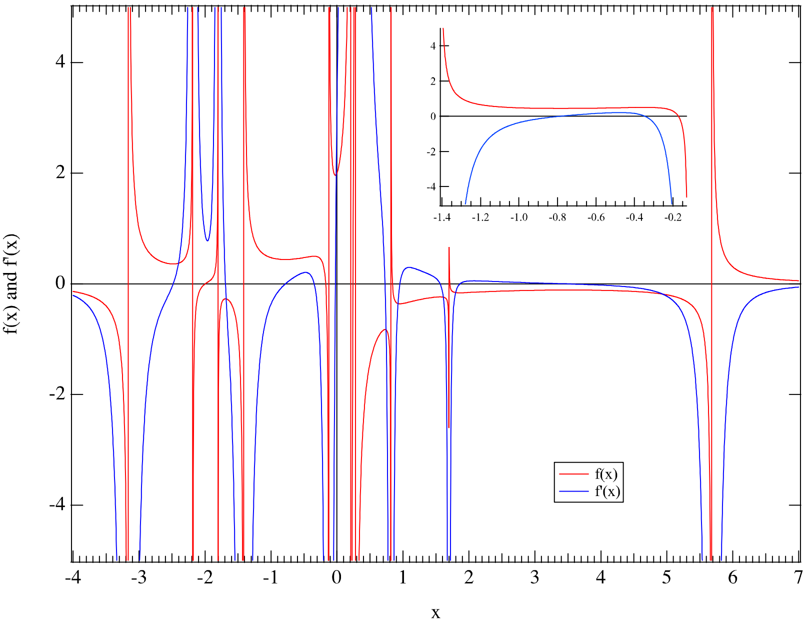


Figure 2: The partial fraction expansion $f(x)$ and its derivative $f'(x)$ versus $x$. The insert shows a region where three eigenvalues occur when $\xi > 1.1$ and all three are real only when $2.004 < \xi < 2.269$.

while the exact sequence is

$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(1)} = (-0.0240595, 1.96212)$$
$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(2)} = (-0.346894, 0.498787)$$
$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(3)} = (-0.783158, 0.440652)$$
$$(\widetilde{\eta}, f(\widetilde{\eta}))_{(4)} = (-2.48316, 0.359844)$$

The first complex eigenvalue, created just above $\xi_c = \xi_1 = \frac{1}{1.96212} = 0.5096$ at $\eta = -0.0240595$, lies between $\lambda_5 = 0.2116$ and $\lambda_6 = -0.1213$. When $\xi$ increases roughly around $\xi = 1.1$, the real part of that complex (conjugate) pair is smaller than $\lambda_6$ (at $\xi = 1.1$, the complex eigenvalue is $\eta = -0.129641 \pm 0.148389i$ and at $\xi = 2$, we find $\eta = -0.345242 \pm 0.0222726i$). At $\xi_2 = \frac{1}{0.498787} = 2.00486$, the complex pair again becomes a double real eigenvalue at $\eta = -0.346894$. The next change occurs at $\xi_3 = \frac{1}{0.440652} = 2.26937$ at a double real eigenvalue $\eta = -0.783158$ and for $\xi \in (\xi_2, \xi_3)$, three *real* eigenvalues exists in between $\lambda_6 = -0.1213$ and $\lambda_7 = -1.4127$. When $\xi > \xi_3$, again a complex pair is created that moves to more negative real part values, for example, $\eta = -0.788374 \pm 0.105644i$ at $\xi = 2.3$ and becomes smaller than $\lambda_6$ around $\xi = 4.75$. Figure 3 draws $|f(x+iy)|$, specified in (18), in the complex plane and the eigenvalues $\eta$ of $A'$ are also solutions of (13), but not vice versa.

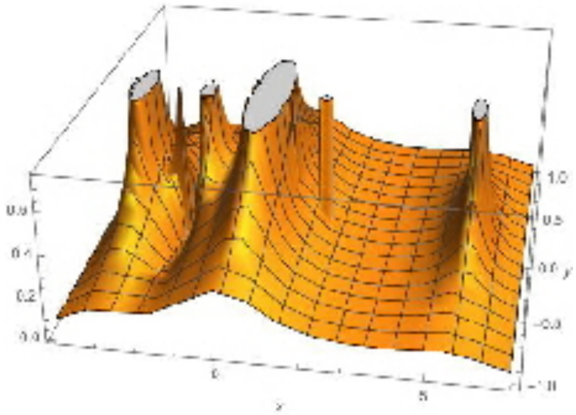

Figure 3: The absolute value of $f(x+iy)$ in (18) for $-4 \leq x \leq 7$ and $-1 \leq y \leq 1$.

## 3 The probability distribution of $\xi_c = \xi_1$

We have simulated $10^3$ ER graphs with $N = 10$ and link density $p = 0.6$ and in each of them, all possible, open link position for $l_{ij}$ were investigated as a function of $\xi$. Fig. 4 shows the probability density function $f_{\xi_1}(x)$ of $\xi_1$, the link weight strength of the added link $l_{ij}$ that creates the first complex zero in the adjacency matrix $A'$. We observe remarkable high peaks in $f_{\xi_1}(x)$ values at integer values of $x$. The probability $\Pr[\xi_1 > 1]$ in Fig. 4 that $\xi_1$ exceeds unity is about 56% and $\Pr[\xi_1 > 1]$ equals the probability that, after adding a link $l_{ij}$ with weight $\xi = 1$ to the graph $G$, all eigenvalues of the adjacency matrix $A'$ of the directed graph $G'$ with one directed link $l_{ij}$ are real.

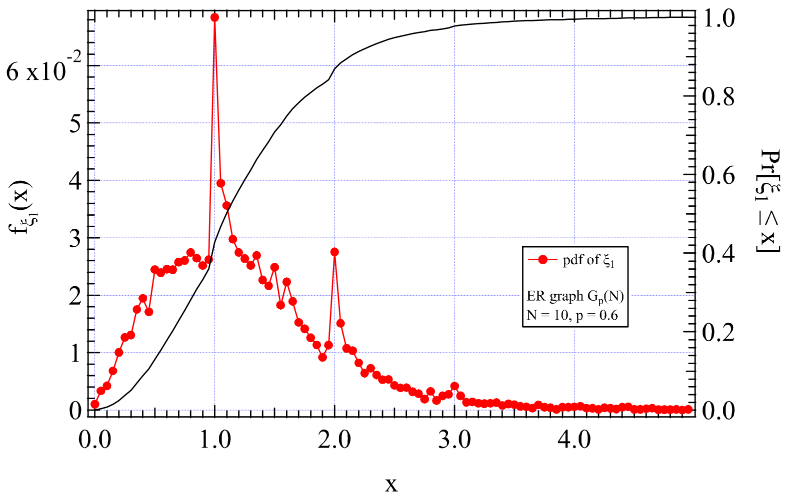


Figure 4: The probability density funtion $f_{\xi_1}(x)$ of the link weight strength $\xi_1$ that creates the first complex eigenvalue in $A'$ in $G_{0.6}(10)$. The binsize is 0.05 with 100 bins in total.

Fig. 5 shows the probability density function $f_{\xi_1}(x)$ of $\xi_1$ in 200 instances of an ER random graph

$G_{0.6}(20)$ and all possible link positions.

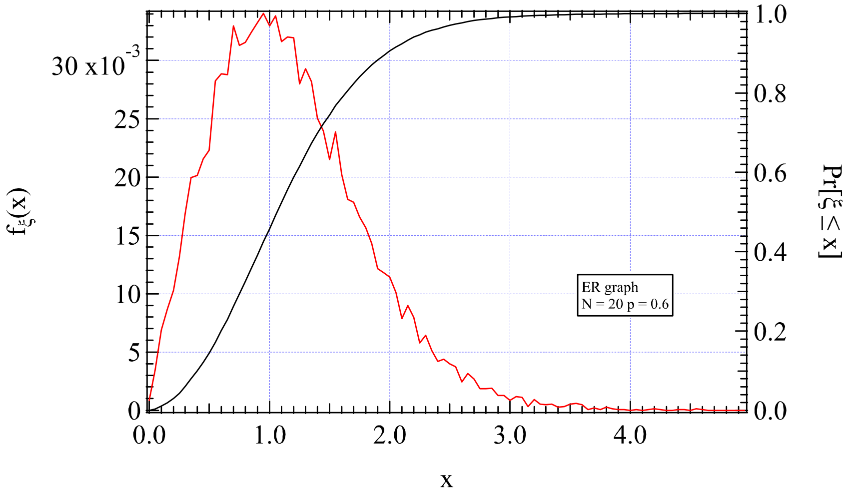


Figure 5: The probability density funtion $f_{\xi_1}(x)$ of the link weight strength $\xi_1$ that creates the first complex eigenvalue in $A'$ in $G_{0.6}(20)$. The binsize is 0.05 with 100 bins in total.

The probability $\Pr[\xi_1 > 1]$ in Fig. 5 that $\xi_1$ exceeds unity is about 55% for $N = 20$, close to the corresponding value for $N = 10$. The discrete peaks at integer values of $x$ have disappeared and $f_{\xi_1}(x)$ for ER random graphs $G_p(N)$ is likely a Gamma distribution (with exponential tail as shown in Fig. 6 and Fig. 7). Fig. 6 and Fig. 7 are based on $10^4$ independent realizations of $G_p(N)$ and only one directed link per graph was considered (as opposed to Fig. 4 and Fig. 5, where all possible directed links in $10^3$ and $2.10^2$ random graphs were taken into account and $p = 0.6$). At first glance, the correlation in the former figures hardly seems to play a significant role. The exponential tails decay fast with increasing link density $p$ in $G_p(N)$, whereas the probability that all eigenvalues of $A'$ are real, i.e. $\Pr[\xi_1 > 1]$, decreases with $p$ (and only signifantly when $p$ approaches 1, in which case a peak around $x = 1$ remains visible for $N = 20$).

Fig. 8 illustrates that $\xi_1$ converges very rapidly in $N$ to an asymptotic distribution in ER graphs $G_p(N)$ with a same and constant $p$. The fit (in thick line in orange) illustrates that the asymptotic distribution may be close to a Gamma distribution with probability density function $f_\Gamma(x;\alpha,\beta) = \frac{\alpha(\alpha x)^{\beta-1}}{\Gamma(\beta)}e^{-\alpha x}$, with $\beta \approx 3$ and $\alpha \approx 2.4$. It would be desirable to have a mathematical proof of this observation.

Plots of $\xi_1$ versus the effective resistance $\omega_{ij}$, the degree product $d_i d_j$, the node betweenness product $b_i b_j$ (computed before adding the link $l_{ij}$) do not seem to exhibit any correlation (same conclusion for $N = 20$). We also found that the approximate $(\xi_1)_a = \frac{1}{(\Xi(\eta_{\min}))_{(1)}}$ is mostly larger than $\xi_1$, but that $\eta_a = (\eta_{\min})_{(1)}$ reasonably well approximates $\eta$, the first complex eigenvalue at link weight strength $\xi_1$ as illustrated in Fig. 9 in a total of 17999 instances.

The correlation between the approximate $(\xi_1)_a$ and $\xi_1$ seems better (i.e. more on a line) for $N = 20$ than for $N = 10$, and similarly, also the correlation between $\eta_a$ and $\eta$ seems to increase with $N$.

Fig. 10 illustrates that a first complex eigenvalue $\eta$ of $A'$ lies between $\lambda_N \le \eta \le \lambda_2$. It seems

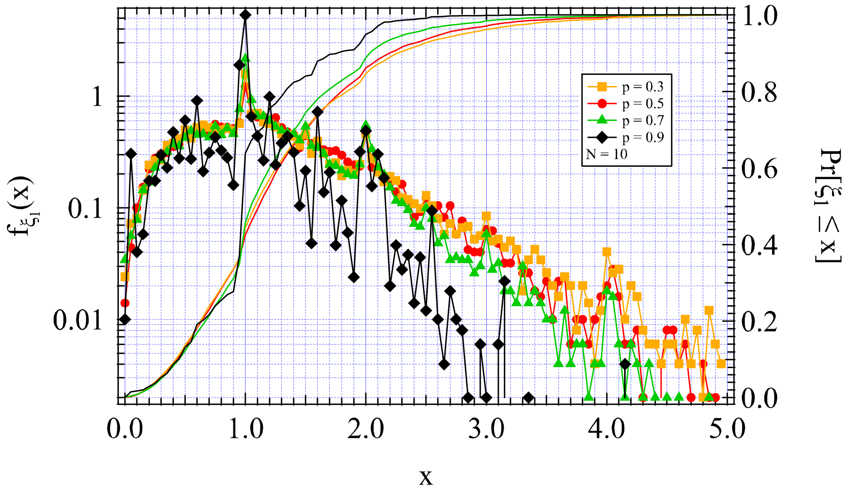


Figure 6: Probability and probability density function of $\xi_1$ for 4 link densities $p$ in an ER-graph on $N = 10$.

that, only in the bulk of the Wigner semi-circle law, complex eigenvalue can be born (and not in the spectral gap between $\lambda_1$ and $\lambda_2$).

## 4 Agenda of future research

On the agenda of future research stands (a) the investigation of the link weight strength $\xi_m$ with $m > 1$ as well as the determination of the total number $n_c$ of complex eigenvalues when $\xi$ ranges over all real, positive values. A study of the link weight strength $\xi$ and complex eigenvalues when the adjacency $A$ possesses multiple eigenvalues is here omitted. Further (b), what is the effect of the addition of more than one directed and weighted link?

We only considered Erdős-Rényi random graphs, so that (c) an extension to other graphs might be desirable. In particular, are there graphs that allow an exact analytical computation? Finally (d), we lack at the moment strong and general mathematical results about properties of complex eigenvalues in directed graphs.

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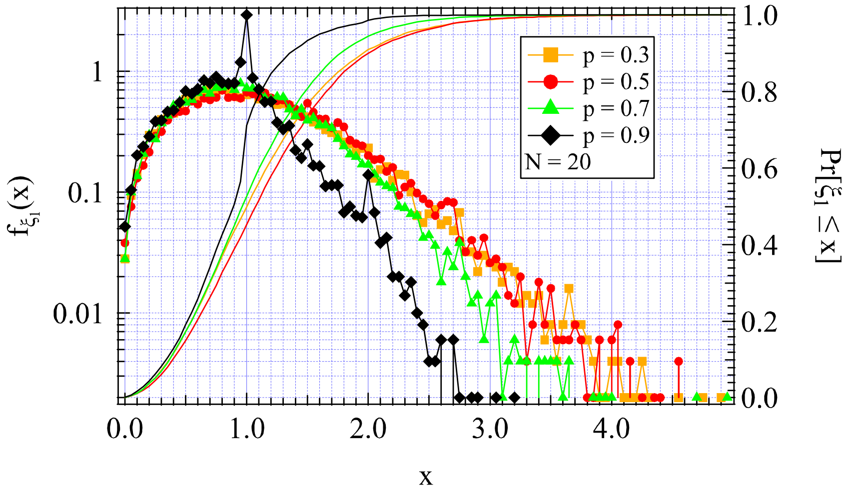


Figure 7: Probability and probability density function of $\xi_1$ for 4 link densities $p$ in an ER-graph on $N = 20$.


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# A Proof of Theorem 3

We provide two different proofs.

(A) First, we determine when a zero $\eta_0$ of $\Xi(\eta)$ occurs that lies in between $\lambda_m < \eta < \lambda_{m-1}$. We write (16) as

$$\Xi(\eta) = \frac{\{\alpha_{m-1}\lambda_m + \alpha_m\lambda_{m-1}\} - \{\alpha_{m-1} + \alpha_m\}\eta}{(\lambda_{m-1} - \eta)(\eta - \lambda_m)}$$

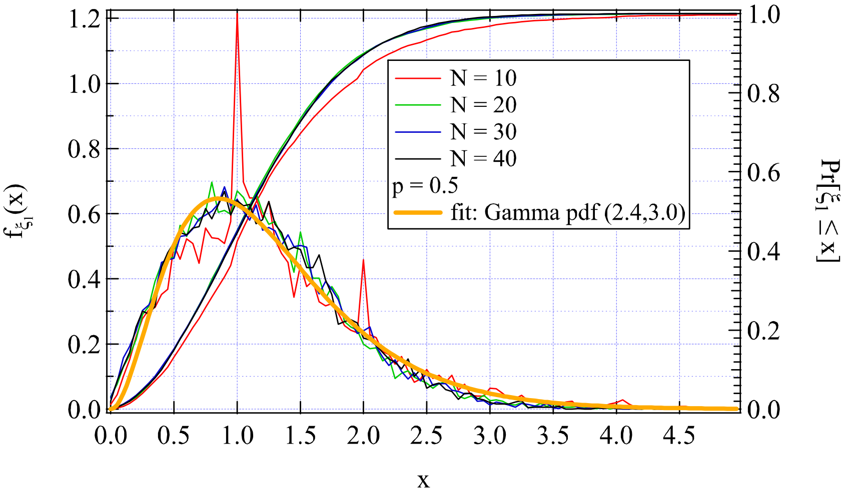


Figure 8: The pdf and distribution of $\xi_1$ in ER graphs $G_{0.5}(N)$ for various $N$, together with a fit of the Gamma distribution with parameter $\alpha = 2.4$ and $\beta = 3.0$.

which possesses a zero at

$$\eta_0 = \left(1 - \frac{\alpha_m}{\alpha_{m-1} + \alpha_m}\right)\lambda_m + \left(\frac{\alpha_m}{\alpha_{m-1} + \alpha_m}\right)\lambda_{m-1}$$

Only if $0 < \frac{\alpha_m}{\alpha_{m-1}+\alpha_m} < 1$, then the right-hand side shows that $\eta_0$ indeed lies between $\lambda_m$ and $\lambda_{m-1}$. If $\alpha_{m-1}+\alpha_m > 0$, then the condition $\frac{\alpha_m}{\alpha_{m-1}+\alpha_m} < 1$, means that $0 < \alpha_{m-1}$, while $0 < \frac{\alpha_m}{\alpha_{m-1}+\alpha_m}$ indicates that also $\alpha_m > 0$. On the other hand, if $\alpha_{m-1} + \alpha_m < 0$, then the condition $\frac{\alpha_m}{\alpha_{m-1}+\alpha_m} < 1$, means that $0 > \alpha_{m-1}$, while $0 < \frac{\alpha_m}{\alpha_{m-1}+\alpha_m}$ indicates that also $\alpha_m < 0$. Thus, $\Xi$ has a zero between $\lambda_m$ and $\lambda_{m-1}$, provided both $\alpha_m$ and $\alpha_{m-1}$ have the same sign.

Next, $\Xi$ has a minimum at $\eta_{\min} \in (\lambda_m, \lambda_{m-1})$ that obeys $\left.\frac{d\Xi}{d\eta}\right|_{\eta=\eta_{\min}} = 0$ and $\left.\frac{d^2\Xi}{d\eta^2}\right|_{\eta=\eta_{\min}} > 0$. Thus, the condition for an extremum, $\frac{d\Xi}{d\eta} = 0$, translates to

$$\frac{(\eta - \lambda_m)^2}{(\eta - \lambda_{m-1})^2} = -\frac{\alpha_m}{\alpha_{m-1}}$$

which requires that $\alpha_m$ and $\alpha_{m-1}$ have a different sign! Assume that $\alpha_m > 0 > \alpha_{m-1}$, then the condition for a minimum $\eta_{\min} \in (\lambda_m, \lambda_{m-1})$ reads

$$\begin{aligned}
0 &= \alpha_m (\eta_{\min} - \lambda_{m-1})^2 + \alpha_{m-1} (\eta_{\min} - \lambda_m)^2 \\
&= \left(\sqrt{\alpha_m}(\lambda_{m-1} - \eta_{\min})\right)^2 - \left(\sqrt{-\alpha_{m-1}}(\eta_{\min} - \lambda_m)\right)^2 \\
&= \left(\sqrt{\alpha_m}(\lambda_{m-1} - \eta_{\min}) + \sqrt{-\alpha_{m-1}}(\eta_{\min} - \lambda_m)\right)\left(\sqrt{\alpha_m}(\lambda_{m-1} - \eta_{\min}) - \sqrt{-\alpha_{m-1}}(\eta_{\min} - \lambda_m)\right)
\end{aligned}$$

Since $\sqrt{\alpha_m}(\lambda_{m-1} - \eta_{\min}) + \sqrt{-\alpha_{m-1}}(\eta_{\min} - \lambda_m) > 0$ due to the assumption $\alpha_m > 0 > \alpha_{m-1}$, a solution is only possible if

$$\sqrt{\alpha_m}(\lambda_{m-1} - \eta_{\min}) = \sqrt{-\alpha_{m-1}}(\eta_{\min} - \lambda_m)$$

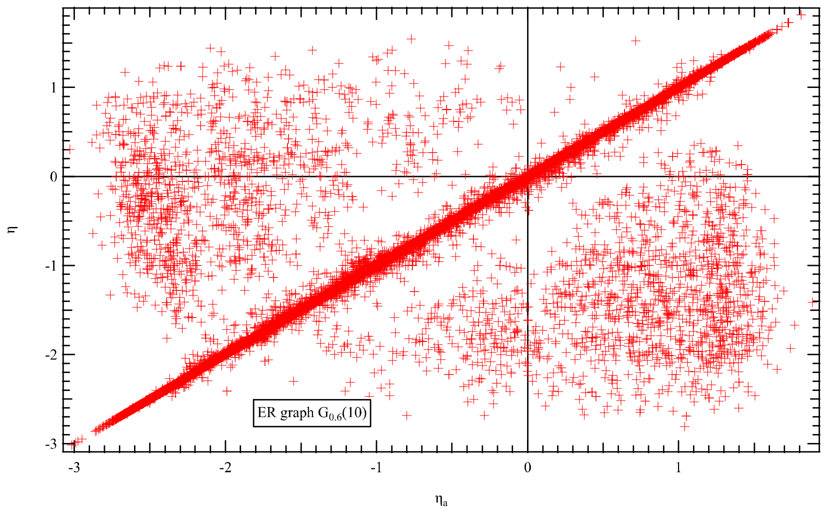


Figure 9: The first complex eigenvalue $\eta$ versus the approximate $\eta_a$ in (17).

from which we find

$$\eta_{\min} = \frac{\sqrt{\alpha_m}\lambda_{m-1} + \sqrt{-\alpha_{m-1}}\lambda_m}{\sqrt{\alpha_m} + \sqrt{-\alpha_{m-1}}}$$

which lies in between $\lambda_m < \eta < \lambda_{m-1}$ (and which agrees with the quadratic solution below). Moreover, the value of $\Xi(\eta_{\min})$ equals

$$\Xi(\eta_{\min}) = \frac{\left(\sqrt{\alpha_m} + \sqrt{-\alpha_{m-1}}\right)^2}{\lambda_{m-1} - \lambda_m}$$

indicating that $\Xi(\eta_{\min}) > 0$ and that a zero $\eta_0 \in (\lambda_m, \lambda_{m-1})$ of $\Xi(\eta)$ is not possible when $\alpha_m > 0 > \alpha_{m-1}$. We verify that the extremum is indeed a minimum for $\lambda_m < \eta < \lambda_{m-1}$,

$$\begin{aligned}\left.\frac{d^2\Xi}{d\eta^2}\right|_{\eta=\eta_{\min}} &= \frac{2\alpha_{m-1}}{(\eta_{\min} - \lambda_{m-1})^3} + \frac{2\alpha_m}{(\eta_{\min} - \lambda_m)^3} \\ &= \frac{2\alpha_m}{(\eta_{\min} - \lambda_m)^3}\left\{1 + \frac{\sqrt{\alpha_m}}{\sqrt{-\alpha_{m-1}}}\right\} > 0\end{aligned}$$

Similarly, if $\alpha_m < 0 < \alpha_{m-1}$, then we will find a maximum $\eta_{\max} \in (\lambda_m, \lambda_{m-1})$,

$$\begin{aligned}0 &= \alpha_m(\eta_{\max} - \lambda_{m-1})^2 + \alpha_{m-1}(\eta_{\max} - \lambda_m)^2 \\ &= \left(\sqrt{\alpha_{m-1}}(\eta_{\max} - \lambda_m) + \sqrt{-\alpha_m}(\lambda_{m-1} - \eta_{\max})\right)\left(\sqrt{\alpha_{m-1}}(\eta_{\max} - \lambda_m) - \sqrt{-\alpha_m}(\lambda_{m-1} - \eta_{\max})\right)\end{aligned}$$

Since $\sqrt{\alpha_{m-1}}(\eta_{\max} - \lambda_m) + \sqrt{-\alpha_m}(\lambda_{m-1} - \eta_{\max}) > 0$ due to the assumption $\alpha_m < 0 < \alpha_{m-1}$, a solution is only possible if

$$\sqrt{\alpha_{m-1}}(\eta_{\max} - \lambda_m) = \sqrt{-\alpha_m}(\lambda_{m-1} - \eta_{\max})$$

from which we find

$$\eta_{\max} = \frac{\sqrt{-\alpha_m}\lambda_{m-1} + \sqrt{\alpha_{m-1}}\lambda_m}{\sqrt{-\alpha_m} + \sqrt{\alpha_{m-1}}}$$

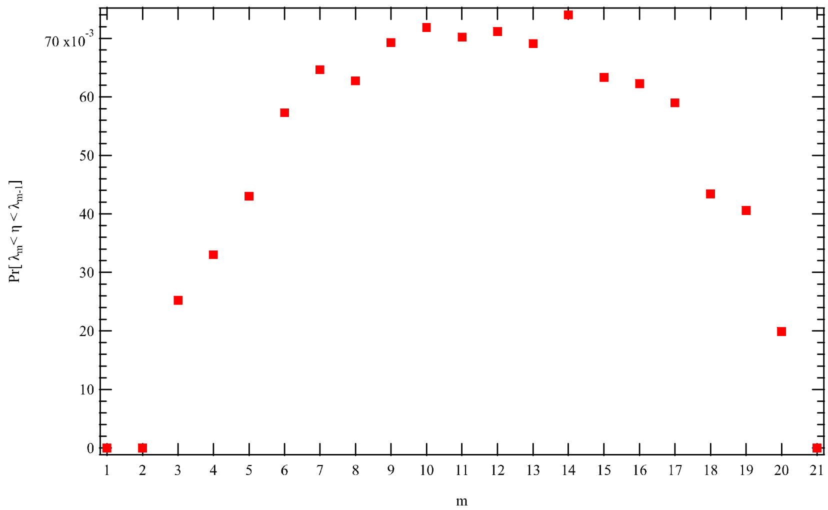


Figure 10: The probability that a complex eigenvalue $\eta$ lies between the eigenvalues $\lambda_{m-1}$ and $\lambda_m$ of the original matrix $A$. If $m = 1$, then $\eta > \lambda_1$ and $m = 21$ means that $\eta < \lambda_N$, both do not occur.

which lies in between $\lambda_m < \eta < \lambda_{m-1}$ (and which agrees with the quadratic solution below in (B)). Moreover, the value of $\Xi(\eta_{\max})$ equals

$$\Xi(\eta_{\max}) = -\frac{\left(\sqrt{-\alpha_m} + \sqrt{\alpha_{m-1}}\right)^2}{(\lambda_{m-1} - \lambda_m)}$$

indicating that $\Xi(\eta_{\max}) < 0$ and that a zero $\eta_0 \in (\lambda_m, \lambda_{m-1})$ of $\Xi(\eta)$ is not possible when $\alpha_m < 0 < \alpha_{m-1}$ due to the fact that $\Xi(\eta_{\max})$ is a maximum for $\lambda_m < \eta < \lambda_{m-1}$. Indeed,

$$\left.\frac{d^2\Xi}{d\eta^2}\right|_{\eta=\eta_{\max}} = \frac{2\alpha_m}{(\eta_{\max} - \lambda_m)^3} \left\{1 + \frac{\sqrt{-\alpha_m}}{\sqrt{\alpha_{m-1}}\,(\alpha_{m-1})}\right\} < 0$$

(B) An alternative method determine the zeros of the function $\Xi(\eta)$ in (15) between $\lambda_m < \eta < \lambda_{m-1}$, by translating (15) into a quadratic equation in $\eta$

$$\Xi\eta^2 - \{\Xi(\lambda_{m-1} + \lambda_m) + \alpha_{m-1} + \alpha_m\}\,\eta + \Xi\lambda_{m-1}\lambda_m + \alpha_{m-1}\lambda_m + \alpha_m\lambda_{m-1} = 0$$

with roots

$$\eta_{1,2} = \frac{\{\Xi(\lambda_{m-1} + \lambda_m) + \alpha_{m-1} + \alpha_m\} \pm \sqrt{\Delta}}{2\Xi}$$

where the discriminant $\Delta$ is negative (leading to complex conjugate eigenvalues $\eta_1$ and $\eta_2 = \eta_1^*$) if

$$\{\Xi(\lambda_{m-1} + \lambda_m) + \alpha_{m-1} + \alpha_m\}^2 < 4\Xi(\Xi\lambda_{m-1}\lambda_m + \alpha_{m-1}\lambda_m + \alpha_m\lambda_{m-1})$$

or

$$\Xi^2(\lambda_{m-1} - \lambda_m)^2 - 2\Xi(\lambda_{m-1} - \lambda_m)(\alpha_m - \alpha_{m-1}) + (\alpha_{m-1} + \alpha_m)^2 < 0$$

Let $f = \Xi(\lambda_{m-1} - \lambda_m)$, then

$$f^2 - 2(\alpha_m - \alpha_{m-1})\,f + (\alpha_{m-1} + \alpha_m)^2 < 0$$

which indicates that the roots have the same sign (because their product $(\alpha_{m-1}+\alpha_m)^2$ is always positive) and

$$\begin{aligned} f_{1,2} &= (\alpha_m - \alpha_{m-1}) \pm \sqrt{(\alpha_m - \alpha_{m-1})^2 - (\alpha_{m-1}+\alpha_m)^2} \\ &= (\alpha_m - \alpha_{m-1}) \pm 2\sqrt{-\alpha_{m-1}\alpha_m} \end{aligned}$$

which requires also that $\alpha_{m-1}\alpha_m < 0$, because $f = \Xi(\lambda_{m-1} - \lambda_m)$ must be real and none of $\alpha_m = 0$ nor $\alpha_{m-1} = 0$. To proceed further, we distinguish between two cases: (a) $\alpha_m > 0 > \alpha_{m-1}$ and thus $\Xi > 0$ as follows[5] from (16) for $\lambda_m < \eta < \lambda_{m-1}$ and (b) $\alpha_m < 0 < \alpha_{m-1}$ (but then $\Xi < 0$) and the roots satisfy

$$\begin{cases} f_{1,2} = \left(\sqrt{\alpha_m} \pm \sqrt{-\alpha_{m-1}}\right)^2 & \text{if } \Xi > 0 \\ f_{1,2} = -\left(\sqrt{-\alpha_m} \pm \sqrt{\alpha_{m-1}}\right)^2 & \text{if } \Xi < 0 \end{cases}$$

Hence, the condition for $\Xi$ to create a complex $\eta$ is

$$\begin{cases} 0 < \frac{\left(\sqrt{\alpha_m} - \sqrt{-\alpha_{m-1}}\right)^2}{(\lambda_{m-1}-\lambda_m)} \leq \Xi \leq \frac{\left(\sqrt{\alpha_m} + \sqrt{-\alpha_{m-1}}\right)^2}{(\lambda_{m-1}-\lambda_m)} & \text{if } \Xi > 0 \\ -\frac{\left(\sqrt{-\alpha_m} + \sqrt{\alpha_{m-1}}\right)^2}{(\lambda_{m-1}-\lambda_m)} \leq \Xi \leq -\frac{\left(\sqrt{-\alpha_m} - \sqrt{\alpha_{m-1}}\right)^2}{(\lambda_{m-1}-\lambda_m)} < 0 & \text{if } \Xi < 0 \end{cases} \tag{19}$$

Just before a complex root arises, the discriminant is zero (equivalent with equality sign in the condition on $\Xi$) and the double eigenvalue then is

$$\eta_{1,2} = \frac{\{\Xi(\lambda_{m-1}+\lambda_m) + \alpha_{m-1} + \alpha_m\}}{2\Xi} = \frac{\lambda_{m-1}+\lambda_m}{2} + \frac{\alpha_{m-1}+\alpha_m}{2\Xi} \tag{20}$$

(a) Introducing in (20) the largest $\Xi > 0$, corresponding to the smallest $\xi > 0$ in the resolvent eigenvalue equation (2), from (19) yields

$$\eta_{1,2} = \frac{\lambda_{m-1}+\lambda_m}{2} + \frac{\alpha_m + \alpha_{m-1}}{\left(\sqrt{\alpha_m} + \sqrt{-\alpha_{m-1}}\right)^2}\left(\frac{\lambda_{m-1}-\lambda_m}{2}\right)$$

and, finally,

$$\eta_{1,2} = \lambda_{m-1}\left(\frac{\sqrt{\alpha_m}}{\sqrt{\alpha_m} + \sqrt{-\alpha_{m-1}}}\right) + \lambda_m\left(1 - \frac{\sqrt{\alpha_m}}{\sqrt{\alpha_m} + \sqrt{-\alpha_{m-1}}}\right)$$

illustrating that $\eta_{1,2}$ indeed lies between $\lambda_m < \eta < \lambda_{m-1}$ and equivalent (17). (b) The case where $\Xi < 0$, is analogous and omitted.

These two demonstrations A and B prove Theorem 3. $\square$

[5] If $\Xi = \frac{\alpha_{m-1}}{\eta-\lambda_{m-1}} + \frac{\alpha_m}{\eta-\lambda_m} < 0$ in (16) for $\lambda_m < \eta < \lambda_{m-1}$, then $\frac{\alpha_m}{\eta-\lambda_m} < \frac{\alpha_{m-1}}{\lambda_{m-1}-\eta}$, which is possible if $\alpha_m < 0 < \alpha_{m-1}$! On the other hand, $\Xi > 0$ if $\frac{\alpha_{m-1}}{\eta-\lambda_{m-1}} + \frac{\alpha_m}{\eta-\lambda_m} > 0$ or $\frac{\alpha_m}{\eta-\lambda_m} > \frac{\alpha_{m-1}}{\lambda_{m-1}-\eta}$ implying that $\alpha_m > 0 > \alpha_{m-1}$, because we know that $\alpha_m\alpha_{m-1} < 0$.

## B Theorems on eigenvalues of an $n \times n$ matrix

We list two interesting theorems and present Theorem 6 for adjacency matrices.

A circuit (closed walk) with length $k$ consists of the ordered links $(j_1 \to j_2), \ldots, (j_{k-1} \to j_k), (j_k \to j_1)$. Hence, the indices $j_1, j_2, \ldots, j_k$ form a circuit if and only if the product $a_{j_1 j_2} a_{j_2 j_3} \ldots a_{j_k j_1} \neq 0$.

**Theorem 4** *Let $\lambda_1$ denote the largest eigenvalue of the asymmetric adjacency matrix $A$ of a directed graph $G$ and let $m$ denote the length (in hops) of the longest circuit in $G$. If $m = 2$, then all eigenvalues of $A$ are real. If $m > 2$, then any eigenvalue $\lambda$ of $A$ satisfies*

$$\operatorname{Re}(\lambda) + |\operatorname{Im}(\lambda)| \tan \frac{\pi}{m} \leq \lambda_1 \tag{21}$$

**Proof:** see [3, p. 210]. □

**Theorem 5 (Schur, 1909)** *If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of a complex $n \times n$ matrix $A$, then*

$$\sum_{k=1}^{n} |\lambda_k|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \tag{22}$$

$$\sum_{k=1}^{n} (\operatorname{Re} \lambda_k)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{a_{ij} + a_{ji}^*}{2} \right|^2 \tag{23}$$

$$\sum_{k=1}^{n} (\operatorname{Im} \lambda_k)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{a_{ij} - a_{ji}^*}{2} \right|^2 \tag{24}$$

*Equality in any one of these relations implies equality in all three and equality occurs if and only if $A$ is normal, which obeys $(A^*)^T A = A (A^*)^T$ also written as $A^H A = A A^H$.*

**Proof:** see Mirsky [7, p. 310]. □

The proof applies the trace-formula $\sum_{k=1}^{n} \lambda_k^2 = \operatorname{trace}(A^2)$ to Schur's famous theorem that every matrix $A$ is unitarily similar to a triangular matrix $T$, i.e. $A = UTU^{-1} = UTU^H$ (because a unitary matrix obeys $U^H U = U U^H = I$). Equality follows because a matrix $A$ is unitarily similar to a diagonal matrix if and only if $A$ is normal.

Schur's inequality (24) provides the bound on the imaginary part of eigenvalues of $A' = A + \xi e_i e_j^T$,

$$\sum_{k=1}^{N} |\operatorname{Im} \eta_k|^2 \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \frac{a'_{ij} - a'_{ji}}{2} \right|^2 = \frac{\xi^2}{4} \tag{25}$$

In particular, after the birth of the first complex conjugate pair at $\xi_c = \xi_1$, we have that $2 |\operatorname{Im} \eta|^2 \leq \sum_{k=1}^{N} |\operatorname{Im} \eta_k|^2$, untill $\xi < \xi_2$, so that

$$|\operatorname{Im} \eta| \leq \frac{\xi_2}{2\sqrt{2}}$$

In the general trace-formula [10, art 138, p. 212]

$$\sum_{k=1}^{n} \lambda_k^m = \sum_{j=1}^{n} (A^m)_{jj} = \operatorname{trace}(A^m)$$

we compute

$$\lambda_k^m = (\operatorname{Re}\lambda_k + i\operatorname{Im}\lambda_k)^m = \sum_{q=0}^{m}\binom{m}{q}(\operatorname{Re}\lambda_k)^{m-q}\, i^q\, (\operatorname{Im}\lambda_k)^q$$

We split the sum into odd and even terms, using

$$\sum_{q=0}^{m} p(q) = \sum_{q=0}^{\left[\frac{m}{2}\right]} p(2q) + \sum_{q=1}^{\left[\frac{m+1}{2}\right]} p(2q-1)$$

valid for any function $p(.)$ and obtain

$$\lambda_k^m = \sum_{q=0}^{\left[\frac{m}{2}\right]}\binom{m}{2q}(\operatorname{Re}\lambda_k)^{m-2q}(-1)^q(\operatorname{Im}\lambda_k)^{2q} + i\sum_{q=1}^{\left[\frac{m+1}{2}\right]}\binom{m}{2q-1}(\operatorname{Re}\lambda_k)^{m+1-2q}(-1)^{q-1}(\operatorname{Im}\lambda_k)^{2q-1}$$

Since trace$(A^m)$ is real, the imaginary part must vanish and we arrive at

$$\operatorname{trace}(A^m) = \sum_{q=0}^{\left[\frac{m}{2}\right]}\binom{m}{2q}(-1)^q\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^{m-2q}(\operatorname{Im}\lambda_k)^{2q} \tag{26}$$

Explicitly, for $m = 2$, 3 and 4, (26) is

$$\begin{aligned}
\operatorname{trace}\left(A^2\right) &= \sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^2 - \sum_{k=1}^{n}(\operatorname{Im}\lambda_k)^2 \\
\operatorname{trace}\left(A^3\right) &= \sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^3 - 3\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)(\operatorname{Im}\lambda_k)^2 \\
\operatorname{trace}\left(A^4\right) &= \sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^4 - 6\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^2(\operatorname{Im}\lambda_k)^2 + \sum_{k=1}^{n}(\operatorname{Im}\lambda_k)^4
\end{aligned}$$

**Theorem 6** *If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of a real $n \times n$ adjacency matrix $A$ of a graph $G$ without self-loops (i.e. $a_{jj} = 0$), with $L_{directed}$ links and in total $L = L_{directed} + 2L_{bidirected}$ links, then*

$$\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^2 - \sum_{k=1}^{n}(\operatorname{Im}\lambda_k)^2 = L - L_{directed} = 2L_{bidirected} \tag{27}$$

**Proof:** From (26) for $m = 2$ and $\left(A^2\right)_{jj} = \sum_{k=1}^{n} a_{jk}a_{kj}$, the trace$\left(A^2\right)$ formula equals

$$\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^2 - \sum_{k=1}^{n}(\operatorname{Im}\lambda_k)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji}$$

With $a_{ij}a_{ji} = \frac{a_{ji}^2 + a_{ij}^2 - (a_{ji} - a_{ij})^2}{2}$, we find the general relation

$$\sum_{k=1}^{n}(\operatorname{Re}\lambda_k)^2 - \sum_{k=1}^{n}(\operatorname{Im}\lambda_k)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - a_{ji})^2 \tag{28}$$

In particular, if $A$ is an adjacency matrix of a graph $G$ with a total of $L$ links and $L_{\text{directed}}$ links (but without self-loops, i.e. $a_{jj} = 0$), then $a_{ij}^2 = a_{ij}$ and $(a_{ij} - a_{ji})^2 = 0$ for a bidirectional or undirected link, while $(a_{ij} - a_{ji})^2 = 1$ for a directed link. For such an asymmetric adjacency matrix, (28) translates to (27). $\square$

Schur's inequality (24) tells us $\sum_{k=1}^{n} (\operatorname{Im} \lambda_k)^2 \leq \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - a_{ji})^2 = \frac{L_{\text{directed}}}{2}$, so that (27) can be bounded

$$L - L_{\text{directed}} \leq \sum_{k=1}^{n} (\operatorname{Re} \lambda_k)^2 \leq L - \frac{L_{\text{directed}}}{2}$$

Since $L - L_{\text{directed}} = 2L_{\text{bidirected}}$, we observe that $\sum_{k=1}^{n} \lambda_k^2 = 2L_{\text{bidirected}}$ is always even and that $\sum_{k=1}^{n} (\operatorname{Re} \lambda_k)^2$ is larger for an asymmetric than for symmetric matrix, because equality of the lower bound holds for a symmetric matrix. Simulations indeed confirm that Schur's inequality (25) is not sharp, at least when one link is added.