The Mittag-Leffler function

P. Van Mieghem*

in honor of Gösta Mittag-Leffler
for Huijuan

v1. 23 May 2020
v2. 22 August 2020
v3. 25 September 2021
v4. 4 September 2022
v5. 18 February 2024

Abstract

We review the function theoretical properties of the Mittag-Leffler function $E_{a,b}(z)$ in a self-contained manner, but also add new results; more than half is new!

1 Introduction

We investigate the Mittag-Leffler function,

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)} \tag{1}$$

introduced by Gösta Mittag-Leffler [34, 33] in 1903 with $b = 1$, which he denoted as $E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+ak)} \triangleq E_{a,1}(z)$.

We consider the broader definition $E_{a,b}(z)$ and not $E_a(z)$, because the functional relations for $E_{a,b}(z)$ are closed and expressed in terms of $E_{a,b}(z)$, whereas confinement to $E_a(z)$ only, deprives the analysis from a complete and more elegant picture. There exist generalizations[1] of the Mittag-Leffler function $E_{a,b}(z)$, which are beyond the present scope, but discussed by Haubold *et al.* [25] and also covered in the recent book by Gorenflo *et al.* [18] on Mittag-Leffler functions and their applications. The Mittag-Leffler function $E_a(z)$ is treated by Erdelyi *et al.* [10, Sec. 18.1 on p. 206-211] and by Sansone and Gerretsen [44, Sec. 6.13 on p. 345-348].

I believe that there are, at least, three compelling reasons that justifies a study of the Mittag-Leffler function $E_{a,b}(z)$. *First*, the Mittag-Leffler function $E_{a,b}(z)$ naturally arises in fractional calculus

*Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, P.O Box 5031, 2600 GA Delft, The Netherlands; *email*: P.F.A.VanMieghem@tudelft.nl

[1]The generalized Mittag-Leffler function is defined as $E_{a,b,c}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(c+k)}{\Gamma(c)} \frac{z^k}{\Gamma(b+ak)}$, where $E_{a,b,0}(z) = E_{a,b}(z)$.

as shown in Appendix D. The analytic solution (202) is undoubtedly the most important driver towards the increasing appearance of the Mittag-Leffler function $E_{a,b}(z)$. The application of the solution (202) of a fractional order integral or differential equations is illustrated in [25], [18], [30] and [43]; for example, in the fractional generalization of the heat equation, random walks, Lévy flights, superdiffusive transport and viscoelasticity, and fractional Ohm's Law. Abel's integral equation, whose solution involves $E_{a,b}(z)$, is treated in [17, Chapter 7]. A "fractional" generalization of the Poisson renewal process, discussed in [17, Sec. 9.4], consists of replacing the exponential interarrival time between events by a Mittag-Leffler distribution $E_{a,1}(-t)$ with real $t \geq 0$ and $0 < a \leq 1$. *Second*, a Mittag-Leffler random variable is heavy-tailed and plays a role in so-called stable distributions. Many observed properties in real-world networks are power-law distributed and the Mittag-Leffler random variable may model such power-law like properties, although none of its moments exists, which is a rather complicating, but at the same time fascinating factor. *Third* and the main focus here, the Mittag-Leffler function $E_{a,b}(z)$ in (1) is an entire function in the complex variable z in two real parameters $a > 0$ and b and constitutes a broad class of entire functions such as the exponential function $E_{1,1}(z) = e^z$ and many exponential-like functions such as the cosine $E_{2,1}\left(-z^2\right) = \cos z$ and $E_{1,2}(z) = \frac{e^z-1}{z}$ and many more.

Our aim[2] here is to deduce the most relevant functional properties of the Mittag-Leffler function $E_{a,b}(z)$ defined in (1). Since about half of the results have been established before, the manuscript in the form of articles (**art.**) as in our book [53], is more a review, without detailed historical citations as in [18], but enriched with new results: **art.** 5, 8, 9, 12, 13 14, 15, 16, 17, 21, 23, 24, 25, 32, 34, 36, 38, 45, 46, 47, 49, 50, 51, 52, 54, 55, 71, 72 and part of **art.** 18, 28, 33, 37, 39, 41, 42.

1.1 Outline

Section 2 briefly summarizes the properties of entire functions that are defined for any complex number z by their Taylor series such as (1) for the Mittag-Leffler function $E_{a,b}(z)$. Section 3 starts with the Taylor series in (1) and deduces unique properties of the Mittag-Leffler function $E_{a,b}(z)$ from that Taylor series (1). We have created a separate Section 4, that only focuses on the logarithm of Mittag-Leffler function $E_{a,b}(z)$. Section 5 explores integrals that contain the Mittag-Leffler function $E_{a,b}(z)$ as integrand and that can be evaluated analytically. One of the most important integrals is the Laplace transform (54) of the Mittag-Leffler function $E_{a,b}(z)$. We continue in Section 6 with complex integrals for the Mittag-Leffler function $E_{a,b}(z)$, some are deduced from the inverse Laplace transform (54) and others are complex representations of the Taylor series (1). Section 7 is devoted to the Mittag-Leffler function $E_{a,b}(z)$ in probability theory. **Art**. 39 presents a different proof of the monotonicity of the Mittag-Leffler function $E_{a,b}(z)$, while **art**. 41 and **art**. 42 focus on the Mittag-Leffler random variable. Section 8 covers various, unrelated topics such as expansions of the Mittag-Leffler function $E_{a,b}(z)$ in powers of a in **art**. 45, the deduction of the integral (84) from the Taylor series (1) and the Taylor series of $\frac{1}{\Gamma(z)}$ in **art**. 46, a product form for $E_{a,b}(z)$ in **art**. 47, Mobius inversion in **art**. 49, Apelblat series in **art**. 51 and the Mittag-Leffler function $E_{a,b}(z)$ as a limit in **art**. 52-55. Section 9 is new and

[2] During my sabattical at Stanford in 2015, I encountered the rather exotic Mittag-Leffler distribution, which I intended to include in my Performance Analysis book [50] as another example of a power law-like distribution. However, the functional properties of the Mittag-Leffler function $E_{a,b}(z)$ require attention first, before one can turn to probability theory.

explores the related integral

$$I_{a,b}(z) = \int_0^\infty \frac{z^u}{\Gamma(b+au)}\,du \tag{2}$$

that naturally appears in the Euler-Maclaurin summation for $E_{a,b}(z)$ in **art**. 24. Section 10 concludes this work with some open problems. For self-consistency, we have included an appendix A on the Gamma function $\Gamma(z)$, whose properties are essential for the Mittag-Leffler function $E_{a,b}(z)$. Appendix B evaluates a Cauchy-type integral and Mellin transforms of product of Gamma functions. Appendix C investigates the inverse Laplace transform, separated in real and imaginary part. The last Appendix D emphasizes the key role of the Mittag-Leffler function $E_{a,b}(z)$ in fractional calculus.

2 Complex function theory: entire functions

Since the Mittag-Leffler function $E_{a,b}(z)$ is defined in (1) as a power series in z, a first concern is its validity range in z. The radius R of convergence of the power series $f(z) = \sum_{k=0}^\infty f_k\, z^k$ of a function f satisfies $\frac{1}{R} = \limsup_{k\to\infty} |f_k|^{1/k}$ or $\frac{1}{R} = \lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|$ when the latter exists [47]. Using $\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}$ in [1, 6.1.47] when $z \to \infty$, the radius of convergence of the power series in (1) is

$$\frac{1}{R} = \lim_{k\to\infty} \left|\frac{\Gamma(b+ak)}{\Gamma(b+a+ak)}\right| = \lim_{k\to\infty} |ak|^{-a} = 0$$

for $\mathrm{Re}(a) > 0$ and all complex b. For $\mathrm{Re}(a) < 0$, the radius of convergence is $R = 0$ and the Taylor series (1) of the Mittag-Leffler function $E_{a,b}(z)$ does not converge for any b and any complex number $z \neq 0$. If $a = 0$, then the Mittag-Leffler function $E_{0,b}(z)$ reduces to the geometric series and $E_{0,b}(z) = \frac{1}{\Gamma(b)}\frac{1}{1-z}$ for $|z| < 1$, which is the only case where the radius of convergence is finite, i.e. $R = 1$.

An entire function has a power series with infinite radius of convergence and is, thus, analytic in the entire complex plane with an essential singularity at infinity. Associated to entire complex functions is the concept of the order ρ, which is defined [47, p. 248] for any $\varepsilon > 0$ as $f(z) = O\left(e^{|z|^{\rho+\varepsilon}}\right)$ when $|z| \to \infty$. A necessary and sufficient condition [47, p. 253] that $f(z) = \sum_{k=0}^\infty f_k\, z^k$ should be an entire function of order ρ is that

$$\frac{1}{\rho} = \lim_{k\to\infty} \frac{-\log|f_k|}{k\log k}$$

Applied to the power series in (1) of the Mittag-Leffler function $E_{a,b}(z)$, after invoking Stirling's asymptotic formula [1, 6.1.39] that follows from (171) in Appendix A,

$$\Gamma(ak+b) \sim \sqrt{2\pi} e^{-ak} (ak)^{ak+b-\frac{1}{2}}$$

yields

$$\frac{1}{\rho} = \lim_{k\to\infty} \frac{\log|\Gamma(b+ak)|}{k\log k} = \lim_{k\to\infty} \frac{\log\sqrt{2\pi} - ak + \left(ak+b-\frac{1}{2}\right)\log(ak)}{k\log k} = a$$

Hence, we find one of the important properties that the Mittag-Leffler function $E_{a,b}(z)$ is an entire complex function in z of order $\rho = \frac{1}{a}$ for $\mathrm{Re}(a) > 0$ and any b. For real a and b, the Mittag-Leffler function $E_{a,b}(z)$ is real on the real axis. Moreover, for real positive a and b, the Mittag-Leffler function $E_{a,b}(z)$ attains its maximum on the real positive axis, because $\left|E_{a,b}\left(re^{i\theta}\right)\right| \leq \sum_{k=0}^\infty \frac{r^k\left|e^{ik\theta}\right|}{|\Gamma(b+ak)|} = \sum_{k=0}^\infty \frac{r^k}{\Gamma(b+ak)} = E_{a,b}(r)$.

The number $n(r)$ of zeros $z_1, z_2, \ldots$ of an entire function $f(z)$ of order ρ, for which $|z_n| \leq r$ is non-decreasing in r, is $n(r) = O(r^{\rho+\varepsilon})$. Roughly stated [47, p. 249], the higher the order ρ of an entire function, the more zeros it may have in a given region of the complex plane. Moreover, if the modulus of a zero z_n is $r_n = |z_n|$, then

$$\sum_{n=1}^{\infty} \frac{1}{(r_n)^{\alpha}} \text{ converges if } \alpha > \rho$$

and the lower bound of α is the exponent ρ_1 of convergence; thus $\rho_1 \leq \rho$. If $\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{p+1}$ converges for an integer p, then $p + 1 > \rho_1$ and the smallest integer p is called the genus of $f(z)$. In any case, $p \leq \rho_1 \leq \rho$. We may have $\rho_1 < \rho$, for example for $f(z) = e^z$, whose order is $\rho = 1$, but the exponent of convergence $\rho_1 = 0$, because e^z does not have zeros. Applied to the Mittag-Leffler function $E_{a,b}(z)$ of order $\rho = \frac{1}{a}$, the theory indicates that more zeros are expected for small a than for large a, which seems contradictory to the monotonicity of $E_{a,b}(z)$ for $0 < a < 1$ on the negative real axis in **art**. 39. The determination of the zeros of $E_{a,b}(z)$ is generally difficult [58], [17, Sec. 4.6], [40] and omitted here.

For some special values of the parameter a, the Taylor series (1) reduces to known functions, such as $E_{1,1}(z) = e^z$ and $E_{2,1}(z) = \cosh(\sqrt{z})$. From the incomplete Gamma function [1, 6.5.29], we have

$$\begin{aligned} E_{1,b}(z) &= e^z \gamma^*(b-1, z) = \frac{e^z}{\Gamma(b-1)} \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,(b-1+k)} \\ &= z^{1-b} e^z \left(1 - \frac{1}{\Gamma(b-1)} \int_z^{\infty} e^{-t} t^{b-2} dt\right) \end{aligned} \tag{3}$$

which can also be written in terms of Kummer's confluent hypergeometric function [1, 13.1.2]

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{z^k}{k!}$$

as

$$E_{1,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+k)} = \frac{M(1, b, z)}{\Gamma(b)}$$

Generalizations of (3) to $E_{\frac{1}{n},b}(z)$ for fractional $a = \frac{1}{n}$, where n is an integer, are derived in (23) in **art**. 7 below. Many analytic functions can be expressed in terms of the hypergeometric function, defined by Gauss's series [1, 15.1.1]

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k \qquad \text{convergent for } |z| < 1 \tag{4}$$

where the argument of the Gamma functions is of the form $\alpha k + \beta$ with $\alpha = 1$, in contrast to the Mittag-Leffler function in (1) where $\alpha = a$ is real positive. Just the fact that a is real and not an integer colors the theory of the Mittag-Leffler function $E_{a,b}(z)$ and causes its main challenges.

3 Deductions from the definition (1) of $E_{a,b}(z)$

1. *Special values of* a *and* b. If $b=0$, then, since $\lim_{z\to 0}\frac{1}{\Gamma(z)}=0$, we have

$$E_{a,0}(z)=\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(ak)}=\sum_{k=0}^{\infty}\frac{z^{k+1}}{\Gamma(ak+a)}$$

and, hence,

$$E_{a,0}(z)=zE_{a,a}(z) \tag{5}$$

If $a=0$, then

$$E_{0,b}(z)=\frac{1}{\Gamma(b)}\frac{1}{1-z}\qquad\text{for }|z|<1$$

From $E_{a,b}(z)=\frac{1}{\Gamma(b)}+\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(b+ak)}$, we observe that $\lim_{a\to\infty}E_{a,b}(z)=\frac{1}{\Gamma(b)}$.

2. After splitting odd and even indices in the k-sum of (1), we obtain

$$E_{a,b}(-z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{\Gamma(b+ak)}=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(b+2ak)}-\sum_{k=0}^{\infty}\frac{z^{2k+1}}{\Gamma(b+a+2ak)}$$

and

$$E_{a,b}(-z)=E_{2a,b}\left(z^2\right)-zE_{2a,b+a}\left(z^2\right) \tag{6}$$

Property (6) cannot be expressed for $E_a(-z)$ in terms of itself and motivates our viewpoint that the complex function theory of the Mittag-Leffler function should focus on $E_{a,b}(z)$, rather than on $E_a(z)$. The differentiation rule in **art**. 6 below is the more fundamental motivation.

Adding $E_{a,b}(z)=E_{2a,b}\left(z^2\right)+zE_{2a,b+a}\left(z^2\right)$ to (6) leads to

$$E_{2a,b}\left(z^2\right)=\frac{E_{a,b}(z)+E_{a,b}(-z)}{2} \tag{7}$$

and, similarly,

$$E_{2a,b+a}\left(z^2\right)=\frac{E_{a,b}(z)-E_{a,b}(-z)}{2z} \tag{8}$$

Examples From $E_{1,1}(z)=E_1(z)=e^z$, the relation (7) indicates that $E_2(z)=\cosh(\sqrt{z})$ and next

$$E_4(z)=\frac{1}{2}\left\{\cosh\left(z^{\frac{1}{4}}\right)+\cos\left(z^{\frac{1}{4}}\right)\right\}$$

The odd variant (8) gives $E_{2,2}\left(z^2\right)=\frac{\sinh z}{z}$.

3. *Cyclotomic property.* When introducing the identity $\sum_{r=0}^{m-1}e^{i\frac{2\pi kr}{m}}=\frac{1-e^{2\pi ki}}{1-e^{i\frac{2\pi k}{m}}}=m1_{m|k}$ into (1),

$$\sum_{r=0}^{m-1}E_{a,b}\left(e^{i\frac{2\pi r}{m}}z\right)=\sum_{k=0}^{\infty}\frac{\sum_{r=0}^{m-1}e^{i\frac{2\pi kr}{m}}z^k}{\Gamma(b+ak)}=m\sum_{k=0}^{\infty}\frac{1_{m|k}z^k}{\Gamma(b+ak)}=m\sum_{l=0}^{\infty}\frac{z^{ml}}{\Gamma(b+aml)}$$

we obtain

$$E_{am,b}(z^m)=\frac{1}{m}\sum_{r=0}^{m-1}E_{a,b}\left(ze^{i\frac{2\pi r}{m}}\right) \tag{9}$$

For $m = 2$ in (9),

$$2E_{2a,b}\left(z^{2}\right) = E_{a,b}\left(z\right) + E_{a,b}\left(-z\right)$$

we retrieve (7), because (9) essentially follows by multisectioning of a power series [42, Section 4.3] of which splitting in odd and even terms is the obvious case in $m = 2$ sections.

Example The case $a = b = 1$ in (9)

$$E_{m,1}\left(z\right) = E_{m}\left(z\right) = \frac{1}{m}\sum_{r=0}^{m-1} e^{z^{\frac{1}{m}}e^{i\frac{2\pi r}{m}}} \tag{10}$$

can be extended to certain integer values of b. Indeed, using

$$E_{1,n}\left(z\right) = \sum_{k=0}^{\infty}\frac{z^{k}}{\left(k+n-1\right)!} = \sum_{k=n-1}^{\infty}\frac{z^{k-(n-1)}}{k!} = z^{-(n-1)}\left(e^{z} - \sum_{j=0}^{n-2}\frac{z^{j}}{j!}\right)$$

in (9) yields

$$E_{m,n}\left(z\right) = \frac{1}{m}\sum_{r=0}^{m-1} E_{1,n}\left(z^{\frac{1}{m}}e^{i\frac{2\pi r}{m}}\right) = \frac{z^{-\frac{n-1}{m}}}{m}\sum_{r=0}^{m-1} e^{-i\frac{2\pi r}{m}(n-1)}\left(e^{z^{\frac{1}{m}}e^{i\frac{2\pi r}{m}}} - \sum_{j=0}^{n-2}\frac{z^{\frac{j}{m}}e^{i\frac{2\pi r}{m}j}}{j!}\right)$$

The last double sum

$$\sum_{r=0}^{m-1}\sum_{j=0}^{n-2} e^{-i\frac{2\pi r}{m}(n-1-j)}\frac{z^{\frac{j}{m}}}{j!} = \sum_{q=1}^{n-1}\frac{z^{\frac{n-1-q}{m}}}{(n-1-q)!}\sum_{r=0}^{m-1} e^{-i\frac{2\pi qr}{m}} = \sum_{q=1}^{n-1}\frac{z^{\frac{n-1-q}{m}} m 1_{m|q}}{(n-1-q)!}$$

vanishes for all integers $n \leq m$. Thus, for integer $0 \leq n \leq m$, we arrive at

$$E_{m,n}\left(z\right) = \frac{z^{-\frac{n-1}{m}}}{m}\sum_{r=0}^{m-1} e^{-i\frac{2\pi r}{m}(n-1)} e^{z^{\frac{1}{m}}e^{i\frac{2\pi r}{m}}} \tag{11}$$

4. *Mittag-Leffler function with $b = \beta + am$ where $m \in \mathbb{Z}$.* Another rewriting of the definition (1) of $E_{a,b}\left(z\right)$,

$$E_{a,b}\left(z\right) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma\left(b+ak\right)} = \sum_{k=1}^{\infty}\frac{z^{k-1}}{\Gamma\left(b-a+ak\right)} = \frac{1}{z}\left(\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma\left(b-a+ak\right)} - \frac{1}{\Gamma\left(b-a\right)}\right)$$

leads to "the shift down of b by a" formula

$$E_{a,b}\left(z\right) = \frac{1}{z}\left(E_{a,b-a}\left(z\right) - \frac{1}{\Gamma\left(b-a\right)}\right) \tag{12}$$

or, similarly after $b \rightarrow a + b$,

$$E_{a,b}\left(z\right) = \frac{1}{\Gamma\left(b\right)} + zE_{a,b+a}\left(z\right)$$

from which, for $b = 0$, we find again (5). If $b = \beta + ma$ in (12), then we obtain a recursion in m

$$E_{a,\beta+ma}\left(z\right) = \frac{1}{z}\left(E_{a,\beta+(m-1)a}\left(z\right) - \frac{1}{\Gamma\left(\beta+\left(m-1\right)a\right)}\right) \tag{13}$$

After iteration of (13), we find

$$z^m E_{a,\beta+ma}(z) = E_{a,\beta}(z) - \sum_{l=0}^{m-1} \frac{z^l}{\Gamma(\beta+la)} \tag{14}$$

The case for $m = 1$ in (14) is again (12). Relation (14) is directly retrieved from the definition (1) of $E_{a,b}(z)$,

$$E_{a,\beta+ma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+a(m+k))} = \sum_{k=m}^{\infty} \frac{z^{k-m}}{\Gamma(\beta+ak)} = z^{-m}\left(\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+ak)} - \sum_{k=0}^{m-1} \frac{z^k}{\Gamma(\beta+ak)}\right)$$

Similarly,

$$E_{a,\beta-ma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+a(k-m))} = \sum_{k=-m}^{\infty} \frac{z^{k+m}}{\Gamma(\beta+ak)} = z^m\left(\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+ak)} + \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\beta-ak)}\right)$$

and, hence,

$$z^{-m} E_{a,\beta-ma}(z) = E_{a,\beta}(z) + \sum_{k=1}^{m} \frac{\left(\frac{1}{z}\right)^k}{\Gamma(\beta-ak)} \tag{15}$$

Subtracting (15) from (14) yields[3]

$$z^m E_{a,\beta+ma}(z) - z^{-m} E_{a,\beta-ma}(z) = -\sum_{l=-m}^{m-1} \frac{z^l}{\Gamma(\beta+la)}$$

illustrating for real $a > 0$, $x \geq 0$ and positive $\beta > ma$ that $x^{2m} E_{a,\beta+ma}(x) < E_{a,\beta-ma}(x)$.

Example If $a = 1$ and $\beta = 1$ in (15), the $\frac{1}{\Gamma(\beta-ak)} = \frac{1}{\Gamma(1-k)} = 0$ for $k \geq 1$ and, with $E_{1,1}(z) = e^z$, we find (also from (3)), for integers $m \geq 0$, that

$$E_{1,1-m}(z) = z^m e^z \tag{16}$$

5. *Differentiation with respect to z.* The derivative of the definition (1) with respect to z is

$$\begin{aligned}
\frac{d}{dz} E_{a,b}(z) &= \sum_{k=1}^{\infty} \frac{k z^{k-1}}{\Gamma(b+ak)} = \sum_{k=0}^{\infty} \frac{(k+1) z^k}{\Gamma(b+a+ak)} \\
&= \frac{1}{a} \sum_{k=0}^{\infty} \frac{(ak+b+a-1) z^k}{\Gamma(b+a+ak)} - \frac{b-1}{a} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+a+ak)} \\
&= \frac{1}{a} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+a-1+ak)} - \frac{b-1}{a} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+a+ak)}
\end{aligned}$$

---

[3] which we can write as

$$\begin{aligned}
-\sum_{l=-m}^{m-1} \frac{z^l}{\Gamma(\beta+la)} &= \int_{-m}^{m} \frac{d}{du}\left\{z^u E_{a,\beta+ua}(z)\right\} du = \int_{-m}^{m} \sum_{k=0}^{\infty} \frac{d}{du} \frac{z^{k+u}}{\Gamma(\beta+a(u+k))} du \\
&= \log z \int_{-m}^{m} z^u E_{a,\beta+ua}(z)\, du - a \int_{-m}^{m} z^u \sum_{k=0}^{\infty} \frac{z^k \psi(\beta+a(u+k))}{\Gamma(\beta+a(u+k))} du
\end{aligned}$$

where $\psi(z) = \frac{d \log \Gamma(z)}{dz}$ is the digamma function [1, 6.3.1].

and, with the definition (1),

$$\frac{d}{dz}E_{a,b}(z)=\frac{1}{a}E_{a,b+a-1}(z)-\frac{b-1}{a}E_{a,b+a}(z)$$

Using (12), $E_{a,b+a}(z)=\frac{1}{z}\left(E_{a,b}(z)-\frac{1}{\Gamma(b)}\right)$, simplifies to

$$az\frac{d}{dz}E_{a,b}(z)=E_{a,b-1}(z)-(b-1)E_{a,b}(z) \tag{17}$$

The m-derivative

$$a\frac{d^m}{dz^m}E_{a,b}(z)=\frac{d^{m-1}}{dz^{m-1}}E_{a,b+a-1}(z)-(b-1)\frac{d^{m-1}}{dz^{m-1}}E_{a,b+a}(z)$$

has a recursive structure, when denoting $h_m(b)=\frac{d^m}{dz^m}E_{a,b}(z)$ and $h_0(b)=E_{a,b}(z)$,

$$ah_m(b)=h_{m-1}(b+a-1)-(b-1)h_{m-1}(b+a)$$

which can be iterated resulting in

$$a^m\frac{d^m}{dz^m}E_{a,b}(z)=\sum_{j=0}^{m}q_j(a,b,m)E_{a,b+ma-j}(z)$$

where the coefficients $q_m(a,b,m)=1$, $q_{m-1}(a,b,m)=-\left(mb+\frac{m(m-1)}{2}a-\frac{m(m+1)}{2}\right)$ and $q_0(a,b,m)=\prod_{k=0}^{m-1}(b-1+ka)$. In general, $q_j(a,b,m)$ are polynomials in a and b of order $m-j$. Unfortunately[4], it is not easy to write all coefficients $q_j(a,b,m)$ in closed form. With (14), we have

$$(az)^m\frac{d^m}{dz^m}E_{a,b}(z)=\sum_{j=0}^{m}q_j(a,b,m)\left(E_{a,b-j}(z)-\sum_{l=0}^{m-1}\frac{z^l}{\Gamma(b-j+la)}\right) \tag{18}$$

For $z=0$, we obtain from (18) with $\frac{d^m}{dz^m}E_{a,b}(z)\big|_{z=0}=\frac{1}{\Gamma(b+am)}$,

$$\frac{1}{\Gamma(b+am)}=\frac{1}{a^m}\sum_{j=0}^{m}\frac{q_j(a,b,m)}{\Gamma(b-j+ma)}$$

and, with $\frac{\Gamma(b+am)}{\Gamma(b-j+ma)}=\prod_{q=1}^{j}(b+am-q)$, the polynomial nature of $q_j(a,b,m)$ is apparent:

$$\sum_{j=1}^{m}\prod_{q=1}^{j}(b+am-q)\,q_j(a,b,m)=a^m-\prod_{k=0}^{m-1}(b-1+ka)$$

[4]For example,

$$a^2\frac{d^2E_{a,b}(z)}{dz^2}=E_{a,b+2a-2}(z)-(2b+a-3)E_{a,b+2a-1}(z)+(b-1)(b+a-1)E_{a,b+2a}(z)$$

$$\begin{aligned}a^3\frac{d^3E_{a,b}(z)}{dz^3}&=E_{a,b+3a-3}(z)-(3b+3a-6)E_{a,b+3a-2}(z)\\&+\{(2b+a-3)(b+2a-2)+(b-1)(b+a-1)\}E_{a,b+3a-1}(z)-(b-1)(b-1+a)(b-1+2a)E_{a,b+3a}(z)\end{aligned}$$

$$\begin{aligned}a^4\frac{d^4E_{a,b}(z)}{dz^4}&=E_{a,b+4a-4}(z)-(4b+6a-10)E_{a,b+4a-3}(z)\\&+\{(2b+a-3)(b+2a-2)+(b-1)(b+a-1)+3(b+a-2)(b+3a-3)\}E_{a,b+4a-2}(z)\\&-\{(b+3a-2)(2b+a-3)(b+2a-2)+(b-1)(b+a-1)(b+3a-2)+(b-1)(b-1+a)(b-1+2a)\}E_{a,b+4a-1}(z)\\&+(b-1)(b-1+a)(b-1+2a)(b-1+3a)E_{a,b+4a}(z)\end{aligned}$$

Art. 9 below will present a closed form for $\frac{d^m}{dz^m}E_{a,b}(z)$.

6. *Differentiation recursion.* Using the functional equation of the Gamma function, $\Gamma(z+1) = z\Gamma(z)$, we write

$$E_{a,b}(xz) = \sum_{k=0}^{\infty} \frac{x^k z^k}{\Gamma(b+ak)} = \sum_{k=0}^{\infty} \frac{x^k z^k}{(b-1+ak)\,\Gamma(b-1+ak)}$$

Thus,

$$z^{b-1}E_{a,b}(xz^a) = \sum_{k=0}^{\infty} \frac{x^k z^{b-1+ak}}{(b-1+ak)\,\Gamma(b-1+ak)}$$

Differentiation with respect to z gives us

$$\frac{d}{dz}\left\{z^{b-1}E_{a,b}(xz^a)\right\} = \sum_{k=0}^{\infty} \frac{x^k z^{b-2+ak}}{\Gamma(b-1+ak)} = z^{b-2}\sum_{k=0}^{\infty} \frac{(xz^a)^k}{\Gamma(b-1+ak)}$$

With the definition (1), we arrive for any x at the differentiation recursion in b,

$$\frac{d}{dz}\left\{z^{b-1}E_{a,b}(xz^a)\right\} = z^{b-2}E_{a,b-1}(xz^a) \tag{19}$$

Differentiating (19) again m-times and using the recursion (19) yields

$$\frac{d^m}{dz^m}\left\{z^{b-1}E_{a,b}(xz^a)\right\} = z^{b-1-m}E_{a,b-m}(xz^a) \tag{20}$$

7. *Fractional values of* a. Let $a = \frac{m}{n}$ where $m \leq n$ are integers, then the differentiation formula (20) becomes, for $x = 1$,

$$\begin{aligned}\frac{d^m}{dz^m}\left\{z^{b-1}E_{\frac{m}{n},b}\left(z^{\frac{m}{n}}\right)\right\} &= z^{b-1}\sum_{k=0}^{\infty} \frac{z^{\frac{m}{n}k-m}}{\Gamma\left(b-m+\frac{m}{n}k\right)} = z^{b-1}\sum_{k=0}^{\infty} \frac{z^{\frac{m}{n}(k-n)}}{\Gamma\left(b+\frac{m}{n}(k-n)\right)}\\ &= z^{b-1}\sum_{k=-n}^{\infty} \frac{z^{\frac{m}{n}k}}{\Gamma\left(b+\frac{m}{n}k\right)} = z^{b-1}\sum_{k=1}^{n} \frac{z^{-\frac{m}{n}k}}{\Gamma\left(b-\frac{m}{n}k\right)} + z^{b-1}\sum_{k=0}^{\infty} \frac{z^{\frac{m}{n}k}}{\Gamma\left(b+\frac{m}{n}k\right)}\end{aligned}$$

and we find that

$$\frac{d^m}{dz^m}\left\{z^{b-1}E_{\frac{m}{n},b}\left(z^{\frac{m}{n}}\right)\right\} = z^{b-1}\sum_{k=1}^{n} \frac{z^{-\frac{m}{n}k}}{\Gamma\left(b-\frac{m}{n}k\right)} + z^{b-1}E_{\frac{m}{n},b}\left(z^{\frac{m}{n}}\right) \tag{21}$$

For $n = 1$, (21) is

$$\frac{d^m}{dz^m}\left\{z^{b-1}E_{m,b}(z^m)\right\} = \frac{z^{b-1-m}}{\Gamma(b-m)} + z^{b-1}E_{m,b}(z^m)$$

In particular, when $b = 1$, then it holds for $m \geq 1$ that $\frac{d^m}{dz^m}E_m(z^m) = E_m(z^m)$, which illustrates that $y = E_m(z^m)$, explicitly given in (10), is a solution of the differential equation $\frac{d^m y}{dz^m} = y$.

For $m = 1$, (21) reduces to

$$\frac{d}{dz}\left\{z^{b-1}E_{\frac{1}{n},b}\left(z^{\frac{1}{n}}\right)\right\} = \sum_{k=1}^{n} \frac{z^{b-1-\frac{1}{n}k}}{\Gamma\left(b-\frac{1}{n}k\right)} + z^{b-1}E_{\frac{1}{n},b}\left(z^{\frac{1}{n}}\right)$$

from which

$$\frac{d}{dz}\left\{e^{-z}z^{b-1}E_{\frac{1}{n},b}\left(z^{\frac{1}{n}}\right)\right\} = e^{-z}\sum_{k=1}^{n} \frac{z^{b-1-\frac{1}{n}k}}{\Gamma\left(b-\frac{1}{n}k\right)}$$

Integrating both sides from 0 to z yields, for $b \geq 1$,

$$E_{\frac{1}{n},b}\left(z^{\frac{1}{n}}\right) = z^{1-b}e^{z}\left\{1_{\{b=1\}} + \sum_{k=1}^{n} \frac{1}{\Gamma\left(b-\frac{1}{n}k\right)} \int_0^z t^{b-1-\frac{1}{n}k}e^{-t}dt\right\} \tag{22}$$

where the indicator function 1_x equals one if the condition x is true, else $1_x = 0$. After letting $x = z^{\frac{1}{n}}$ in (22) and replacing $j = n-k$ in the summation, we obtain[5]

$$E_{\frac{1}{n},b}(x) = x^{(1-b)n}e^{x^n}\left\{1_{\{b=1\}} + \sum_{j=0}^{n-1} \frac{1}{\Gamma\left(b-1+\frac{j}{n}\right)} \int_0^{x^n} t^{\left(b-1+\frac{j}{n}\right)-1}e^{-t}dt\right\} \tag{23}$$

that reduces, for $n=1$, to (3) and for $b=1$, to Wiman's form in [57]

$$E_{\frac{1}{n}}(x) = e^{x^n}\left\{1 + \int_0^{x^n} e^{-t}\sum_{j=1}^{n-1} \frac{t^{\frac{j}{n}-1}}{\Gamma\left(\frac{j}{n}\right)}dt\right\}$$

For positive real x, it holds that $\int_0^{x^n} t^{\left(b-1+\frac{j}{n}\right)-1}e^{-t}dt < \int_0^{\infty} t^{\left(b-1+\frac{j}{n}\right)-1}e^{-t}dt = \Gamma\left(b-1+\frac{j}{n}\right)$ and (23) shows that $E_{\frac{1}{n},b}(x) < x^{(1-b)n}e^{x^n}\left\{1_{\{b=1\}} + n - 1_{\{b=1\}}\right\}$. Thus,

$$E_{\frac{1}{n},b}(x) < nx^{(1-b)n}e^{x^n}$$

illustrates for $a = \frac{1}{n}$ that the entire function $E_{\frac{1}{n},b}(z)$ has order $\rho = \frac{1}{a} = n$. This bound also reappears in **art**. 31. Moreover, it is interesting to compare (11) formally, written for an integer $b < m$,

$$E_{m,b}(z) = \frac{z^{\frac{1-b}{m}}}{m}\left\{e^{z^{\frac{1}{m}}} + \sum_{r=1}^{m-1} e^{-i\frac{2\pi r}{m}(b-1)}e^{z^{\frac{1}{m}}e^{i\frac{2\pi r}{m}}}\right\}$$

with Bieberbach's integral (141) for $E_{a,b}(z)$ and with the bound, where $\frac{1}{n}$ is replaced by m,

$$E_{m,b}(z) < \frac{z^{\frac{1-b}{m}}}{m}e^{z^{\frac{1}{m}}}$$

We return to the relation between $E_{a,b}(z)$ and $E_{\frac{1}{a},b}(z)$ later in **art**. 31.

Example If $n=1$ in (22), we retrieve (3) and when $n=2$ in (22), we find for $b=1$

$$E_{\frac{1}{2}}\left(z^{\frac{1}{2}}\right) = e^{z}\left\{1 + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^z t^{-\frac{1}{2}}e^{-t}dt\right\} = e^{z}\left\{1 + \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{z}} e^{-u^2}du\right\}$$

---

[5] Using the recursion $P(a,x) = P(a+1,x) + \frac{x^a}{\Gamma(a+1)}e^{-x}$ in [1, 6.5.21] for the incomplete Gamma function $P(a,x) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1}e^{-t}dt$ in (23), $E_{\frac{1}{n},b}(x) = x^{(1-b)n}e^{x^n}\left\{1_{\{b=1\}} + \sum_{j=0}^{n-1} P\left(b-1+\frac{j}{n}, x^n\right)\right\}$, shows that

$$E_{\frac{1}{n},b}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma\left(b+\frac{j}{n}\right)} = x^{(1-b)n}e^{x^n}\left\{1_{\{b=1\}} + \sum_{j=0}^{n-1} P\left(b+\frac{j}{n}, x^n\right)\right\} + \sum_{j=0}^{n-1} \frac{x^j}{\Gamma\left(b+\frac{j}{n}\right)}$$

which agrees with the Wiman-recursion (23) when $b \to b+1$,

$$\sum_{j=n}^{\infty} \frac{x^j}{\Gamma\left(b+\frac{j}{n}\right)} = x^n E_{\frac{1}{n},b+1}(x) = x^n x^{(1-(b+1))n}e^{x^n}\left\{1_{\{b=0\}} + \sum_{j=0}^{n-1} P\left(b+\frac{j}{n}, x^n\right)\right\}$$

so that, with the definition [1, 7.1.1] of the error function $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du$,

$$E_{\frac{1}{2}}(z)=e^{z^2}\left\{1+\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du\right\}=e^{z^2}\left\{1+\operatorname{erf}(z)\right\} \tag{24}$$

8. *Hadamard's series* $\sum_{k=0}^{\infty}\frac{z^k}{(k!)^a}$ *and* $E_a(z)$. Sharp bounds of the Gamma function for $ak+b>0$, that follow from (171) in Appendix A, are

$$\sqrt{2\pi}\,(ak+b)^{ak+b-\frac{1}{2}}\,e^{-(ak+b)}\leq\Gamma(ak+b)\leq\sqrt{2\pi}\,(ak+b)^{ak+b-\frac{1}{2}}\,e^{-(ak+b)}e^{\frac{1}{12(ak+b)}}$$

Now, $\sqrt{2\pi}\,(ak+b)^{ak+b-\frac{1}{2}}\,e^{-(ak+b)}=\sqrt{2\pi}a^{a\left(k+\frac{b}{a}\right)-\frac{1}{2}}\left(k+\frac{b}{a}\right)^{\frac{a-1}{2}}\left(\left(k+\frac{b}{a}\right)^{\left(k+\frac{b}{a}-\frac{1}{2}\right)}e^{-\left(k+\frac{b}{a}\right)}\right)^a$ and again using (171),

$$\sqrt{2\pi}\,(ak+b)^{ak+b-\frac{1}{2}}\,e^{-(ak+b)}\leq\sqrt{2\pi}a^{a\left(k+\frac{b}{a}\right)-\frac{1}{2}}\left(k+\frac{b}{a}\right)^{\frac{a-1}{2}}\left(\frac{\Gamma\left(k+\frac{b}{a}\right)}{\sqrt{2\pi}}\right)^a$$

and replacing the inequality by an order estimate, we have for large $ak+b$,

$$\Gamma(ak+b)=\frac{\left(\sqrt{2\pi}\right)^{1-a}a^{b-\frac{1}{2}}}{\sqrt{k+\frac{b}{a}}}\left(a^k\sqrt{k+\frac{b}{a}}\Gamma\left(k+\frac{b}{a}\right)\right)^a\left(1+O\left(\frac{1}{ak+b}\right)\right)$$

If $b=1$, then for $ak+1>0$, the above reduces to

$$\Gamma(ak+1)\approx\frac{\sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}}{\sqrt{k+\frac{1}{a}}}\left(a^k\sqrt{k+\frac{1}{a}}\Gamma\left(k+\frac{1}{a}\right)\right)^a<\frac{\sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}}{\sqrt{k}}\left(a^k\Gamma\left(k+\frac{1}{a}+1\right)\right)^a$$

For large a, we approximate as

$$\Gamma(ak+1)\approx\frac{\sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}}{\sqrt{k}}\left(a^k\Gamma(k+1)\right)^a \tag{25}$$

In **art**. 66, Gauss's multiplication formula is written as $\Gamma(nz+1)=(2\pi)^{-\frac{n-1}{2}}\,n^{nz+\frac{1}{2}}\prod_{j=1}^{n}\Gamma\left(z+\frac{j}{n}\right)$ and indicates, assuming that $n=a$ is an integer, that

$$\Gamma(ak+1)=\sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}a^{ak}\prod_{j=1}^{a}\Gamma\left(k+\frac{j}{a}\right)$$

from which, with $\Gamma(k) < \Gamma\left(k+\frac{j}{a}\right) < \Gamma(k+1)$, it holds[6] for $k \geq 2$,

$$\sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}\left(a^k\Gamma(k)\right)^a < \Gamma(ak+1) < \sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}\left(a^k\Gamma(k+1)\right)^a$$

After introducing (25) in the definition (1) of the Mittag-Leffler function for real, positive x, we approximately obtain

$$E_a(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(ak+1)} \approx \frac{\left(\sqrt{2\pi}\right)^{a-1}}{\sqrt{a}}\sum_{k=0}^{\infty}\frac{\sqrt{k}\left(\frac{x}{a}\right)^k}{(k!)^a}$$

while Gauss's multiplication formula produces the bounds

$$\frac{\left(\sqrt{2\pi}\right)^{a-1}}{\sqrt{a}}\frac{x}{a}\sum_{k=1}^{\infty}\frac{\left(\frac{x}{a}\right)^k}{(k!)^a} > E_a(x) - 1 - \frac{x}{\Gamma(a+1)} > \frac{\left(\sqrt{2\pi}\right)^{a-1}}{\sqrt{a}}\sum_{k=2}^{\infty}\frac{\left(\frac{x}{a}\right)^k}{(k!)^a} \tag{26}$$

About 10 years before Mittag-Leffler has introduced his function $E_a(x)$, Hadamard [22, p. 180] suggests in his study of entire functions that $E_a(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(ak+1)} \sim \frac{\left(\sqrt{2\pi}\right)^{a-1}}{\sqrt{a}}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{a}\right)^k}{(k!)^a}$ for large x. Hadamard derives an exact a-fold integral for the last series, from which he deduces that $\sum_{k=0}^{\infty}\frac{x^k}{(k!)^a} < e^{ax^{\frac{1}{a}}}$. **Art**. 7 shows that $E_a(x) < \frac{1}{a}e^{x^{\frac{1}{a}}}$. Combined with (26) leads to

$$\sum_{k=0}^{\infty}\frac{x^k}{(k!)^a} < \frac{\left(\sqrt{2\pi}\right)^{1-a}}{\sqrt{a}}e^{a^{\frac{1}{a}}x^{\frac{1}{a}}}$$

which is considerably sharper for $a > 1$ than Hadamard's [22, p. 180] bound.

We also give Hadamard's [22, p. 180] nice argument, starting from

$$e^{x^{\frac{1}{a}}} = \sum_{k=0}^{\infty}\frac{x^{\frac{k}{a}}}{\Gamma(k+1)}$$

and letting $m' = \frac{k}{a}$, which runs over fractions for $a > 1$, so that $e^{x^{\frac{1}{a}}} = \sum_{m'=0}^{\infty}\frac{x^{m'}}{\Gamma(m'a+1)}$. Comparing terms with $E_a(x) = \sum_{m=0}^{\infty}\frac{x^m}{\Gamma(am+1)}$, then shows that $E_a(x) < e^{x^{\frac{1}{a}}}$. For $a < 1$, Hadamard states that $E_a(x) < \left[\frac{1}{a}\right]x^{\frac{1}{a}}e^{x^{\frac{1}{a}}}$. However, the bounds presented here based on the theory of the Mittag-Leffler function are sharper than Hadamard's estimates.

---

[6] Now, $(\Gamma(k))^{\frac{a}{2}} < \prod_{j=1}^{\frac{a}{2}}\Gamma\left(k+\frac{j}{a}\right) < \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{\frac{a}{2}}$ and $\left(\Gamma\left(k+\frac{1}{2}\right)\right)^{\frac{a}{2}} < \prod_{j=\frac{a}{2}}^{a}\Gamma\left(k+\frac{j}{a}\right) < (\Gamma(k+1))^{\frac{a}{2}}$ and, hence

$$(\Gamma(k))^{\frac{a}{2}}\left(\Gamma\left(k+\frac{1}{2}\right)\right)^{\frac{a}{2}} < \prod_{j=1}^{\frac{a}{2}}\Gamma\left(k+\frac{j}{a}\right)\prod_{j=\frac{a}{2}}^{a}\Gamma\left(k+\frac{j}{a}\right) < \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{\frac{a}{2}}(\Gamma(k+1))^{\frac{a}{2}}$$

The duplication formula of the Gamma function, $\Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ indicates that $\Gamma\left(k+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2k-1}}\frac{(2k-1)!}{(k-1)!}$ so that

$$\sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}a^{ak}\left(\Gamma(k)\Gamma\left(k+\frac{1}{2}\right)\right)^{\frac{a}{2}} < \Gamma(ak+1) < \sqrt{a}\left(\sqrt{2\pi}\right)^{1-a}a^{ak}\left(\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)\right)^{\frac{a}{2}}$$

and a sharper upper and lower bound is

$$\frac{\sqrt{2\pi a}}{2^{ka}\pi^{\frac{a}{4}}}a^{ak}\left((2k-1)!\right)^{\frac{a}{2}} < \Gamma(ak+1) < \frac{\sqrt{2\pi a}}{2^{ka}\pi^{\frac{a}{4}}}k^{\frac{a}{2}}a^{ak}\left((2k-1)!\right)^{\frac{a}{2}}$$

9. *Taylor expansion around* z_0. Since the Mittag-Leffler function $E_{a,b}(z)$ is an entire function, any Taylor expansion around an arbitrary (finite) point z_0 has infinite radius of convergence,

$$E_{a,b}(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \left. \frac{d^m E_{a,b}(z)}{dx^m} \right|_{x=z_0} (z - z_0)^m$$

With the m-th derivative (18), we find

$$E_{a,b}(z) = \sum_{m=0}^{\infty} \left\{ \sum_{j=0}^{m} q_j(a,b,m) \left(E_{a,b-j}(z_0) - \sum_{l=0}^{m-1} \frac{z_0^l}{\Gamma(b-j+la)} \right) \right\} \frac{\left(\frac{z-z_0}{az_0}\right)^m}{m!}$$

but, unfortunately (see **art**. 5), the coefficients $q_j(a,b,m)$ are not available in closed form. However, the closed form (20), which is a rather fundamental property of $E_{a,b}(z)$, opens a new avenue. The Taylor expansion around z_0 of

$$z^{b-1} E_{a,b}(z^a) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dx^m} \left. \left\{ x^{b-1} E_{a,b}(x^a) \right\} \right|_{x=z_0} (z - z_0)^m$$

becomes with the differentiation recursion (20)

$$z^{b-1} E_{a,b}(z^a) = z_0^{b-1} \sum_{m=0}^{\infty} \frac{E_{a,b-m}(z_0^a)}{m!} z_0^{-m} (z - z_0)^m$$

The function $z^{b-1}E_{a,b}(z^a)$ has a branch cut at the negative real axis for real a and b, implying that the radius R of convergence equals $|z_0|$. Using the defining relation of the radius of convergence in Section 2,

$$\frac{1}{R} = \lim_{m\to\infty} \left| \frac{z_0^{-m-1} E_{a,b-m-1}(z_0^a)\, m!}{z_0^{-m} E_{a,b-m}(z_0^a)\,(m+1)!} \right| = z_0^{-1} \lim_{m\to\infty} \left| \frac{E_{a,b-m-1}(z_0^a)}{E_{a,b-m}(z_0^a)\,(m+1)} \right|$$

then indicates for any finite $z \neq 0$, a and b that

$$\left| \frac{E_{a,b-m-1}(z)}{E_{a,b-m}(z)} \right| \sim (m+1) \text{ if } m \to \infty \tag{27}$$

Introducing (33) for $z \neq 0$ that

$$1 = \lim_{m\to\infty} \left| \frac{E_{a,b-m-1}(z)}{E_{a,b-m}(z)\,(m+1)} \right| = \lim_{m\to\infty} \left| \frac{az}{m} \frac{d \log E_{a,b-m}(z)}{dz} - \left(1 - \frac{b}{m}\right) \right|$$

shows that

$$\lim_{m\to\infty} \left| \frac{1}{m} \frac{d \log E_{a,b-m}(z)}{dz} \right| = 0$$

Thus, for large m, the logarithmic derivative for $z \neq 0$ and finite a and b is of order $\frac{d \log E_{a,b-m}(z)}{dz} = O\left(m^{1-\varepsilon}\right)$ for any positive $\varepsilon > 0$.

We proceed by removing real powers of $z^\beta = e^{\beta \log z}$ that destroy analyticity in the complex plane and introduce formally the Taylor series $\left(\frac{z}{z_0}\right)^{1-b} = \left(\frac{z}{z_0} - 1 + 1\right)^{1-b} = \sum_{k=0}^{\infty} \binom{1-b}{k} \left(\frac{z}{z_0} - 1\right)^k$, valid for any b and $|z| < |z_0|$, in

$$E_{a,b}(z^a) = \left(\frac{z}{z_0}\right)^{1-b} \sum_{m=0}^{\infty} \frac{E_{a,b-m}(z_0^a)}{m!} \left(\frac{z}{z_0} - 1\right)^m$$

yielding, after executing the Cauchy product,

$$E_{a,b}\left(z^{a}\right)=\sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}\binom{1-b}{m-k}\frac{E_{a,b-k}\left(z_0^{a}\right)}{k!}\right\}\left(\frac{z}{z_0}-1\right)^{m}\qquad\text{for }\;|z|<|z_0|$$

Next, letting $y=z^{a}$ and $y_0=z_0^{a}$, we first expand in a Taylor series around y_0

$$\left(\left(\frac{y}{y_0}\right)^{\frac{1}{a}}-1\right)^{m}=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^{k}}{dz^{k}}\left(\left(\frac{y}{y_0}\right)^{\frac{1}{a}}-1\right)^{m}\Bigg|_{z=y_0}(y-y_0)^{k}$$

as

$$\begin{aligned}\left(\left(\frac{y}{y_0}\right)^{\frac{1}{a}}-1\right)^{m}&=\sum_{q=0}^{m}\binom{m}{q}\left(\frac{y}{y_0}\right)^{\frac{q}{a}}(-1)^{m-q}=\sum_{q=0}^{m}\binom{m}{q}\left(\frac{y-y_0}{y_0}+1\right)^{\frac{q}{a}}(-1)^{m-q}\\&=\sum_{j=0}^{\infty}\left\{\sum_{q=0}^{m}\binom{m}{q}\binom{\frac{q}{a}}{j}(-1)^{m-q}\right\}\left(\frac{y-y_0}{y_0}\right)^{j}\end{aligned}$$

Substitution and reversing the m- and k-sum yields

$$\begin{aligned}E_{a,b}(y)&=\sum_{j=0}^{\infty}\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{1-b}{m-k}\frac{E_{a,b-k}(y_0)}{k!}\left\{\sum_{q=0}^{m}\binom{m}{q}\binom{\frac{q}{a}}{j}(-1)^{m-q}\right\}\right]\left(\frac{y-y_0}{y_0}\right)^{j}\\&=\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\frac{E_{a,b-k}(y_0)}{k!}\sum_{m=k}^{\infty}\binom{1-b}{m-k}\sum_{q=0}^{m}\binom{m}{q}\binom{\frac{q}{a}}{j}(-1)^{m-q}\right]\left(\frac{y-y_0}{y_0}\right)^{j}\end{aligned}$$

The characteristic coefficients [49, Appendix] of a complex function $f(z)$ with Taylor series $f(z)=\sum_{k=0}^{\infty}f_k(z_0)(z-z_0)^{k}$, defined by $s[k,m]|_f(z_0)=\frac{1}{m!}\frac{d^m}{dz^m}\left(f(z)-f(z_0)^{k}\right)\Big|_{z=z_0}$, possesses a general form

$$s[k,m]|_f(z_0)=\sum_{\sum_{i=1}^{k}j_i=m;j_i>0}\prod_{i=1}^{k}f_{j_i}(z_0)\tag{28}$$

and obeys $s[k,m]|_f(z_0)=0$ if $k<0$ and $k>m$. Also, $s[1,m]|_f(z_0)=f_m(z_0)$, while (28) indicates that $s[m,m]|_f(z_0)=f_1^{m}(z_0)$. We can show [48] that

$$s[k,m]|_{(1+z)^{\alpha}}=\sum_{j=1}^{k}(-1)^{j+k}\binom{k}{j}\binom{\alpha j}{m}=\frac{k!}{m!}\sum_{j=k}^{m}S_m^{(j)}\mathcal{S}_j^{(k)}\,\alpha^{j}\tag{29}$$

where $S_m^{(k)}$ and $\mathcal{S}_m^{(k)}$ are the Stirling numbers of the first and second kind [1, Sec. 24.1.3 and 24.1.4], respectively. We apply these properties to the Taylor series

$$\begin{aligned}E_{a,b}(y)&=\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\frac{E_{a,b-k}(y_0)}{k!}\sum_{m=k}^{\infty}\binom{1-b}{m-k}\,s[m,j]|_{(1+z)^{\frac{1}{a}}}\right]\left(\frac{y-y_0}{y_0}\right)^{j}\\&=\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\frac{E_{a,b-k}(y_0)}{k!}\sum_{m=k}^{j}\binom{1-b}{m-k}\,s[m,j]|_{(1+z)^{\frac{1}{a}}}\right]\left(\frac{y-y_0}{y_0}\right)^{j}\end{aligned}$$

Finally, we arrive with (29) at the Taylor series of $E_{a,b}(y)$ around y_0,

$$E_{a,b}(y)=\sum_{j=0}^{\infty}\left[\frac{y_0^{-j}}{j!}\sum_{k=0}^{j}E_{a,b-k}(y_0)\sum_{m=k}^{j}\binom{m}{k}\frac{\Gamma(2-b)}{\Gamma(2-b-m+k)}\sum_{q=m}^{j}S_j^{(q)}\mathcal{S}_q^{(m)}\frac{1}{a^q}\right](y-y_0)^j \tag{30}$$

from which the closed form of the j-th derivative of the Mittag-Leffler function, evaluated at $y_0=w$, follows as

$$\left.\frac{d^jE_{a,b}(z)}{dz^j}\right|_{z=w}=w^{-j}\sum_{k=0}^{j}E_{a,b-k}(w)\sum_{m=k}^{j}\binom{m}{k}\frac{\Gamma(2-b)}{\Gamma(2-b-m+k)}\sum_{q=m}^{j}S_j^{(q)}\mathcal{S}_q^{(m)}\frac{1}{a^q} \tag{31}$$

Since the Mittag-Leffler function $E_{a,b}(z)$ is an entire function, the j-th Taylor coefficient around w decreases faster than any power of $(z-w)^j$ for large j and any z, we deduce that $\left.\frac{d^jE_{a,b}(z)}{dz^j}\right|_{z=w}=o\left(j!w^j\right)$. Thus, the j-th Taylor coefficient $\frac{1}{j!}\left.\frac{d^jE_{a,b}(z)}{dz^j}\right|_{z=w}$ increases at most as a polynomial in w of order j. In contrast to the relatively simple Taylor series of $E_{a,b}(z)$ in (1) around the origin, the general form (31) emphasizes the complicated nature of the Mittag-Leffler function $E_{a,b}(z)$ elsewhere in the complex plane.

As a check for $a=b=1$, the orthogonality condition of the Stirling numbers

$$\sum_{k=n}^{m}S_k^{(n)}\mathcal{S}_m^{(k)}=\delta_{nm} \tag{32}$$

shows that (30) simplifies to

$$E_{a,b}(y)=\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}E_{1,1-k}(y_0)\sum_{m=k}^{j}\binom{m}{k}\frac{\Gamma(2-b)}{\Gamma(2-b-m+k)}\delta_{jm}\right]\frac{\left(\frac{y-y_0}{y_0}\right)^j}{j!}$$

With (16), we find

$$E_{1,1}(y)=e^{y_0}\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}y_0^{k-j}\frac{\binom{j}{k}}{\Gamma(1-j+k)}\right]\frac{(y-y_0)^j}{j!}=e^{y_0}\sum_{j=0}^{\infty}\frac{(y-y_0)^j}{j!}=e^{y_0}e^{y-y_0}=e^y$$

4 Logarithm of the Mittag-Leffler function

10. *Logarithmic derivative.* The logarithmic derivative $\frac{d}{dz}\log E_{a,b}(z)=\frac{\frac{d}{dz}E_{a,b}(z)}{E_{a,b}(z)}$ follows directly from (17) as

$$\frac{d\log E_{a,b}(z)}{dz}=\frac{1}{az}\left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)}-(b-1)\right) \tag{33}$$

Similarly, invoking (19), we find the companion logarithmic derivative

$$\frac{d}{dz}\log\left(z^{b-1}E_{a,b}(xz^a)\right)=\frac{\frac{d}{dz}\left\{z^{b-1}E_{a,b}(xz^a)\right\}}{z^{b-1}E_{a,b}(xz^a)}=\frac{1}{z}\frac{E_{a,b-1}(xz^a)}{E_{a,b}(xz^a)} \tag{34}$$

Since $b-1+ak>b-1$ for $k\geq 1$ because $a>0$, we have for positive real z and $b>1$,

$$E_{a,b}(z)=\sum_{k=0}^{\infty}\frac{z^k}{(b-1+ak)\Gamma(b-1+ak)}<\frac{1}{b-1}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(b-1+ak)}=\frac{E_{a,b-1}(z)}{b-1}$$

and that $(b-1) < \frac{E_{a,b-1}(z)}{E_{a,b}(z)}$. The logarithmic derivative (33) becomes $\frac{d\log E_{a,b}(z)}{dz} > 0$, illustrating that $\log E_{a,b}(z)$ is increasing for real $z \geq 0$ and $b > 1$. More precisely, with $b-1+ak > b-1+a$ for $k \geq 1$, we have for real positive z and for $b > 1-a$, because then $\Gamma(b-1+ak) > 0$ for $k \geq 1$,

$$\begin{aligned} E_{a,b}(z) &= \frac{1}{\Gamma(b)} + \sum_{k=1}^{\infty} \frac{z^k}{(b-1+ak)\,\Gamma(b-1+ak)} \\ &< \frac{1}{\Gamma(b)} + \frac{1}{b-1+a}\left(\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b-1+ak)} - \frac{1}{\Gamma(b-1)}\right) \end{aligned}$$

and

$$E_{a,b}(z) < \frac{1}{(b-1+a)}\left(\frac{a}{\Gamma(b)} + E_{a,b-1}(z)\right)$$

Thus, for real positive z and for $b > 1-a$, the inequality is equivalent to

$$a\left(1 - \frac{1}{\Gamma(b)\,E_{a,b}(z)}\right) < \frac{E_{a,b-1}(z)}{E_{a,b}(z)} - (b-1)$$

and the logarithmic derivative (33) is lower bounded by

$$\frac{d\log E_{a,b}(z)}{dz} > \frac{1}{z}\left(1 - \frac{1}{\Gamma(b)\,E_{a,b}(z)}\right)$$

Example If $b=1$ and $a=1$, then $E_{1,1}(z) = e^z$ and the above inequality for real positive z results in the well-known bound (see e.g. [50, p. 103]) that $e^{-z} > 1-z$, which holds for all real z.

11. *Second-order logarithmic derivative.* Similarly as in **art**. 10, we can directly differentiate (33) again,

$$\frac{d^2 \log E_{a,b}(z)}{dz^2} = -\frac{1}{az^2}\left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)} - (b-1)\right) + \frac{1}{az}\frac{d}{dz}\left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)}\right)$$

The derivative at the right-hand side is computed by using (17),

$$\frac{d}{dz}\left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)}\right) = \frac{1}{az}\left\{\frac{E_{a,b-2}(z)}{E_{a,b}(z)} + \frac{E_{a,b-1}(z)}{E_{a,b}(z)} - \left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)}\right)^2\right\} \tag{35}$$

After substitution of the latter into the former, we arrive at the second-order logarithmic derivative

$$\frac{d^2 \log E_{a,b}(z)}{dz^2} = \frac{1}{(az)^2}\left\{a(b-1) + \frac{E_{a,b-2}(z)}{E_{a,b}(z)} - (a-1)\frac{E_{a,b-1}(z)}{E_{a,b}(z)} - \left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)}\right)^2\right\} \tag{36}$$

As already observed in **art**. 5, higher-order derivatives will become less wieldy, which suggests us to consider the companion differential rule (19) in **art**. 14.

12. *The Taylor series of* $\log E_{a,b}(z)$ *around* $z=0$. From the power series definition (1) of $E_{a,b}(z)$, we have

$$\begin{aligned} \lim_{z\to 0} \frac{d\log E_{a,b}(z)}{dz} &= \lim_{z\to 0} \frac{1}{\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)}} \lim_{z\to 0} \frac{1}{az}\left(\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b-1+ak)} - \sum_{k=0}^{\infty} \frac{(b-1)\,z^k}{\Gamma(b+ak)}\right) \\ &= \Gamma(b) \lim_{z\to 0} \sum_{k=1}^{\infty} \frac{k z^{k-1}}{\Gamma(b+ak)} \end{aligned}$$

and find

$$\lim_{z\to 0}\frac{d\log E_{a,b}(z)}{dz}=\frac{\Gamma(b)}{\Gamma(b+a)}$$

Proceeding in this way to higher-order derivatives becomes cumbersome. The general theory of characteristic coefficients [49, Appendix] provides us with

$$\log f(z)=\log f_0(z_0)+\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\frac{(-1)^{k-1}}{k\,f_0^k(z_0)}\,s[k,m](z_0)\right)(z-z_0)^m \tag{37}$$

Confining to $z_0=0$, the characteristic coefficients (28) of the Mittag-Leffler function $E_{a,b}(z)$ are

$$s[k,m]|_{E_{a,b}(z)}=\sum_{\sum_{i=1}^{k}j_i=m;j_i>0}\prod_{i=1}^{k}\frac{1}{\Gamma(b+j_ia)}$$

and the Taylor series of $\log E_{a,b}(z)$ around $z_0=0$ follows as

$$\log E_{a,b}(z)=\sum_{m=0}^{\infty}c_m\,z^m$$

where we define $c_0=\log(-\Gamma(b))$ and the coefficients c_m for $m>0$

$$\begin{aligned}c_m&=-\sum_{k=1}^{m}\frac{(-\Gamma(b))^k}{k}\;s[k,m]|_{E_{a,b}(z)}\\&=\frac{\Gamma(b)}{\Gamma(b+ma)}+\sum_{k=2}^{m-1}\frac{(-1)^{k-1}}{k}\sum_{\sum_{i=1}^{k}j_i=m;j_i>0}\prod_{i=1}^{k}\frac{\Gamma(b)}{\Gamma(b+j_ia)}+\frac{(-1)^{m-1}}{m}\left(\frac{\Gamma(b)}{\Gamma(b+a)}\right)^m\end{aligned}$$

The list of the coefficients c_m for $m=1$ up to $m=5$ is

$$\begin{aligned}c_1&=\frac{\Gamma(b)}{\Gamma(b+a)}\\c_2&=\frac{\Gamma(b)}{\Gamma(b+2a)}-\frac{1}{2}\left(\frac{\Gamma(b)}{\Gamma(b+a)}\right)^2\\c_3&=\frac{\Gamma(b)}{\Gamma(b+3a)}+\frac{\Gamma^2(b)}{\Gamma(b+2a)\,\Gamma(b+a)}+\frac{1}{3}\left(\frac{\Gamma(b)}{\Gamma(b+a)}\right)^3\\c_4&=\frac{\Gamma(b)}{\Gamma(b+4a)}-\frac{\Gamma^2(b)}{\Gamma(b+3a)\,\Gamma(b+a)}-\frac{1}{2}\left(\frac{\Gamma(b)}{\Gamma(b+2a)}\right)^2+\frac{\Gamma^3(b)}{\Gamma(b+2a)\,\Gamma^2(b+a)}-\frac{1}{4}\left(\frac{\Gamma(b)}{\Gamma(b+a)}\right)^4\\c_5&=\frac{\Gamma(b)}{\Gamma(b+5a)}-\frac{\Gamma^2(b)}{\Gamma(b+4a)\,\Gamma(b+a)}-\frac{\Gamma^2(b)}{\Gamma(b+3a)\,\Gamma(b+2a)}+\frac{\Gamma^3(b)}{\Gamma(b+3a)\,\Gamma^2(b+a)}+\frac{\Gamma^3(b)}{\Gamma(b+a)\,\Gamma^2(b+2a)}\\&\quad-\frac{\Gamma^4(b)}{\Gamma^3(b+a)\,\Gamma(b+2a)}+\frac{1}{5}\left(\frac{\Gamma(b)}{\Gamma(b+a)}\right)^5\end{aligned}$$

Unfortunately, it seems difficult to further sum the terms in c_m. The closest zero to $z_0=0$ lies at a distance from z_0 equal to the radius R of convergence of $\log E_{a,b}(z)$, which is

$$\frac{1}{R}=\lim_{m\to\infty}\left|\frac{c_{m+1}}{c_m}\right|=\lim_{m\to\infty}\frac{\left|\sum_{k=1}^{m+1}\frac{(-\Gamma(b))^k}{k}\;s[k,m+1]|_{E_{a,b}(z)}\right|}{\left|\sum_{k=1}^{m}\frac{(-\Gamma(b))^k}{k}\;s[k,m]|_{E_{a,b}(z)}\right|}$$

The generalization towards the closest zero to z_0 requires the Taylor coefficients of (30).

13. *An expansion of* $\log E_{a,b}(z)$ *in powers of* $\log z$. Integration of (33) leads to

$$\log \frac{E_{a,b}(z)}{E_{a,b}(z_0)} = \frac{(1-b)}{a} \log \frac{z}{z_0} + \frac{1}{a} \int_{z_0}^{z} \frac{E_{a,b-1}(w)}{E_{a,b}(w)} \frac{dw}{w}$$

Using (35), partial integration shows that

$$\int_{z_0}^{z} \frac{E_{a,b-1}(w)}{E_{a,b}(w)} \frac{dw}{w} = \frac{E_{a,b-1}(w)}{E_{a,b}(w)} \log w \Big|_{z_0}^{z} - \frac{1}{a} \int_{z_0}^{z} \frac{\log w}{w} \left\{ \frac{E_{a,b-2}(w)}{E_{a,b}(w)} + \frac{E_{a,b-1}(w)}{E_{a,b}(w)} - \left(\frac{E_{a,b-1}(w)}{E_{a,b}(w)} \right)^2 \right\} dw$$

and

$$\begin{aligned} \log \frac{E_{a,b}(z)}{E_{a,b}(z_0)} &= \frac{(1-b)}{a} \log \frac{z}{z_0} + \frac{1}{a} \left(\frac{E_{a,b-1}(z)}{E_{a,b}(z)} \log z - \frac{E_{a,b-1}(z_0)}{E_{a,b}(z_0)} \log z_0 \right) \\ &\quad - \frac{1}{a^2} \int_{z_0}^{z} \frac{\log w}{w} \left\{ \frac{E_{a,b-2}(w)}{E_{a,b}(w)} + \frac{E_{a,b-1}(w)}{E_{a,b}(w)} - \left(\frac{E_{a,b-1}(w)}{E_{a,b}(w)} \right)^2 \right\} dw \end{aligned}$$

Since $\frac{\log w}{w} dw = \frac{1}{2} d\left(\log^2 w\right)$, again partial integration of the right-hand side integral R leads to

$$\begin{aligned} R &= \int_{z_0}^{z} \frac{\log w}{w} \left\{ \frac{E_{a,b-2}(w)}{E_{a,b}(w)} + \frac{E_{a,b-1}(w)}{E_{a,b}(w)} - \left(\frac{E_{a,b-1}(w)}{E_{a,b}(w)} \right)^2 \right\} dw \\ &= \left\{ \frac{E_{a,b-2}(w)}{E_{a,b}(w)} + \frac{E_{a,b-1}(w)}{E_{a,b}(w)} - \left(\frac{E_{a,b-1}(w)}{E_{a,b}(w)} \right)^2 \right\} \frac{\log^2 w}{2} \Big|_{z_0}^{z} \\ &\quad - \int_{z_0}^{z} \frac{\log^2 w}{2} \frac{d}{dw} \left(\left\{ \frac{E_{a,b-2}(w)}{E_{a,b}(w)} + \frac{E_{a,b-1}(w)}{E_{a,b}(w)} - \left(\frac{E_{a,b-1}(w)}{E_{a,b}(w)} \right)^2 \right\} \right) dw \end{aligned}$$

Each derivative of a fraction of Mittag-Leffler functions can be computed with (35), which indicates that again a factor $\frac{1}{aw}$ will appear so that $\frac{\log^2 w}{2w} dw = \frac{1}{3!} d\left(\log^3 w\right)$ which enables a next integration. Continuing partial integration and choosing $z_0 = 1$ to simplify the sums will lead to an expansion of the form

$$\log \frac{E_{a,b}(z)}{E_{a,b}(1)} = \frac{(1-b)}{a} \log z + \sum_{k=1}^{K} \frac{\log^k z}{k!} \digamma_k(z) - \int_{z_0}^{z} \frac{\log^k w}{k!} \frac{d}{dw} \digamma_k(w)\, dw$$

where $K > 1$ is an integer and $\digamma_k(z)$ consists of sum of fractions of Mittag-Leffler functions, whose explicit evaluation is possible, but tedious and omitted. We have computed above,

$$\begin{aligned} \digamma_1(z) &= \frac{1}{a} \frac{E_{a,b-1}(z)}{E_{a,b}(z)} \\ \digamma_2(z) &= -\frac{1}{a^2} \left\{ \frac{E_{a,b-2}(w)}{E_{a,b}(w)} + \frac{E_{a,b-1}(w)}{E_{a,b}(w)} - \left(\frac{E_{a,b-1}(w)}{E_{a,b}(w)} \right)^2 \right\} \end{aligned}$$

14. *Higher-order logarithmic derivatives.* The second-order logarithmic derivative of the companion in (34) is

$$\frac{d^2}{dz^2} \log \left(z^{b-1} E_{a,b}(xz^a) \right) = \frac{d}{dz} \left(\frac{1}{z} \frac{E_{a,b-1}(xz^a)}{E_{a,b}(xz^a)} \right) = \frac{d}{dz} \left(\frac{z^{b-2} E_{a,b-1}(xz^a)}{z^{b-1} E_{a,b}(xz^a)} \right)$$

Invoking (19) yields

$$\frac{d}{dz}\left(\frac{z^{b-2}E_{a,b-1}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)=\frac{z^{b-3}E_{a,b-2}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}-\left(\frac{z^{b-2}E_{a,b-1}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)^{2}$$

and

$$\frac{d^{2}}{dz^{2}}\log\left(z^{b-1}E_{a,b}\left(xz^{a}\right)\right)=\frac{1}{z^{2}}\left(\frac{E_{a,b-2}\left(xz^{a}\right)}{E_{a,b}\left(xz^{a}\right)}-\left(\frac{E_{a,b-1}\left(xz^{a}\right)}{E_{a,b}\left(xz^{a}\right)}\right)^{2}\right) \tag{38}$$

whose structure is more pleasing than that of (36).

A next differentiation yields

$$\frac{d^{3}}{dz^{3}}\log\left(z^{b-1}E_{a,b}\left(xz^{a}\right)\right)=\frac{d}{dz}\left(\frac{z^{b-3}E_{a,b-2}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}-\left(\frac{z^{b-2}E_{a,b-1}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)^{2}\right)$$

The first derivative equals

$$\frac{d}{dz}\left(\frac{z^{b-3}E_{a,b-2}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)=\frac{1}{z^{3}}\left(\frac{E_{a,b-3}\left(xz^{a}\right)}{E_{a,b}\left(xz^{a}\right)}-\frac{E_{a,b-2}\left(xz^{a}\right)E_{a,b-1}\left(xz^{a}\right)}{\left(E_{a,b}\left(xz^{a}\right)\right)^{2}}\right)$$

The second is

$$\frac{d}{dz}\left(\frac{z^{b-2}E_{a,b-1}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)^{2}=\frac{2}{z^{3}}\left(\frac{E_{a,b-1}\left(xz^{a}\right)E_{a,b-2}\left(xz^{a}\right)}{\left(E_{a,b}\left(xz^{a}\right)\right)^{2}}-\left(\frac{E_{a,b-1}\left(xz^{a}\right)}{E_{a,b}\left(xz^{a}\right)}\right)^{3}\right)$$

Combining results in

$$\frac{d^{3}}{dz^{3}}\log\left(z^{b-1}E_{a,b}\left(xz^{a}\right)\right)=\frac{1}{z^{3}}\left(\frac{E_{a,b-3}\left(xz^{a}\right)}{E_{a,b}\left(xz^{a}\right)}-\frac{3E_{a,b-1}\left(xz^{a}\right)E_{a,b-2}\left(xz^{a}\right)}{\left(E_{a,b}\left(xz^{a}\right)\right)^{2}}+2\left(\frac{E_{a,b-1}\left(xz^{a}\right)}{E_{a,b}\left(xz^{a}\right)}\right)^{3}\right) \tag{39}$$

Rather than continuing step-wise differentiation, we consider

$$\frac{d^{m+1}}{dz^{m+1}}\log\left(z^{b-1}E_{a,b}\left(xz^{a}\right)\right)=\frac{d^{m}}{dz^{m}}\left(\frac{z^{b-2}E_{a,b-1}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)$$

and invoke Leibniz' rule

$$\frac{d^{m}}{dz^{m}}\left(\frac{z^{b-2}E_{a,b-1}\left(xz^{a}\right)}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)=\sum_{k=0}^{m}\binom{m}{k}\frac{d^{k}}{dz^{k}}\left(\frac{1}{z^{b-1}E_{a,b}\left(xz^{a}\right)}\right)\frac{d^{m-k}}{dz^{m-k}}\left(z^{b-2}E_{a,b-1}\left(xz^{a}\right)\right)$$

With (20), we obtain

$$\frac{d^{m-k}}{dz^{m-k}}\left\{z^{b-2}E_{a,b-1}\left(xz^{a}\right)\right\}=z^{b-2-m+k}E_{a,b-1-m+k}\left(xz^{a}\right)=\frac{d^{m+1-k}}{dz^{m+1-k}}\left\{z^{b-1}E_{a,b}\left(xz^{a}\right)\right\}$$

The Taylor series of $\frac{1}{f(z)}$ around z_0, in terms of our characteristic coefficients [48],

$$\frac{1}{f(z)}=\frac{1}{f_{0}\left(z_{0}\right)}+\sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\frac{(-1)^{k}}{f_{0}^{k+1}\left(z_{0}\right)}s[k,m]\left(z_{0}\right)\right]\left(z-z_{0}\right)^{m} \tag{40}$$

where the characteristic coefficient $s[k,m]\left(z_0\right)$ is defined in (28), illustrates for $m\geq 1$ that

$$\left.\frac{1}{m!}\frac{d^{m}}{dz^{m}}\left(\frac{1}{f\left(z\right)}\right)\right|_{z=z_{0}}=\sum_{k=1}^{m}\frac{(-1)^{k}}{f_{0}^{k+1}\left(z_{0}\right)}s[k,m]\left(z_{0}\right)$$

Applying (40) and replacing z_0 by z specifies, for $k \geq 1$,

$$\frac{1}{k!}\frac{d^k}{dz^k}\left(\frac{1}{z^{b-1}E_{a,b}\left(xz^a\right)}\right) = \sum_{n=1}^{k} \frac{(-1)^n}{\left(z^{b-1}E_{a,b}\left(xz^a\right)\right)^{n+1}} \; s[n,k]|\,(z)$$

where the explicit form (28) of the characteristic coefficient indicates that

$$s[n,k]|\,(z) = \sum_{\sum_{i=1}^{n} j_i = k; j_i > 0} \prod_{i=1}^{n} \frac{1}{j_i!}\frac{d^{j_i}}{dz^{j_i}}\left\{z^{b-1}E_{a,b}\left(xz^a\right)\right\} = \sum_{\sum_{i=1}^{n} j_i = k; j_i > 0} \prod_{i=1}^{n} \frac{z^{b-1-j_i}E_{a,b-j_i}\left(xz^a\right)}{j_i!}$$

Since $\prod_{i=1}^{n} \frac{z^{b-1-j_i}E_{a,b-j_i}(xz^a)}{j_i!} = z^{\sum_{i=1}^{n}(b-1-j_i)} \prod_{i=1}^{n} \frac{E_{a,b-j_i}(xz^a)}{j_i!} = z^{n(b-1)-\sum_{i=1}^{n} j_i} \prod_{i=1}^{n} \frac{E_{a,b-j_i}(xz^a)}{j_i!}$ and $\sum_{i=1}^{n} j_i = k$, we find

$$s[n,k]|\,(z) = z^{n(b-1)-k} \sum_{\sum_{i=1}^{n} j_i = k; j_i > 0} \prod_{i=1}^{n} \frac{E_{a,b-j_i}\left(xz^a\right)}{j_i!}$$

so that

$$\frac{1}{k!}\frac{d^k}{dz^k}\left(\frac{1}{z^{b-1}E_{a,b}\left(xz^a\right)}\right) = z^{-(b-1)-k} \sum_{n=1}^{k} \frac{(-1)^n}{\left(E_{a,b}\left(xz^a\right)\right)^{n+1}} \sum_{\sum_{i=1}^{n} j_i = k; j_i > 0} \prod_{i=1}^{n} \frac{E_{a,b-j_i}\left(xz^a\right)}{j_i!}$$

for $k \geq 1$. Substituting these results into Leibniz's formula yields the logarithmic derivatives $l_m = \frac{d^m}{dz^m}\log\left(z^{b-1}E_{a,b}\left(xz^a\right)\right)$

$$\begin{aligned} l_{m+1} &= \frac{z^{b-2-m}E_{a,b-1-m}\left(xz^a\right)}{z^{b-1}E_{a,b}\left(xz^a\right)} + \sum_{k=1}^{m} \binom{m}{k} \frac{d^k}{dz^k}\left(\frac{1}{z^{b-1}E_{a,b}\left(xz^a\right)}\right) z^{b-2-m+k} E_{a,b-1-m+k}\left(xz^a\right) \\ &= \frac{1}{z^{m+1}}\left(\frac{E_{a,b-1-m}\left(xz^a\right)}{E_{a,b}\left(xz^a\right)} + m! \sum_{k=1}^{m}\left\{\sum_{n=1}^{k} \frac{(-1)^n \sum_{\sum_{i=1}^{n} j_i = k; j_i > 0} \prod_{i=1}^{n} \frac{E_{a,b-j_i}(xz^a)}{j_i!}}{\left(E_{a,b}\left(xz^a\right)\right)^{n+1}}\right\} \frac{E_{a,b-1-m+k}\left(xz^a\right)}{(m-k)!}\right) \end{aligned}$$

which shows that $l'_m = z^m l_m$ is a sum of m fractions of Mittag-Leffler functions. In addition to (34), (38) and (39), we list the scaled logarithmic derivatives $l'_m = z^m \frac{d^m}{dz^m}\log\left(z^{b-1}E_{a,b}\left(xz^a\right)\right)$ for $m = 4, 5$

and 6,

$$
\begin{aligned}
l_4' &= \frac{E_{a,b-4}(xz^a)}{E_{a,b}(xz^a)} - \frac{4E_{a,b-3}(xz^a)\,E_{a,b-1}(xz^a) + 3\left(E_{a,b-2}(xz^a)\right)^2}{\left(E_{a,b}(xz^a)\right)^2} + \frac{12E_{a,b-2}(xz^a)\left(E_{a,b-1}(xz^a)\right)^2}{\left(E_{a,b}(xz^a)\right)^3} \\
&\quad - 6\left(\frac{E_{a,b-1}(xz^a)}{E_{a,b}(xz^a)}\right)^4 \\
l_5' &= \frac{E_{a,b-5}(xz^a)}{E_{a,b}(xz^a)} - \frac{10E_{a,b-3}(xz^a)\,E_{a,b-2}(xz^a) + 5E_{a,b-4}(xz^a)\,E_{a,b-1}(xz^a)}{\left(E_{a,b}(xz^a)\right)^2} \\
&\quad + \frac{20E_{a,b-3}(xz^a)\left(E_{a,b-1}(xz^a)\right)^2 + 30E_{a,b-1}(xz^a)\left(E_{a,b-2}(xz^a)\right)^2}{\left(E_{a,b}(xz^a)\right)^3} \\
&\quad - \frac{60E_{a,b-2}(xz^a)\left(E_{a,b-1}(xz^a)\right)^3}{\left(E_{a,b}(xz^a)\right)^4} + 24\left(\frac{E_{a,b-1}(xz^a)}{E_{a,b}(xz^a)}\right)^5 \\
l_6' &= \frac{E_{a,b-6}(xz^a)}{E_{a,b}(xz^a)} - \frac{10\left(E_{a,b-3}(xz^a)\right)^3 + 15E_{a,b-4}(xz^a)\,E_{a,b-2}(xz^a) + 6E_{a,b-5}(xz^a)\,E_{a,b-1}(xz^a)}{\left(E_{a,b}(xz^a)\right)^2} \\
&\quad + \frac{30\left(E_{a,b-2}(xz^a)\right)^3 + 30E_{a,b-4}(xz^a)\left(E_{a,b-1}(xz^a)\right)^2 + 120E_{a,b-3}(xz^a)\,E_{a,b-2}(xz^a)\,E_{a,b-1}(xz^a)}{\left(E_{a,b}(xz^a)\right)^3} \\
&\quad - \frac{120E_{a,b-3}(xz^a)\left(E_{a,b-1}(xz^a)\right)^3 + 270\left(E_{a,b-2}(xz^a)\right)^2\left(E_{a,b-1}(xz^a)\right)^2}{\left(E_{a,b}(xz^a)\right)^4} \\
&\quad + \frac{360E_{a,b-2}(xz^a)\left(E_{a,b-1}(xz^a)\right)^4}{\left(E_{a,b}(xz^a)\right)^5} - 120\left(\frac{E_{a,b-1}(xz^a)}{E_{a,b}(xz^a)}\right)^6
\end{aligned}
$$

The recursion of the characteristic coefficients [48]

$$
\begin{aligned}
s[1,m] &= f_m \\
s[k,m] &= \sum_{j=k}^{m} f_{m-j+1}\; s[k-1,j-1] \qquad (k>1) \\
&= \sum_{j=1}^{m-k+1} f_j\; s[k-1,m-j] \qquad (k>1)
\end{aligned}
\tag{41}
$$

enables exact computation to any desired (finite) m by symbolic software.

In summary, the explicit derivatives of $l_m = \frac{d^m}{dz^m}\log\left(z^{b-1}E_{a,b}(xz^a)\right)$ for any z, a and b provides us with the Taylor series around any complex z_0

$$
\log\left(E_{a,b}(xz^a)\right) = (1-b)\log\left(\frac{z}{z_0}\right) + \log\left(E_{a,b}(xz_0^a)\right) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{d^m}{dz^m}\log\left(z^{b-1}E_{a,b}(xz^a)\right)\right|_{z=z_0}(z-z_0)^m
\tag{42}
$$

The Taylor series around $z_0 = 0$ in **art**. 12 is limited to a region around the origin in the complex plane. Taylor series (42) possesses a radius $R(z_0)$ of convergence around z_0 that equals the distance between z_0 and the nearest zero of $E_{a,b}(xz^a)$ to z_0.

5 Integrals containing $E_{a,b}(z)$

15. *Integral duplication formula for* $E_{a,b}(-z)$. Using the duplication formula of the Gamma function,

$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z) \Gamma\left(z+\frac{1}{2}\right)$, the definition (1) is rewritten as

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)} = \frac{2^{2b-1}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+b+ak\right)\left(2^{2a}z\right)^k}{\Gamma(2b+2ak)}$$

Invoking the Euler integral $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $\mathrm{Re}(s) > 0$,

$$\begin{aligned} E_{a,b}(-z) &= \frac{2^{2b-1}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(-2^{2a}z\right)^k}{\Gamma(2b+2ak)} \int_0^\infty t^{b-\frac{1}{2}} t^{ak} e^{-t} dt \\ &= \frac{2^{2b-1}}{\sqrt{\pi}} \int_0^\infty t^{b-\frac{1}{2}} e^{-t} \sum_{k=0}^{\infty} \frac{\left(-z(4t)^a\right)^k}{\Gamma(2b+2ak)} dt \end{aligned}$$

and the definition (1), we find an integral duplication formula for $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(b) \geq -\frac{1}{2}$,

$$E_{a,b}(-z) = \frac{2^{2b-1}}{\sqrt{\pi}} \int_0^\infty t^{b-\frac{1}{2}} e^{-t} E_{2a,2b}\left(-z(4t)^a\right) dt \tag{43}$$

After substituting the Gamma duplication formula in the slightly rewritten power series

$$\begin{aligned} E_{a,b}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)} = \sum_{k=0}^{\infty} \frac{z^k}{(b-1+ak)\Gamma(b-1+ak)} \\ &= \frac{2^{2b-2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(4^a z)^k}{\Gamma(2b-1+2ak)} \Gamma\left(b+ak-\frac{1}{2}\right) \end{aligned}$$

we find alternatively,

$$\begin{aligned} E_{a,b}(-z) &= \frac{2^{2b-2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-4^a z)^k}{\Gamma(2b-1+2ak)} \int_0^\infty t^{b-\frac{3}{2}} t^{ak} e^{-t} dt \\ &= \frac{2^{2b-2}}{\sqrt{\pi}} \int_0^\infty t^{b-\frac{3}{2}} e^{-t} \sum_{k=0}^{\infty} \frac{\left(-z(4t)^a\right)^k}{\Gamma(2b-1+2ak)} dt \end{aligned}$$

for $\mathrm{Re}(b) \geq \frac{1}{2}$ and $\mathrm{Re}(z)$,

$$E_{a,b}(-z) = \frac{2^{2(b-1)}}{\sqrt{\pi}} \int_0^\infty t^{b-\frac{3}{2}} e^{-t} E_{2a,2b-1}\left(-z(4t)^a\right) dt \tag{44}$$

which also applies to $E_a(z) = E_{a,1}(z)$ after choosing $b = 1$.

Let $u^a = z(4t)^a$ or $u = 4z^{\frac{1}{a}}t$, then (44) is, for real, nonnegative z,

$$E_{a,b}(-z) = \frac{2^{2b-2}}{\sqrt{\pi}} \left(\frac{1}{4}z^{-\frac{1}{a}}\right)^{b-\frac{1}{2}} \int_0^\infty u^{b-\frac{3}{2}} e^{-\left(\frac{1}{4}z^{-\frac{1}{a}}\right)u} E_{2a,2b-1}(-u^a)\, du$$

Let $s = \frac{1}{4}z^{-\frac{1}{a}}$ or $z = (4s)^{-a}$, then we arrive at the Laplace transform

$$\int_0^\infty e^{-su} u^{b-\frac{3}{2}} E_{2a,2b-1}(-u^a)\, du = \frac{\sqrt{\pi}}{2^{2b-2} s^{b-\frac{1}{2}}} E_{a,b}\left(-(4s)^{-a}\right) \tag{45}$$

Example For $b = 1$ and $a = \frac{1}{2}$, the Laplace transform (45) becomes

$$\frac{\sqrt{\pi}}{\sqrt{s}} E_{\frac{1}{2}}\left(-\frac{1}{2\sqrt{s}}\right) = \int_0^\infty u^{-\frac{1}{2}} e^{-su-\sqrt{u}} du = 2\int_0^\infty e^{-st^2-t} dt$$

With $st^2+t=s\left(t+\frac{1}{2s}\right)^2-\frac{1}{4s}$, we have

$$\frac{\sqrt{\pi}}{\sqrt{s}}E_{\frac{1}{2}}\left(-\frac{1}{2\sqrt{s}}\right)=2e^{\frac{1}{4s}}\int_0^\infty e^{-s\left(t+\frac{1}{2s}\right)^2}dt=2e^{\frac{1}{4s}}\int_{\frac{1}{2s}}^\infty e^{-su^2}du=e^{\frac{1}{4s}}\frac{2}{\sqrt{s}}\int_{\frac{1}{2\sqrt{s}}}^\infty e^{-t^2}dt$$

Simplified with $\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du=1-\operatorname{erf}(x)$ and $\operatorname{erf}(-x)=\frac{2}{\sqrt{\pi}}\int_0^{-x}e^{-u^2}du=-\operatorname{erf}(x)$

$$E_{\frac{1}{2}}(-x)=e^{x^2}\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt=e^{x^2}\operatorname{erfc}(x)$$

is again (24), because $\operatorname{erf}(-x)=\frac{2}{\sqrt{\pi}}\int_0^{-x}e^{-u^2}du=-\operatorname{erf}(x)$.

16. *Integral multiplication formula for $E_{a,b}(z)$.* The method of **art.** 15 is readily generalized. Invoking Gauss's multiplication formula (162) into the definition (1) yields

$$E_{a,b}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(b+ak)}=(2\pi)^{\frac{1}{2}(1-n)}n^{nb-\frac{1}{2}}\sum_{k=0}^\infty\frac{\prod_{j=1}^{n-1}\Gamma\left(b+ak+\frac{j}{n}\right)}{\Gamma(nb+nak)}(n^{na}z)^k$$

We introduce the Mellin transform (184) of a product of Gamma functions for $\operatorname{Re}(s)>0$,

$$\prod_{j=1}^{n-1}\Gamma\left(s+\frac{j}{n}\right)=\int_0^\infty u^{s-1}h_n(u)\,du$$

where the inverse function $h_n(u)$ is specified in (187) in **art.** 72 as a Taylor series in u, and we obtain

$$E_{a,b}(z)=(2\pi)^{\frac{1}{2}(1-n)}n^{nb-\frac{1}{2}}\int_0^\infty h_n(u)\,u^{b-1}\sum_{k=0}^\infty\frac{(u^an^{na}z)^k}{\Gamma(nb+nak)}du$$

Thus, for any integer n, we arrive at an integral multiplication formula for the Mittag-Leffler functions,

$$E_{a,b}(z)=(2\pi)^{\frac{1}{2}(1-n)}n^{nb-\frac{1}{2}}\int_0^\infty h_n(u)\,u^{b-1}E_{na,nb}(u^an^{na}z)\,du \tag{46}$$

The companion of (46) follows similarly from $E_{a,b}(z)=\sum_{k=0}^\infty\frac{z^k}{(b-1+ak)\Gamma(b-1+ak)}$ as

$$E_{a,b}(z)=(2\pi)^{\frac{1}{2}(1-n)}n^{n(b-1)+\frac{1}{2}}\int_0^\infty h_n(u)\,u^{b-2}E_{na,n(b-1)+1}(u^an^{na}z)\,du \tag{47}$$

which directly reduces for $b=1$ to the Mittag-Leffler function $E_a(z)=E_{a,1}(z)$,

$$E_a(z)=(2\pi)^{\frac{1}{2}(1-n)}n^{\frac{1}{2}}\int_0^\infty h_n(u)\,u^{-1}E_{na}(u^an^{na}z)\,du$$

17. *Special cases of the integral multiplication formula for $E_{a,b}(z)$.* The case $n=3$ in (46) with $h_3(u)$ in (191) expressed in terms of the modified Bessel function $K_\nu(z)$, defined in (189), becomes

$$E_{a,b}(z)=\frac{3^{3b-\frac{1}{2}}}{\pi}\int_0^\infty K_{\frac{1}{3}}\left(2\sqrt{u}\right)u^{b-\frac{1}{2}}E_{3a,3b}\left(3^{3a}u^az\right)du \tag{48}$$

The companion of (48) follows from (47) as

$$E_{a,b}(z)=\frac{3^{3(b-1)+\frac{1}{2}}}{\pi}\int_0^\infty K_{\frac{1}{3}}\left(2\sqrt{u}\right)u^{b-\frac{3}{2}}E_{3a,3b-2}\left(3^{3a}u^az\right)du \tag{49}$$

In [2, eq. (25), (28) and (31)], Apelblat has recently presented three remarkable integral functional relations,

$$E_a\left(t^a\right) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{u^2}{4t}} E_{2a}\left(u^{2a}\right) du \tag{50}$$

$$E_a\left(t^a\right) = \frac{1}{\pi} \int_0^\infty \sqrt{\frac{u}{t}} K_{\frac{1}{3}}\left(\frac{2u^{\frac{3}{2}}}{\sqrt{27t}}\right) E_{3a}\left(u^{3a}\right) du \tag{51}$$

$$\frac{E_a\left(t^a\right) - 1}{\sqrt{t}} = \int_0^\infty \frac{J_1\left(2\sqrt{tu}\right)}{\sqrt{u}} E_a\left(u^a\right) du \tag{52}$$

which he has skillfully derived by manipulations of Laplace transforms. Apart from the last relation (52), where $J_p(z)$ is the Bessel function of order p, the first two are special cases of the multiplication formula for $E_{a,b}(z)$ in **art.** 16. Indeed, after substitution of $x = \frac{1}{4st}u^2$ in the first integral (50), we obtain

$$E_{a,1}\left(t^a\right) = \frac{s^{\frac{1}{2}}}{\sqrt{\pi}} \int_0^\infty x^{-\frac{1}{2}} e^{-sx} E_{2a,1}\left((4st)^a x^a\right) dx$$

which is a special case of (45), more easily noticed from its generalization (55) below, for $a \to 2a$, $b = 1$ and $x = (4st)^a$. With $z = t^a$ and after substituting $x = \frac{u^3}{27t}$ in (49), we obtain

$$E_{a,b}\left(t^a\right) = \frac{t^{\frac{1}{2}-b}}{\pi} \int_0^\infty K_{\frac{1}{3}}\left(2\sqrt{\frac{u^3}{27t}}\right) u^{3b-\frac{5}{2}} E_{3a,3b-2}\left(u^{3a}\right) dx$$

which leads to (51) by choosing $b = 1$.

18. *Laplace transform of* $E_{a,b}\left(xz^\beta\right)$. The Laplace transform of $t^{\gamma-1}E_{a,b}\left(xt^\beta\right)$ is

$$\int_0^\infty e^{-st} t^{\gamma-1} E_{a,b}\left(xt^\beta\right) dt = \int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{x^k t^{\gamma-1+\beta k}}{\Gamma(b+ak)} dt = \sum_{k=0}^\infty \frac{x^k}{\Gamma(b+ak)} \int_0^\infty e^{-st} t^{\gamma+\beta k-1} dt$$

for $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\beta) > 0$. Fubini's theorem[7] states that the summation and integration can be reversed, leading to

$$\int_0^\infty e^{-st} t^{\gamma-1} E_{a,b}\left(xt^\beta\right) dt = \sum_{k=0}^\infty \frac{\Gamma(\gamma+\beta k)}{\Gamma(b+ak)} \frac{x^k}{s^{\gamma+k\beta}} \tag{53}$$

provided the integrals $\int_0^\infty e^{-st} t^{\gamma-1} E_{a,b}\left(xt^\beta\right) dt$ and $\int_0^\infty e^{-st} t^{\gamma+\beta k-1} dt = \frac{\Gamma(\gamma+\beta k)}{s^{\gamma+k\beta}}$ exist and the resulting series at the right-hand side of (53) converges. Stirling's formula (173) indicates for large k that, to first order,

$$\frac{\Gamma(\gamma+\beta k)}{\Gamma(b+ak)} \sim k^{(\beta-a)k} \left(\frac{\beta^\beta}{a^a}\right)^k$$

Hence, the series in (53) diverges if $\beta > a$ and converges for all $\frac{x}{s^\beta}$ if $\beta < a$. However, if $\beta = a$, then ((176) shows, for any complex number $z = re^{i\theta}$ with large modulus $r = |z|$, that $\frac{|\Gamma(a+z)|}{|\Gamma(b+z)|} = r^{a-b}\left(1 + O\left(\frac{1}{r}\right)\right)$ and thus that $\frac{\Gamma(\gamma+ak)}{\Gamma(b+ak)} \sim (ak)^{\gamma-b}$ for large $k > k_0 >> 1$. The convergence of the series $\frac{1}{s^\gamma}\sum_{k=k_0}^\infty \frac{\Gamma(\gamma+ak)}{\Gamma(b+ak)}\left(\frac{x}{s^a}\right)^k \sim \frac{a^{\gamma-b}}{s^\gamma}\sum_{k=k_0}^\infty k^{\gamma-b}\left(\frac{x}{s^a}\right)^k$ in (53) requires that $\left|\frac{x}{s^a}\right| < 1$, for any finite γ and b.

[7] Another argument is that a Taylor series can be integrated within its range of convergence.

A first choice is $\beta = a$ and $\gamma = b$ in (53), which restricts s so that $|s^a| > |x|$, is, for $\operatorname{Re}(b) > 0$ and $\operatorname{Re}(a) > 0$,

$$\int_0^\infty e^{-st} t^{b-1} E_{a,b}(xt^a)\, dt = \frac{1}{s^b} \sum_{k=0}^\infty \left(\frac{x}{s^a}\right)^k = \frac{s^{a-b}}{s^a - x} \tag{54}$$

The Laplace transform (54) plays a key role in the theory of the Mittag-Leffler function.

Two other choices in (53) follow after the introduction of the duplication formula of the Gamma function

$$\frac{\Gamma(\gamma + \beta k)}{\Gamma(b + ak)} = \frac{\sqrt{\pi}}{2^{b-1}} \frac{\Gamma(\gamma + \beta k)}{2^{ak} \Gamma\left(\frac{b}{2} + \frac{a}{2}k\right) \Gamma\left(\frac{b+1}{2} + \frac{a}{2}k\right)}$$

as $\gamma = \frac{b}{2}$ and $\beta = \frac{a}{2}$ and $\gamma = \frac{b+1}{2}$ and $\beta = \frac{a}{2}$, respectively. If $\gamma = \frac{b}{2}$ and $\beta = \frac{a}{2}$, then we find

$$\int_0^\infty e^{-st} t^{\frac{b}{2}-1} E_{a,b}\left(xt^{\frac{a}{2}}\right) dt = \frac{\sqrt{\pi}}{2^{b-1} s^{\frac{b}{2}}} \sum_{k=0}^\infty \frac{1}{\Gamma\left(\frac{b+1}{2} + \frac{a}{2}k\right)} \left(\frac{x}{2^a s^{\frac{a}{2}}}\right)^k$$

resulting, with the definition (1), in

$$\int_0^\infty e^{-st} t^{\frac{b}{2}-1} E_{a,b}\left(xt^{\frac{a}{2}}\right) dt = \frac{\sqrt{\pi}}{2^{b-1} s^{\frac{b}{2}}} E_{\frac{a}{2},\frac{b+1}{2}}\left(\frac{x}{(4s)^{\frac{a}{2}}}\right) \tag{55}$$

while the third choice $\gamma = \frac{b+1}{2}$ and $\beta = \frac{a}{2}$ leads to

$$\int_0^\infty e^{-st} t^{\frac{b+1}{2}-1} E_{a,b}\left(xt^{\frac{a}{2}}\right) dt = \frac{\sqrt{\pi}}{2^{b-1} s^{\frac{b+1}{2}}} E_{\frac{a}{2},\frac{b}{2}}\left(\frac{x}{(4s)^{\frac{a}{2}}}\right) \tag{56}$$

Both (55) and (56) are slightly more general than and reduce to (45) and (43), respectively, for $x = -1$.

19. *Generalized integration.* By using a variation of the integral of the Beta-function [1, 6.2], $\int_0^x u^{z-1} (x-u)^{w-1}\, du = x^{z+w-1} \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ for real $x > 0$, $\operatorname{Re}(z) > 0$ and $\operatorname{Re}(w) > 0$, we obtain a generalized integral variant of the Mittag-Leffler function $E_{a,b}(z)$ in (1),

$$\begin{aligned}\frac{1}{\Gamma(w)} \int_0^x (x-u)^{w-1} u^{\gamma-1} E_{a,b}\left(\lambda u^\beta\right) du &= \frac{1}{\Gamma(w)} \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(b+ak)} \int_0^x u^{\beta k + \gamma - 1} (x-u)^{w-1}\, du \\ &= x^{\gamma+w-1} \sum_{k=0}^\infty \frac{\lambda^k x^{\beta k}}{\Gamma(b+ak)} \frac{\Gamma(\gamma+\beta k)}{\Gamma(\gamma+w+\beta k)}\end{aligned}$$

which reduces for $\beta = a$ and $\gamma = b$ to

$$\frac{1}{\Gamma(w)} \int_0^x (x-u)^{w-1} u^{b-1} E_{a,b}(\lambda u^a)\, du = x^{b-1+w} E_{a,b+w}(\lambda x^a) \tag{57}$$

The m-fold integral[8] (57) for $w = m$ and $\lambda = 1$ possesses the same form as the m-fold differentiation in (20)

$$\frac{d^m}{dx^m} \left\{x^{b-1} E_{a,b}(x^a)\right\} = x^{b-1-m} E_{a,b-m}(x^a)$$

[8]The n-fold integral is

$$F_n(x,a) = \int_a^x du_1 \int_a^{u_1} du_2 \ldots \int_a^{u_{n-1}} du_n f(u_n) = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)\, du$$

The generalization towards fractional calculus, where the integer n is extended to a real number, is treated in [19].

that is better recognized with Cauchy's integral for the m-th derivative of an analytic function, $\left.\frac{d^m f(z)}{dz^m}\right|_{z=z_0} = \frac{m!}{2\pi i}\int_{C(z_0)} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{m+1}}$, where $C(z_0)$ is a closed contour around z_0,

$$\frac{(-1)^{m+1} m!}{2\pi i}\int_{C(x)} (x-\omega)^{-m-1}\omega^{b-1}E_{a,b}(\omega^a)\; d\omega = x^{b-1-m}E_{a,b-m}(x^a)$$

Hence, (57) written with the reflection formula (161), suggests that

$$\frac{\frac{\sin \pi w}{\pi}\Gamma(1-w)}{2\pi i}\int_{C(x)} (x-\omega)^{w-1}\omega^{b-1}E_{a,b}(\omega^a)\; d\omega = x^{b-1-w}E_{a,b-w}(x^a)$$

holds for any negative real w, leading to fractional derivatives [52]. The Laplace transform (54) easier connects to fractional derivatives, avoiding the contour integral. Applications of the Mittag-Leffler function to fractional calculus are amply illustrated in [17].

Example For $a = b = 1$ in (57) and, next, $x = 1$, $\lambda = z$ and $w + 1 \to b$, we obtain, for $\mathrm{Re}(b) > 1$,

$$E_{1,b}(z) = \frac{1}{\Gamma(b-1)}\int_0^1 (1-u)^{b-2} e^{zu} du$$

Let $t = 1 - u$, then the incomplete Gamma function appears

$$E_{1,b}(z) = \frac{e^z}{\Gamma(b-1)}\int_0^1 t^{b-2}e^{-zt}dt = \frac{z^{1-b}e^z}{\Gamma(b-1)}\int_0^z u^{b-2}e^{-u}du$$

which is again equal to (3).

20. *Integration of a product of Mittag-Leffler functions.* We extend the idea in **art**. 19 and consider, for $x \geq 0$,

$$\begin{aligned} L &= \frac{1}{\Gamma(w)}\int_0^x (x-u)^{w-1} u^{\beta-1}E_{a,b}(\lambda u^\alpha) E_{c,d}(\mu (x-u)^\gamma)\, du \\ &= \frac{1}{\Gamma(w)}\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{\lambda^k}{\Gamma(b+ak)}\frac{\mu^m}{\Gamma(d+cm)}\int_0^x (x-u)^{w+m\gamma-1} u^{\beta+k\alpha-1}du \end{aligned}$$

Introducing the integral of the Beta-function results in

$$L = \frac{x^{\beta+w-1}}{\Gamma(w)}\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(\lambda x^\alpha)^k}{\Gamma(b+ak)}\frac{(\mu x^\gamma)^m}{\Gamma(d+cm)}\frac{\Gamma(\beta+k\alpha)\Gamma(w+m\gamma)}{\Gamma(\beta+k\alpha+w+m\gamma)}$$

After the choice $\alpha = a$, $\beta = b$, $\gamma = c$ and $d = w$, the double sum simplifies to

$$L = \frac{x^{b+w-1}}{\Gamma(w)}\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(\lambda x^a)^k (\mu x^c)^m}{\Gamma(b+w+ka+mc)}$$

Further computations require the choice $c = a$,

$$L = \frac{x^{b+w-1}}{\Gamma(w)}\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(\lambda x^a)^k (\mu x^a)^m}{\Gamma(b+w+a(k+m))}$$

Let $q = k + m$, then $0 \leq q$ and $m = q - k \geq 0$, while $k \geq 0$, thus

$$L = \frac{x^{b+w-1}}{\Gamma(w)}\sum_{q=0}^\infty\sum_{k=0}^q \frac{\lambda^k\mu^{q-k}x^{aq}}{\Gamma(b+w+aq)} = \frac{x^{b+w-1}}{\Gamma(w)}\sum_{q=0}^\infty \frac{\mu^q x^{aq}}{\Gamma(b+w+aq)}\sum_{k=0}^q\left(\frac{\lambda}{\mu}\right)^k$$

Executing the finite geometric series $\sum_{k=0}^{q}\left(\frac{\lambda}{\mu}\right)^k = \frac{\left(\frac{\lambda}{\mu}\right)^{q+1}-1}{\frac{\lambda}{\mu}-1}$ leads to

$$L = \frac{x^{b+w-1}}{\Gamma(w)(\lambda-\mu)}\left\{\lambda\sum_{q=0}^{\infty}\frac{(\lambda x^a)^q}{\Gamma(b+w+aq)} - \mu\sum_{q=0}^{\infty}\frac{(\mu x^a)^q}{\Gamma(b+w+aq)}\right\}$$

Finally, we arrive for $b > 0$ and $w > 0$ at

$$\int_0^x (x-u)^{w-1}u^{b-1}E_{a,b}(\lambda u^a)E_{a,w}(\mu(x-u)^a)\,du = x^{b+w-1}\frac{\lambda E_{a,b+w}(\lambda x^a) - \mu E_{a,b+w}(\mu x^a)}{\lambda-\mu} \tag{58}$$

Clearly[9], since $\lim_{\mu\to 0}E_{a,w}(\mu(x-u)^a) = \frac{1}{\Gamma(w)}$, then (58) reduces to (57).

21. *An asymptotic result.* We invoke the device[10], used by Gauss in his grand treatise [16, p. 146] on the hypergeometric function [27, p. 74] to deduce the Euler integral for the Gamma function from the Beta-integral based on

$$\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^{xn} = e^{-xt}$$

and start from (57)

$$\int_0^x \left(1-\frac{u}{x}\right)^{w-1}u^{b-1}E_{a,b}(\lambda u^a)\,du = \Gamma(w)\,x^bE_{a,b+w}(\lambda x^a)$$

Let $w = 1+xs$ with $\operatorname{Re}(s) > 0$ and x is real, then

$$\int_0^x \left(1-\frac{u}{x}\right)^{xs}u^{b-1}E_{a,b}(\lambda u^a)\,du = \Gamma(xs+1)\,x^bE_{a,b+xs+1}(\lambda x^a)$$

and, after taking the limit for $x\to\infty$

$$\int_0^\infty e^{-su}u^{b-1}E_{a,b}(\lambda u^a)\,du = \lim_{x\to\infty}\Gamma(xs+1)\,x^bE_{a,b+xs+1}(\lambda x^a)$$

Comparison with the Laplace transform (54) for $|s^a| > |\lambda|$ shows that

$$\lim_{x\to\infty}\Gamma(xs+1)\,x^bE_{a,b+xs+1}(\lambda x^a) = \frac{s^{a-b}}{s^a-\lambda}$$

---

[9] In case $\mu\to\lambda$, then, after using de l'Hospital's rule,

$$\lim_{\mu\to\lambda}\frac{\lambda E_{a,b+w}(\lambda x^a) - \mu E_{a,b+w}(\mu x^a)}{\lambda-\mu} = E_{a,b+w}(\lambda x^a) + \lambda x^a\left.\frac{dE_{a,b+w}(z)}{dz}\right|_{z=\lambda x^a}$$

and the differential rule (17), formula (58) becomes

$$\int_0^x (x-u)^{w-1}u^{b-1}E_{a,b}(\lambda u^\alpha)E_{a,w}(\lambda(x-u)^a)\,du = \frac{x^{b+w-1}}{a}\{E_{a,b+w-1}(\lambda x^a) + (a+1-(b+w))E_{a,b+w}(\lambda x^a)\}$$

[10] Starting from the above Beta-integral, it holds that

$$\Gamma(z)\lim_{x\to\infty}\frac{x^z\Gamma(x+1)}{\Gamma(z+x+1)} = \lim_{x\to\infty}\int_0^x u^{z-1}\left(1-\frac{u}{x}\right)^x du = \int_0^\infty u^{z-1}e^{-u}du = \Gamma(z)$$

from which

$$\frac{\Gamma(z+x)}{\Gamma(x)} \sim x^z$$

and, for large real x,

$$E_{a,b+xs}\left(\lambda x^a\right) \sim \frac{1}{1-\frac{\lambda}{s^a}} \frac{1}{\Gamma\left(xs\right)\left(sx\right)^b}$$

Let $z = \lambda x^a$, obeying $|xs|^a > |z|$, then for real $r = x\,|s| \to \infty$

$$E_{a,b+re^{i\theta}}\left(z\right) \sim \frac{r^{-b}e^{-bi\theta}}{1-zr^{-a}e^{-ai\theta}} \frac{1}{\Gamma\left(re^{i\theta}\right)} \sim \frac{1}{\sqrt{2\pi}} \frac{r^{\frac{1}{2}-b}e^{-bi\theta}}{1-zr^{-a}e^{-ai\theta}} e^{-r(\ln(r)-1)\cos\theta+\theta r\sin\theta}\left(1+O\left(\frac{1}{r}\right)\right)$$

where in the last step (176) is used. Thus for $a > 0$ and for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we evidently find that $E_{a,b}\left(z\right) \to 0$ for $\text{Re}\left(b\right) \to \infty$ and for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ that $E_{a,b}\left(z\right) \to \infty$ for $\text{Re}\left(b\right) \to -\infty$. For $\theta = \pm\frac{\pi}{2}$, it holds that $E_{a,b+ir}\left(z\right) \sim \frac{1}{\sqrt{2\pi}} \frac{r^{\frac{1}{2}-b}e^{-bi\frac{\pi}{2}}}{1-zr^{-a}e^{-ai\frac{\pi}{2}}} e^{\frac{\pi}{2}r}$, from which $\left|E_{a,\pm ib}\left(z\right)\right| \sim \frac{b^{\frac{1}{2}}}{\sqrt{2\pi}}$ for real $b \to \infty$.

22. *Berberan-Santos' integral for the double argument.* Berberan-Santos [4] applied a simplified form (195) of the inverse Laplace contour integral (193), which he deduced in [3], to the Mittag-Leffler function $E_{a,b}\left(z\right)$. First, let the Laplace transform of a real function $g\left(u\right)$ be equal to $E_{a,b}\left(-z\right)$, hence, $\int_0^\infty e^{-su}g\left(u\right)du = E_{a,b}\left(-s\right)$. From the Laplace transform with $s = \sigma + iT$

$$\left|E_{a,b}\left(-s\right)\right| = \left|\int_0^\infty e^{-\sigma u}e^{-iTu}g\left(u\right)du\right| \leq \int_0^\infty e^{-\sigma t}g\left(t\right)dt$$

it follows that $\lim_{s\to\infty}\int_0^\infty e^{-st}g\left(t\right)dt = 0$ for any direction in which s with $\text{Re}\left(s\right) > 0$ tends to infinity. Since $E_{a,b}\left(z\right)$ is entire function of order $\frac{1}{a}$, $E_{a,b}\left(z\right) = O\left(e^{z^{\frac{1}{a}}}\right)$, we have that $E_{a,b}\left(-z\right) = O\left(e^{r^{\frac{1}{a}}e^{i\frac{\theta+\pi}{a}}}\right) = O\left(e^{r^{\frac{1}{a}}\cos\frac{\theta+\pi}{a}}\right)$ and $\left|E_{a,b}\left(-s\right)\right| \to 0$ provided that $\cos\frac{\theta+\pi}{a} < 0$ or $\frac{\pi a}{2} - \pi < \theta < \frac{3\pi a}{2} - \pi$. With $s = re^{i\theta}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ because $\text{Re}\left(s\right) > 0$, the limit $\lim_{s\to\infty}\left|E_{a,b}\left(-s\right)\right| = 0$ requires that $0 \leq a \leq 1$. Incidentally, the definition of c in Appendix C indicates that $c = 0$ and Berberan-Santos' inverse Laplace transform (195) then yields

$$g(t) = \frac{2}{\pi}\int_0^\infty \text{Re}\left(E_{a,b}\left(-iw\right)\right)\cos twdw \qquad \text{for } t > 0$$

Next, Berberan-Santos observed from (6) that $E_{a,b}\left(-iw\right) = E_{2a,b}\left(-w^2\right) - iwE_{2a,b+2a}\left(-w^2\right)$ so that, for real w,

$$\text{Re}\left(E_{a,b}\left(-iw\right)\right) = E_{2a,b}\left(-w^2\right)$$

Hence, the inverse Laplace transform becomes

$$g(t) = \frac{2}{\pi}\int_0^\infty E_{2a,b}\left(-w^2\right)\cos twdw \qquad \text{for } t > 0$$

Finally, taking the Laplace transform of both sides and reversing the integrals

$$E_{a,b}\left(-s\right) = \frac{2}{\pi}\int_0^\infty E_{2a,b}\left(-w^2\right)\left(\int_0^\infty e^{-st}\cos twdt\right)dw$$

results in Berberan-Santos' remarkable integral for the double argument in a (not b)

$$E_{a,b}\left(-s\right) = \frac{2s}{\pi}\int_0^\infty \frac{E_{2a,b}\left(-w^2\right)}{s^2+w^2}dw \qquad \text{for } 0 \leq a \leq 1,\ \text{Re}\left(s\right) > 0 \tag{59}$$

Example For $a = \frac{1}{2}$, $b = 1$ and with $E_1(z) = e^z$, Berberan-Santos' integral (59) yields again (24),

$$E_{\frac{1}{2}}(-x) = \frac{2x}{\pi} \int_0^\infty \frac{e^{-t^2}}{t^2 + x^2} dt = e^{x^2} \operatorname{erfc}(x)$$

23. *Another proof of Berberan-Santos' integral (59).* Application of Theorem 1 in Appendix B to $f(z) = E_{a,b}(s - z)$, only valid for $0 \le a \le 1$ because then $\lim_{r\to\infty} \frac{E_{a,b}(s-ir)}{r^2} = 0$ (see **art.** 22), yields

$$E_{a,b}(-s) = \frac{s}{\pi} \int_{-\infty}^{\infty} \frac{E_{a,b}(-iw)}{s^2 + w^2} dw$$

Using (6), $E_{a,b}(-iw) = E_{2a,b}(-w^2) - iwE_{2a,b+2a}(-w^2)$ and the fact that $wE_{2a,b+2a}(-w^2)$ is odd and $E_{2a,b}(-w^2)$ is even, again leads to (59).

Using $\int_0^\infty e^{-t(s^2+w^2)} dt$ in (59) and reversing the integrals, justified by absolute convergence, gives

$$E_{a,b}(-s) = \frac{2s}{\pi} \int_0^\infty e^{-ts^2} \left\{ \int_0^\infty e^{-tw^2} E_{2a,b}(-w^2) \, dw \right\} dt = \frac{s}{\pi} \int_0^\infty e^{-ts^2} \left\{ \int_0^\infty e^{-tx} x^{-\frac{1}{2}} E_{2a,b}(-x) \, dx \right\} dt$$

Thus, Berberan-Santos' integral (59) is equivalent to the statement that the Laplace transform of the Laplace transform of $x^{-\frac{1}{2}} E_{2a,b}(-x)$ equals $\frac{\pi E_{a,b}(-\sqrt{s})}{\sqrt{s}}$. In fact, the latter property follows from the Stieltjes transform, which is an iteration of the Laplace transform and which is treated in the book by Widder [56, Chapter VIII]. Appendix C discusses Gross' Laplace transform pair [21], in which the inverse Laplace transform (193) is of the same form as the Laplace transform (192) itself.

24. *Euler-Maclaurin summation.* The Euler-Maclaurin summation formula [41, p. 14] is

$$\sum_{n=\alpha+1}^{\beta} f(n) = \int_\alpha^\beta f(t) \, dt + \sum_{n=1}^{N} (-1)^n \frac{B_n}{n!} \left[f^{(n-1)}(\beta) - f^{(n-1)}(\alpha) \right] + R_N \tag{60}$$

with remainder term

$$R_N = \frac{(-1)^{N-1}}{N!} \int_\alpha^\beta B_N(u - [u]) \, f^{(N)}(u) \, du$$

where B_n and $B_n(x)$ are the Bernoulli numbers and the Bernoulli polynomials defined in [1, Chapter 23].

The right-hand side summation in the Euler-Maclaurin summation formula (60) requires higher order derivatives for the function $f(w) = \frac{z^w}{\Gamma(b+aw)}$. We invoke Leibniz' rule

$$f^{(k)}(x) = \left. \frac{d^k}{dw^k} \frac{z^w}{\Gamma(b + aw)} \right|_{w=x} = \sum_{j=0}^{k} \binom{k}{j} (\log z)^{k-j} \, z^x \left. \frac{d^j}{dw^j} \frac{1}{\Gamma(b + aw)} \right|_{w=x}$$

which simplifies, with $\frac{d^j}{dw^j} \frac{1}{\Gamma(b+aw)} = a^j \left. \frac{d^j}{dy^j} \frac{1}{\Gamma(y)} \right|_{y=b+ax}$, to

$$\frac{f^{(k)}(x)}{k!} = z^x \sum_{j=0}^{k} \frac{a^j}{j!} \left. \frac{d^j}{dy^j} \frac{1}{\Gamma(y)} \right|_{y=b+ax} \frac{(\log z)^{k-j}}{(k-j)!}$$

Since $\frac{1}{\Gamma(z)}$ is an entire function, the j-sum converges when $k \to \infty$. Applying the Euler-Maclaurin summation formula (60) to the definition (1) of $E_{a,b}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(b+ak)}$ shows that $\beta \to \infty$, but that we can choose the integer $\alpha = m$. The asymptotic form (176) indicates that $\lim_{\beta\to\infty} \frac{d^j}{dw^j} \frac{1}{\Gamma(b+aw)}\Big|_{w=\beta} = 0$ for all j and (60) becomes

$$\sum_{k=m+1}^{\infty} \frac{z^k}{\Gamma(b+ak)} = \int_m^{\infty} \frac{z^t}{\Gamma(b+at)}\,dt + z^m \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{n+1} \sum_{j=0}^{n} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+am} \frac{(\log z)^{n-j}}{(n-j)!} + R_N \tag{61}$$

The integral is written in terms of the integral $I_{a,b}(z)$, defined in (2), as

$$\int_m^{\infty} \frac{z^t}{\Gamma(b+at)}\,dt = z^m \int_0^{\infty} \frac{z^u}{\Gamma(b+am+au)}\,du = z^m I_{a,b+ma}(z)$$

In summary, the Euler-Maclaurin expansion (61) is, with $\sum_{k=m+1}^{\infty} \frac{z^k}{\Gamma(b+ak)} = \sum_{k=0}^{\infty} \frac{z^{k+m+1}}{\Gamma(b+a(m+1)+ak)}$,

$$z^{m+1} E_{a,b+(m+1)a}(z) = z^m I_{a,b+ma}(z) + z^m S_N + R_N \tag{62}$$

where

$$S_N = \sum_{n=0}^{N-1} (-1)^n \frac{B_{n+1}}{n+1} \sum_{j=0}^{n} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+ma} \frac{(\log z)^{n-j}}{(n-j)!}$$

25. *Euler-Maclaurin sum S_N.* We concentrate on S_N and reverse the summations,

$$S_N = \sum_{j=0}^{N-1} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+ma} \sum_{n=j}^{N-1} (-1)^n \frac{B_{n+1}}{n+1} \frac{(\log z)^{n-j}}{(n-j)!}$$

Let $x = \log z$ and use $\frac{d^j}{dx^j} x^n = \frac{n!}{(n-j)!} x^{n-j}$, then

$$\sum_{n=j}^{N-1} (-1)^n \frac{B_{n+1}}{n+1} \frac{x^{n-j}}{(n-j)!} = \sum_{n=0}^{N-1} (-1)^n \frac{B_{n+1}}{n+1} \frac{1}{n!} \frac{d^j}{dx^j} x^n$$

and

$$\begin{aligned}\lim_{N\to\infty} \sum_{n=j}^{N-1} (-1)^n \frac{B_{n+1}}{n+1} \frac{x^{n-j}}{(n-j)!} &= \frac{d^j}{dx^j} \sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} (-x)^n \\ &= \frac{d^j}{dx^j} \frac{1}{x} \sum_{n=1}^{\infty} \frac{B_n}{n!} (-x)^n = \frac{d^j}{dx^j} \frac{1}{x} \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} (-x)^n - 1 \right)\end{aligned}$$

The generating function (168) of the Bernoulli numbers B_n,

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad \text{convergent for } |t| < 2\pi$$

leads, for $|x| < 2\pi$, to

$$\sum_{n=j}^{\infty} (-1)^n \frac{B_{n+1}}{n+1} \frac{x^{n-j}}{(n-j)!} = \frac{d^j}{dx^j} \frac{1}{x} \left(\frac{-x}{e^{-x} - 1} - 1 \right) = \frac{d^j}{dx^j} \left(\frac{1}{1 - e^{-x}} - x^{-1} \right)$$

For $N \to \infty$ and $|\log z| < 2\pi$, we obtain

$$S_\infty = \sum_{j=0}^{\infty} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+ma} \sum_{n=j}^{\infty} (-1)^n \frac{B_{n+1}}{n+1} \frac{(\log z)^{n-j}}{(n-j)!}$$
$$= \sum_{j=0}^{\infty} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+ma} \frac{d^j}{dx^j} \left(\frac{1}{1-e^{-x}} - x^{-1} \right)\Big|_{x=\log z}$$

Using the Fermi-Dirac integrals in **art**. 45, $F_{-j-1}(y) = \frac{d^j}{dy^j}\left(\frac{1}{1+e^{-y}}\right)$ and $\frac{d^j}{dx^j} x^{-1} = (-1)^j \, j! x^{-1-j}$, we have

$$S_\infty = \sum_{j=0}^{\infty} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+ma} F_{-j-1}(\log z + i\pi) - \frac{1}{\log z} \sum_{j=0}^{\infty} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+ma} \left(-\frac{a}{\log z}\right)^j$$

We recognize from (106) that, for $|z| < 1$,

$$E_{a,b}\left(-ze^{i\pi}\right) = E_{a,b}(z) = \frac{1}{\Gamma(b)} - \sum_{j=0}^{\infty} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b} F_{-j-1}(\log z + i\pi)$$

and from the series (143) for $I_{a,b}(z)$, we obtain, for $\left|\frac{a}{\log z}\right| < 1$,

$$S_\infty = -E_{a,b+ma}(z) + \frac{1}{\Gamma(b+ma)} + I_{a,b+ma}(z) \tag{63}$$

which is thus valid for $|z| < 1$ and $\left|\frac{a}{\log z}\right| < 1$.

The Euler-Maclaurin expansion (62) becomes, for $N \to \infty$ and with (63) and assuming that $R_N \to 0$,

$$z^{m+1} E_{a,b+(m+1)a}(z) + z^m E_{a,b+ma}(z) = 2z^m I_{a,b+ma}(z) + \frac{z^m}{\Gamma(b+ma)}$$

With (14), we arrive, for $|z| < 1$, $\left|\frac{a}{\log z}\right| < 1$ but all b, at

$$E_{a,b}(z) = z^m I_{a,b+ma}(z) + \sum_{l=0}^{m} \frac{z^l}{\Gamma(b+la)} \tag{64}$$

Numerical computations for $m \le 20$ show that (64) is increasingly accurate for increasing m as long as $\left|\frac{a}{\log z}\right| < 1$, irrespective of b. On the other hand, when $\left|\frac{a}{\log z}\right| > 1$, increasing m (up to 20) show a decreasing accuracy. Since $\lim_{m\to\infty} z^m I_{a,b+ma}(z) = 0$, (64) reduces to an identity when $m \to \infty$. Anticipating (140), the Euler-Maclaurin sum (61) becomes

$$\sum_{k=m+1}^{\infty} \frac{z^k}{\Gamma(b+ak)} = \frac{z^{\frac{1-b}{a}}}{a} \left\{ e^{z^{\frac{1}{a}}} + \int_0^\infty \frac{e^{-z^{\frac{1}{a}} x}}{x^{b+ma}} \left(\frac{\frac{\sin(b+ma)\pi}{\pi} \ln x + \cos(b+ma)\pi}{\pi^2 + (\ln x)^2} \right) dx \right\}$$
$$+ z^m \sum_{n=0}^{N-1} (-1)^n \frac{B_{n+1}}{n+1} \sum_{j=0}^{n} \frac{a^j}{j!} \frac{d^j}{dy^j} \frac{1}{\Gamma(y)}\Big|_{y=b+am} \frac{(\log z)^{n-j}}{(n-j)!} + R_N \tag{65}$$

Numerical computations (for $m=-1$) indicate that the n-summation converges for small a and small z, with fast convergence when $z=1$. For $z=1$, (65) simplifies to

$$\sum_{k=m+1}^{\infty}\frac{1}{\Gamma(b+ak)}=\frac{1}{a}\left\{e+\int_0^\infty\frac{e^{-x}}{x^{b+ma}}\left(\frac{\frac{\sin(b+ma)\pi}{\pi}\ln x+\cos(b+ma)\pi}{\pi^2+(\ln x)^2}\right)dx\right\}$$
$$+\sum_{n=0}^{\infty}\frac{B_{n+1}(-a)^n}{(n+1)!}\left.\frac{d^n}{dy^n}\frac{1}{\Gamma(y)}\right|_{y=b+am}$$

where the latter sum converges for $|a|<1$.

6 Complex integrals for $E_{a,b}(z)$

Two different complex integrals for the Mittag-Leffler function $E_{a,b}(z)$ are discussed. The first in Section 6.1, called the basic complex integral (66), follows from the inverse Laplace transform of (54). The second complex integral (78) in Section 6.2 is an instance of a Plana-like summation formula. Section 6.3 derives further forms of the integral (86) for $E_a(-z)$, which is the special for $b=1$ and $0<a<1$ of the integral (85).

6.1 Basic complex integral

26. *Basic complex integral for* $E_{a,b}(z)$. Inverse Laplace transformation (193) of (54) yields, with $c'>|x|^{\frac{1}{a}}$ for $\operatorname{Re}(b)>0$,

$$t^{b-1}E_{a,b}(xt^a)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{s^{a-b}e^{st}}{s^a-x}ds$$

where the integrand is analytic for $\operatorname{Re}(s)>c'$. For real $t\geq 0$, we can move the line of integration to $c>t^a|x|=t^a(c')^a$ and perform an ordinary substitution $w=st$. Let $z=xt^a$, then we arrive, for $\operatorname{Re}(b)>0$ and $\operatorname{Re}(a)>0$, at the *basic complex integral*

$$E_{a,b}(z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{w^{a-b}e^w}{w^a-z}dw\qquad c>|z| \tag{66}$$

The basic complex integral (66) of $E_{a,b}(z)$ can also be deduced from Hankel's deformed integral (180) in the power series of the Mittag-Leffler function (1)

$$E_{a,b}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(b+ak)}=\frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{C_\phi}\left(zw^{-a}\right)^k w^{-b}e^w dw=\frac{1}{2\pi i}\int_{C_\phi}\sum_{k=0}^{\infty}\left(zw^{-a}\right)^k w^{-b}e^w dw$$

Only if $|zw^{-a}|<1$, implying $|z|<|w^a|$ for any w along the contour C_ϕ described in **art**. 70, then we obtain

$$E_{a,b}(z)=\frac{1}{2\pi i}\int_{C_\phi}\frac{w^{-b}e^w}{1-zw^{-a}}dw \tag{67}$$

where the radius ρ of the circle at $w=0$ in the contour C_ϕ (see **art**. 70) can be appropriately chosen to satisfy $|z|<\rho^{\operatorname{Re}a}$ or $|z|^{\frac{1}{\operatorname{Re}a}}<\rho$ for any z. We obtain again (66) by choosing $\phi=\frac{\pi}{2}$ and $c=\rho$.

27. *Mittag-Leffler function $E_{a,b}(z)$ for negative real a.* Although $E_{a,b}(z)$, defined by the power series (1), is not valid for negative $\operatorname{Re}(a)$ values, the Taylor series (106) of $E_{a,b}(z)$ in a around $a_0 = 0$,

$$E_{a,b}(-z) = \frac{1}{\Gamma(b)} - \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{du^m} \frac{1}{\Gamma(u)}\bigg|_{u=b} F_{-m-1}(\log z) a^m$$

and its companion (108)

$$E_{a,b}(-z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{du^m} \frac{1}{\Gamma(u)}\bigg|_{u=b} F_{-m-1}(-\log z)(-a)^m$$

indicate that

$$E_{a,b}\left(-\frac{1}{z}\right) = \frac{1}{\Gamma(b)} - E_{-a,b}(-z)$$

which may be used, by analytic continuation, as a definition of $E_{a,b}(z)$ for $\operatorname{Re}(a) < 0$. Thus, for $\operatorname{Re}(a) < 0$, the Mittag-Leffler function $E_{a,b}(z)$ possesses an essential singularity at $z = 0$ and is not an entire function.

The same result is also deduced in [17, Sec. 4.8, pp. 80-82] from the basic complex integral (67) in **art**. 26

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_{C_\phi} \frac{w^{a-b} e^w}{w^a - z} dw$$

is valid for any complex a, b and z. Introducing the identity

$$\frac{w^{a-b}}{w^a - z} = \frac{1}{w^b - z w^{b-a}} = \frac{1}{w^b} - \frac{1}{w^b - \frac{1}{z} w^{a+b}}$$

leads to

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_{C_\phi} \frac{e^w}{w^b} dw - \frac{1}{2\pi i} \int_{C_\phi} \frac{w^{-a-b} e^w}{w^{-a} - \frac{1}{z}} dw$$

Invoking Hankel's integral (181) to the first integral and the basic complex integral (67) to the last integral leads again to

$$E_{a,b}(z) = \frac{1}{\Gamma(b)} - E_{-a,b}\left(\frac{1}{z}\right) \tag{68}$$

The Taylor series (1) of $E_{a,b}(z)$ then shows that

$$E_{-a,b}(z) = \frac{1}{\Gamma(b)} - E_{a,b}\left(\frac{1}{z}\right) = -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(b + ak)}$$

28. *Evaluation of the basic complex integral for $E_{a,b}(z)$.* We will now evaluate the integral (66) by closing the contour over the negative $\operatorname{Re}(w)$-plane. Consider the closed loop integral

$$\frac{1}{2\pi i} \int_L \frac{w^{a-b} e^w}{w^a - z} dw$$

where the contour L consists of the vertical line at $w = c + it$, the circle segment at infinity turning from $\frac{\pi}{2}$ towards $\pi - \varepsilon$, the line above the negative real axis, the small circle with radius ε turning around the origin from $\pi - \varepsilon$ to $-\pi + \varepsilon$, the line from the origin just below the negative real axis, over the circle segment with infinite radius from $-\pi + \varepsilon$ to $-\frac{\pi}{2}$ and ending at the negative side of the vertical line. The integrand has only singularities of $(w^a - z)^{-1}$, which are simple poles because the zeros of $w^a - z = 0$ are all simple and lie at $w_k = |z|^{\frac{1}{a}} e^{i\frac{\arg z}{a}} e^{i\frac{2\pi k}{a}}$ for each $k \in \mathbb{Z}$. If a is irrational, there are infinitely many zeros. The contour L only encloses the poles with argument $\theta_k = \frac{\arg z}{a} + \frac{2\pi k}{a}$ between $-\pi \le \theta_k \le \pi$. Thus, the integer k ranges from

$$-\left\lfloor \frac{a}{2} + \frac{\arg z}{2\pi} \right\rfloor \le k \le \left\lfloor \frac{a}{2} - \frac{\arg z}{2\pi} \right\rfloor$$

and there are precisely $\lfloor a \rfloor$ enclosed poles, where $\lfloor v \rfloor$ is the integer smaller than or equal to v. Hence, a must be at least equal to 1, else no singularities are enclosed. By Cauchy's residue theorem, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_L \frac{w^{a-b}e^w}{w^a - z}dw &= \sum_{k\in\left[-\left\lfloor \frac{a}{2}+\frac{\arg z}{2\pi}\right\rfloor, \left\lfloor \frac{a}{2}-\frac{\arg z}{2\pi}\right\rfloor\right]} \lim_{w\to w_k} \frac{w - w_k}{w^a - z} w^{a-b}e^w \\ &= \frac{1}{a} \sum_{k\in\left[-\left\lfloor \frac{a}{2}+\frac{\arg z}{2\pi}\right\rfloor, \left\lfloor \frac{a}{2}-\frac{\arg z}{2\pi}\right\rfloor\right]} w_k^{1-b} e^{w_k} \qquad \text{with } w_k = |z|^{\frac{1}{a}} e^{i\frac{\arg z}{a}} e^{i\frac{2\pi k}{a}}\end{aligned}$$

We can always choose $0 < \varepsilon < r$ small enough, provided that $\mathrm{Re}\,(a-b) > -1$. Evaluation of the contour L yields

$$\begin{aligned}\frac{1}{2\pi i}\int_L \frac{w^{a-b}e^w}{w^a - z}dw &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{w^{a-b}e^w}{w^a - z}dw + \frac{1}{2\pi i}\left\{\int_\infty^0 \frac{\left(ye^{i\pi}\right)^{a-b}e^{-y}}{\left(ye^{i\pi}\right)^a - z}d\left(ye^{i\pi}\right) + \int_0^\infty \frac{\left(ye^{-i\pi}\right)^{a-b}e^{-y}}{\left(ye^{-i\pi}\right)^a - z}d\left(ye^{-i\pi}\right)\right\} \\ &= E_{a,b}(z) - \frac{1}{2\pi i}\int_0^\infty \left(\frac{e^{-i\pi(a-b)}}{y^a e^{-i\pi a} - z}dy - \frac{e^{i\pi(a-b)}}{y^a e^{i\pi a} - z}\right) y^{a-b}e^{-y}dy\end{aligned}$$

Hence, for $\mathrm{Re}\,(a-b) > -1$, we have

$$E_{a,b}(z) = \frac{1}{2\pi i}\int_0^\infty \left(\frac{e^{-(a-b)i\pi}}{e^{-ai\pi}y^a - z} - \frac{e^{(a-b)i\pi}}{e^{ai\pi}y^a - z}\right) y^{a-b}e^{-y}dy + \frac{1}{a} \sum_{k\in\left[-\left\lfloor \frac{a}{2}+\frac{\arg z}{2\pi}\right\rfloor, \left\lfloor \frac{a}{2}-\frac{\arg z}{2\pi}\right\rfloor\right]} w_k^{1-b} e^{w_k}$$

Finally, with $\frac{e^{-(a-b)i\pi}}{e^{-ai\pi}y^a - z} - \frac{e^{(a-b)i\pi}}{e^{ai\pi}y^a - z} = 2i\frac{\sin\pi b y^a + z\sin\pi(a-b)}{y^{2a} - 2\cos a\pi z y^a + z^2}$, we arrive for $\mathrm{Re}\,(a) > \mathrm{Re}\,(b) - 1$ at a fundamental formula

$$\begin{aligned}E_{a,b}(z) &= z\frac{\sin(a-b)\pi}{\pi}\int_0^\infty \frac{y^{a-b}e^{-y}dy}{y^{2a} - 2zy^a\cos a\pi + z^2} + \frac{\sin\pi b}{\pi}\int_0^\infty \frac{y^{2a-b}e^{-y}}{y^{2a} - 2zy^a\cos a\pi + z^2}dy \\ &\quad + \frac{z^{\frac{1-b}{a}}}{a} \sum_{k\in\left[-\left\lfloor \frac{a}{2}+\frac{\arg z}{2\pi}\right\rfloor, \left\lfloor \frac{a}{2}-\frac{\arg z}{2\pi}\right\rfloor\right]} e^{i\frac{2\pi k}{a}(1-b)} e^{|z|^{\frac{1}{a}} e^{i\frac{\arg z}{a}} e^{i\frac{2\pi k}{a}}}\end{aligned} \tag{69}$$

The last residue sum illustrates again that $E_{a,b}(z)$ is an entire function of order $\frac{1}{a}$. The sign of $w_k = |z|^{\frac{1}{a}} e^{i\frac{\arg z}{a}} e^{i\frac{2\pi k}{a}}$ is determined by $\cos\left(\frac{\arg z + 2\pi k}{a}\right)$. Consequently for large $|z|$, in the sectors $-\frac{\pi a}{2} - 2\pi k < \arg z < \frac{\pi a}{2} - 2\pi k$, the function $E_{a,b}(z)$ tends to infinity, while in the complementary sectors $\frac{\pi a}{2} - 2\pi k < \arg z < \frac{3\pi a}{2} - 2\pi k$, $E_{a,b}(z) \to 0$.

29. *Discussion of (69).* If $a = m$ and $b = n$ are integers, then $\sin(a-b)\pi = \sin\pi b = 0$, the integrals in (69) disappear and (69) simplifies to

$$E_{m,n}(z) = \frac{z^{\frac{1-n}{m}}}{m} \sum_{k\in\left[-\left\lfloor \frac{m}{2}+\frac{\arg z}{2\pi}\right\rfloor, \left\lfloor \frac{m}{2}-\frac{\arg z}{2\pi}\right\rfloor\right]} e^{i\frac{2\pi k}{m}(1-n)} e^{z^{\frac{1}{m}} e^{i\frac{2\pi k}{m}}}$$

which equals (11). The contribution of the integrals in (69), studied further in **art**. 34-36, can be regarded for a real pair (a,b) as an interpolator between integer pairs (m,n).

For real $z=-x^a=x^a e^{i\pi}$ in (69) and $w_k = xe^{i\frac{(2k+1)\pi}{a}}$, we find

$$E_{a,b}\left(-x^a\right) = -x^a \frac{\sin\left(a-b\right)\pi}{\pi}\int_0^\infty \frac{y^{a-b}e^{-y}dy}{y^{2a}+2x^a y^a\cos a\pi + x^{2a}} + \frac{\sin \pi b}{\pi}\int_0^\infty \frac{y^{2a-b}e^{-y}}{y^{2a}+2x^a y^a \cos a\pi + z^2}dy$$
$$+\frac{x^{1-b}}{a}e^{i\left\{\frac{\pi(1-b)}{a}\right\}}\sum_{k=0}^{\lfloor a\rfloor} e^{x\cos\frac{(2k-1)\pi}{a}} e^{i\left\{-x\sin\frac{(2k-1)\pi}{a}-\frac{2k\pi(1-b)}{a}\right\}}$$

For real x, $b=1$ and $0<a<1$, the condition to enclose a pole is $-a\pi \le \arg z + 2\pi k \le a\pi$. Since $\arg z = \pi \notin [-a\pi, a\pi]$ for $0<a<1$, the residue sum disappears and we obtain

$$E_a\left(-x^a\right) = x^a \frac{\sin a\pi}{\pi}\int_0^\infty \frac{y^{a-1}e^{-y}dy}{y^{2a}+2x^a y^a\cos a\pi + x^{2a}} \qquad \text{for } 0<a<1 \tag{70}$$

from which the asymptotic behavior for large x, with $y^{2a}+2x^a y^a\cos a\pi + x^{2a}\sim x^{2a}$, are

$$E_a\left(-x^a\right)\sim \frac{\sin a\pi}{\pi}\frac{\Gamma\left(a\right)}{x^a}$$

Art. 34 presents another derivation in (86) that is equivalent to (70). **Art**. 46 verifies the integral in (69), thus only for $0<a<1$, via a series approach. For small, real x, the series (1) gives

$$E_a\left(-x^a\right) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{ak}}{\Gamma\left(1+ak\right)} = 1-\frac{x^a}{\Gamma\left(1+a\right)}+O\left(x^{2a}\right) = \exp\left(-\frac{x^a}{\Gamma\left(1+a\right)}\right)+O\left(x^{2a}\right)$$

Hence, $E_a\left(-x^a\right)$ for $0<a<1$ is said to “interpolate” between an exponential (for small x) and a power law (for large x). These two regimes have been studied by Mainardi [28], who illustrated their accuracy with several plots for $a=0.25, 0.5, 0.75, 0.9$ and 0.99.

30. *Mittag-Leffler's contour integral.* In a rather long article of 1905, Mittag-Leffler [35, at p. 133-135] proceeds one step further and substitutes $w=t^{\frac{1}{a}}$ in the basic complex integral (66). The map $w\to t^{\frac{1}{a}}$ is multi-valued. For $w=|w|\,e^{i\theta_w}$ with $-\pi\le\theta_w\le\pi$, the inverse map $t\to w^a$ shows that $t=|t|\,e^{i\theta_t}=|w|^a\,e^{ia\theta_w+2\pi iak}$, from which the argument $\theta_t = a\theta_w + 2\pi ak$ for any integer k. The contour C_ϕ in the w-plane requires that $\frac{\pi}{2}<|\phi|<\pi$, because $e^w\to 0$ for large w at the contour C_ϕ. Similarly, the transformed contour $C_{a\phi}$ in the t-plane requires that $\frac{a\pi}{2}<|\arg t|<a\pi$ in order that $e^{t^{\frac{1}{a}}}\to 0$ along the straight lines towards infinity. Thus, the map $w\to t^{\frac{1}{a}}$ changes the angles from ϕ to $a\phi$ of the straight lines in the contour (Fig. 43 in Bieberbach's book [7, p. 273]). Moreover, we must choose the branch (i.e. the appropriate integer k in $\theta_t = a\theta_w+2\pi ak$) of the function $t^{\frac{1}{a}}$ that is positive for positive t, i.e. along the positive real t-axis, because the same holds along the positive real w-axis. Performing the substitution $w=t^{\frac{1}{a}}$ in (66) leads, for $|z|<|t|$ and $|\arg z|<\frac{a\pi}{2}$, to Mittag-Leffler's contour integral

$$E_{a,b}\left(z\right) = \frac{1}{2\pi ia}\int_{C_{a\phi}}\frac{t^{\frac{1-b}{a}}e^{t^{\frac{1}{a}}}}{t-z}dt \qquad \text{with } \frac{\pi}{2}<|\phi|<\pi \tag{71}$$

Mittag-Leffler's integral (71) for $b=1$ is actually more elegant than the basic complex integral (66) at the expense of a more complicated contour $C_{a\phi}$.

31. *Deductions from Mittag-Leffler's contour integral (71) for $E_{a,b}(z)$.* Based on (71), we follow Bieberbach [7, p. 273], who deduces two bounds for $0 < a < 2$ and $b = 1$. First, using

$$\frac{1}{t-z} = -\frac{1}{z} + \frac{t}{z(t-z)}$$

in (71), we have

$$\begin{aligned} E_{a,b}(z) &= -\frac{1}{z}\frac{1}{2\pi ia}\int_{C_{a\phi}} t^{\frac{1-b}{a}} e^{t^{\frac{1}{a}}} dt + \frac{1}{2\pi ia}\int_{C_{a\phi}} \frac{t^{\frac{1-b+a}{a}} e^{t^{\frac{1}{a}}}}{z(t-z)} dt \\ &= -\frac{1}{z}\frac{1}{2\pi i}\int_{C_{\phi}} w^{a-b} e^{w} dw + \frac{1}{2\pi ia}\int_{C_{a\phi}} \frac{t^{\frac{a-b-1}{a}} e^{t^{\frac{1}{a}}}}{z(t-z)} dt \end{aligned}$$

With Hankel's contour integral (180),

$$E_{a,b}(z) = -\frac{1}{z}\frac{1}{\Gamma(b-a)} + \frac{1}{2\pi ia}\int_{C_{a\phi}} \frac{t^{\frac{a-b+1}{a}} e^{t^{\frac{1}{a}}}}{z(t-z)} dt$$

The remaining integral is upper bounded by

$$\left| \frac{1}{2\pi ia}\int_{C_{a\phi}} \frac{t^{\frac{a-b+1}{a}} e^{t^{\frac{1}{a}}}}{z(t-z)} dt \right| < \frac{1}{2\pi a|z|}\int_{C_{a\phi}} \frac{|t|^{\frac{a-b+1}{a}} \left| e^{t^{\frac{1}{a}}} \right|}{|z|\left|1-\frac{t}{z}\right|} |dt| = \frac{c}{|z|^2}$$

because $\frac{1}{2\pi a}\int_{C_{a\phi}} \frac{|t|^{\frac{a-b-1}{a}}\left|e^{t^{\frac{1}{a}}}\right|}{\left|1-\frac{t}{z}\right|}|dt|$ converges for $t \to \infty$ if $\left|e^{t^{\frac{1}{a}}}\right| = e^{|t|^{\frac{1}{a}}\cos\frac{\arg t}{a}} \to 0$ for large t, which requires that $\frac{\arg t}{a} \geq \frac{\pi}{2}$. In addition, we must prevent that $\frac{t}{z}$ for large z can tend arbitrarily close to 1, which is guaranteed if $|\arg z| \notin \left(\frac{a\pi}{2}, a\pi\right)$, because $|\arg t| \in \left(\frac{a\pi}{2}, a\pi\right)$. Hence, for $|\arg z| \notin \left(\frac{a\pi}{2}, a\pi\right)$, we arrive at

$$\left| E_{a,b}(z) + \frac{1}{z\Gamma(b-a)} \right| < \frac{c}{|z|^2}$$

For the second bound, Bieberbach [7, p. 275] cleverly observes that a similar integration path $C'_{a\phi}$ as in Mittag-Leffler's integral (71) can be followed with the only difference that the circular part of the path now has a radius smaller than $|z|$. In other words, while $|z| < |t|$ in Mittag-Leffler's integral (71), the path $C'_{a\phi}$ now turns over an angle $-a\phi$ to $a\phi$ with the radius smaller than $|z|$. The closed contour (see also [44, p. 346, Fig. 6.13-2]), that first follows the Mittag-Leffler path $C_{a\phi}$ and returns via the path $C'_{a\phi}$, encloses the point $t = z$ as the only singularity, provided $-a\phi < \arg z < a\phi$. Hence, by Cauchy's residue theorem, it holds that

$$\frac{1}{2\pi ia}\int_{C_{a\phi}} \frac{t^{\frac{1-b}{a}} e^{t^{\frac{1}{a}}}}{t-z} dt - \frac{1}{2\pi ia}\int_{C'_{a\phi}} \frac{t^{\frac{1-b}{a}} e^{t^{\frac{1}{a}}}}{t-z} dt = \frac{1}{a} z^{\frac{1-b}{a}} e^{z^{\frac{1}{a}}}$$

and

$$E_{a,b}(z) = \frac{1}{2\pi ia}\int_{C'_{a\phi}} \frac{t^{\frac{1-b}{a}} e^{t^{\frac{1}{a}}}}{t-z} dt + \frac{1}{a} z^{\frac{1-b}{a}} e^{z^{\frac{1}{a}}}$$

from which

$$\left| E_{a,b}(z) - \frac{1}{a} z^{\frac{1-b}{a}} e^{z^{\frac{1}{a}}} \right| \leq \frac{1}{2\pi a|z|}\int_{C'_{a\phi}} \frac{\left|t^{\frac{1-b}{a}}\right| \left|e^{t^{\frac{1}{a}}}\right|}{\left|\frac{t}{z}-1\right|} dt = \frac{c'}{|z|}$$

by the same convergence argument as above, where now on the circular segment $|t| < |z|$ can be chosen small enough. In summary, the second bound for $|\arg z| \in \left(\frac{a\pi}{2}, a\pi\right)$ is

$$\left| E_{a,b}(z) - \frac{1}{a} z^{\frac{1-b}{a}} e^{z^{\frac{1}{a}}} \right| < \frac{c'}{|z|}$$

The derivation illustrates why Bieberbach considers $0 < a < 2$, because $|\arg z| \in (0, 2\pi)$.

The second bound shows that $aE_{a,b}(z^a) \approx z^{1-b} e^z$ is independent of a so that $aE_{a,b}(z^a) \approx \frac{1}{a} E_{\frac{1}{a},b}\left(z^{\frac{1}{a}}\right)$. Hence, we are led for non-negative real z and $|z| > \varepsilon$ to

$$E_{\frac{1}{a},b}(z) \approx a^2 E_{a,b}\left(z^{a^2}\right)$$

whose exact corresponding relation (135) for the associated integral $I_{a,b}(z) = \int_0^\infty \frac{z^u}{\Gamma(b+au)} du$, explored in Section 9, is $I_{\frac{1}{a},b}(z) = a^2 I_{a,b}\left(z^{a^2}\right)$.

32. *Evaluation of the basic complex integral along the line $w = c + it$.* The *basic complex integral* in (66) is evaluated along the straight line $w = c + it$ as

$$E_{a,b}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(c+it)^{a-b} e^{c+it}}{(c+it)^a - z} dt \qquad c > |z|$$

Since $c + it = \sqrt{c^2+t^2} e^{i \arctan \frac{t}{c}}$, we have

$$E_{a,b}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left(c^2+t^2\right)^{\frac{a-b}{2}} e^{i(a-b)\arctan \frac{t}{c}} e^{c+it}}{\left(c^2+t^2\right)^{\frac{a}{2}} e^{ia \arctan \frac{t}{c}} - |z| e^{i\theta}} dt \qquad c > |z|$$

We split the integration interval into two parts

$$E_{a,b}(z) = \frac{1}{2\pi} \int_{-\infty}^{0} \frac{\left(c^2+t^2\right)^{\frac{a-b}{2}} e^{i(a-b)\arctan \frac{t}{c}} e^{c+it}}{\left(c^2+t^2\right)^{\frac{a}{2}} e^{ia \arctan \frac{t}{c}} - |z| e^{i\theta}} dt + \frac{1}{2\pi} \int_{0}^{\infty} \frac{\left(c^2+t^2\right)^{\frac{a-b}{2}} e^{i(a-b)\arctan \frac{t}{c}} e^{c+it}}{\left(c^2+t^2\right)^{\frac{a}{2}} e^{ia \arctan \frac{t}{c}} - |z| e^{i\theta}} dt$$

and change the integration parameter in the first integral from t to $-t$,

$$E_{a,b}(z) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{\left(c^2+t^2\right)^{\frac{a-b}{2}} e^{-i(a-b)\arctan \frac{t}{c}} e^{c-it}}{\left(c^2+t^2\right)^{\frac{a}{2}} e^{-ia \arctan \frac{t}{c}} - |z| e^{i\theta}} dt + \frac{1}{2\pi} \int_{0}^{\infty} \frac{\left(c^2+t^2\right)^{\frac{a-b}{2}} e^{i(a-b)\arctan \frac{t}{c}} e^{c+it}}{\left(c^2+t^2\right)^{\frac{a}{2}} e^{ia \arctan \frac{t}{c}} - |z| e^{i\theta}} dt$$

We simplify the integrand further and we find, for $c > |z|$,

$$E_{a,b}(z) = \frac{e^c}{\pi} \int_0^\infty \left(c^2+t^2\right)^{\frac{a-b}{2}} \left(\frac{\left(c^2+t^2\right)^{\frac{a}{2}} \cos\left(t - b \arctan \frac{t}{c}\right) - z \cos\left(t + (a-b)\arctan \frac{t}{c}\right)}{\left(c^2+t^2\right)^a - 2z \left(c^2+t^2\right)^{\frac{a}{2}} \cos\left(a \arctan \frac{t}{c}\right) + z^2} \right) dt$$

After substitution $u = \frac{t}{c}$, we arrive, for $\operatorname{Re}(b) > 0$ and $\operatorname{Re}(a) > 0$ and for $c > |z|$, at

$$E_{a,b}(z) = \frac{e^c c^{1-b}}{\pi} \int_0^\infty \left(1+u^2\right)^{\frac{a-b}{2}} \left(\frac{\left(1+u^2\right)^{\frac{a}{2}} \cos(cu - b \arctan u) - \frac{z}{c^a} \cos(cu + (a-b)\arctan u)}{\left(1+u^2\right)^a - 2\frac{z}{c^a} \left(1+u^2\right)^{\frac{a}{2}} \cos(a \arctan u) + \left(\frac{z}{c^a}\right)^2} \right) du \tag{72}$$

For $z = 0$ and $c > 0$, (72) simplifies to

$$E_{a,b}(0) = \frac{1}{\Gamma(b)} = \frac{e^c c^{1-b}}{\pi} \int_0^\infty \frac{\cos(cu - b \arctan u)}{\left(1+u^2\right)^{\frac{b}{2}}} du \tag{73}$$

After choosing $c = 1$, we obtain

$$\frac{1}{\Gamma(b)} = \frac{e}{\pi}\int_0^\infty \frac{\cos(t - b\arctan t)}{(1+t^2)^{\frac{b}{2}}}dt$$

The positive real number $c > |z|$, which is a tuneable parameter, makes the integral (72) interesting. For $z \neq 0$, there are several choices for $c > |z|$ in (72). For example, we can choose $c = \beta|z|$ with real $\beta > 1$ or simply $c = |z| + \beta$. The more interesting choice for $z \neq 0$ is either (1) $c = |z|^{\frac{1}{a}} > |z|$, for $0 < a < 1$ and $|z| > 1$ or for $a > 1$ and $|z| < 1$ or (2) $c = |z|^a > |z|$, which holds for $a > 1$ and $|z| > 1$ or for $0 < a < 1$ and $|z| < 1$.

(1) The choice $c = |z|^{\frac{1}{a}}$ in (72), valid for $\{0 < a < 1$ and $|z| > 1\}$ or $\{a > 1$ and $|z| < 1\}$, becomes with $z = |z|e^{i\theta}$

$$E_{a,b}(z) = \frac{e^{|z|^{\frac{1}{a}}}|z|^{\frac{1-b}{a}}}{\pi}\int_0^\infty \frac{(1+u^2)^{\frac{a}{2}}\cos\left(|z|^{\frac{1}{a}}u - b\arctan u\right) - e^{i\theta}\cos\left(|z|^{\frac{1}{a}}u + (a-b)\arctan u\right)}{(1+u^2)^{\frac{b-a}{2}}\left((1+u^2)^a - 2e^{i\theta}(1+u^2)^{\frac{a}{2}}\cos(a\arctan u) + (e^{i\theta})^2\right)}du \tag{74}$$

The prefactor $e^{|z|^{\frac{1}{a}}}|z|^{\frac{1-b}{a}}$ in (74) gives the correct order of magnitude (see Bieberbach's second bound in **art**. 31). A conservative upper bound (74) is

$$|E_{a,b}(z)| \leq \frac{e^{|z|^{\frac{1}{a}}}|z|^{\frac{1-b}{a}}}{\pi}\int_0^\infty \frac{(1+u^2)^{\frac{a}{2}} + 1}{(1+u^2)^{\frac{b-a}{2}}\left((1+u^2)^{\frac{a}{2}} - 1\right)^2}du$$

But, $\int_0^\infty \frac{\left((1+u^2)^{\frac{a}{2}}+1\right)du}{(1+u^2)^{\frac{b-a}{2}}\left((1+u^2)^{\frac{a}{2}}-1\right)^2} > \int_0^\infty \frac{(1+u^2)^{\frac{a}{2}}du}{(1+u^2)^{\frac{b-a}{2}}\left((1+u^2)^{\frac{a}{2}}\right)^2} = \int_0^\infty \frac{du}{(1+u^2)^{\frac{b}{2}}} = \frac{\sqrt{\pi}\Gamma\left(\frac{b-1}{2}\right)}{2\Gamma\left(\frac{b}{2}\right)}$. The last step follows from the Beta-function integral (157): $B(p,q) = \int_0^\infty \frac{t^{p-1}dt}{(1+t)^{p+q}} = 2\int_0^\infty \frac{u^{2p-1}du}{(1+u^2)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ for $p = \frac{1}{2}$ and $q = \frac{b-1}{2}$. Hence, we may approximate $|E_{a,b}(z)| \approx \frac{\Gamma\left(\frac{b-1}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{b}{2}\right)}e^{|z|^{\frac{1}{a}}}|z|^{\frac{1-b}{a}}$ illustrating that the integral in (74) is weakly dependent on a and z, agreeing with Bieberbach's estimate.

(2) The choice $c = |z|^a$ in (72), valid for $\{0 < a < 1$ and $|z| < 1\}$ or $\{a > 1$ and $|z| > 1\}$, becomes

$$E_{a,b}(z) = \frac{e^{|z|^a}|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{(1+u^2)^{\frac{a}{2}}\cos(|z|^a u - b\arctan u) - \frac{z}{|z|^{a^2}}\cos(|z|^a u + (a-b)\arctan u)}{(1+u^2)^{\frac{b-a}{2}}\left((1+u^2)^a - 2\frac{z}{|z|^{a^2}}(1+u^2)^{\frac{a}{2}}\cos(a\arctan u) + \left(\frac{z}{|z|^{a^2}}\right)^2\right)}du \tag{75}$$

The appearing ratio $\left|\frac{z}{|z|^{a^2}}\right|$ is smaller than 1, i.e. $\left|\frac{z}{|z|^{a^2}}\right| < 1$ in both regimes $\{0 < a < 1$ and $|z| < 1\}$ and $\{a > 1$ and $|z| > 1\}$. However, in the regime $\{0 < a < 1$ and $|z| < 1\}$, $|z|^{a^2} \to 1$ and $\frac{z}{|z|^{a^2}} \to z$ for $a \to 0$, whereas in the regime $\{a > 1$ and $|z| > 1\}$, $|z|^{a^2} \to \infty$ and $\frac{z}{|z|^{a^2}} \to 0$ for $a \to \infty$. In the latter regime $\{a > 1$ and $|z| > 1\}$ for sufficiently large a, the denominator is expanded with $\frac{1}{1-x} =$

$1+x+O\left(x^2\right)$ and the integral (75) equals

$$E_{a,b}(z) = \frac{e^{|z|^a}\,|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{\cos\left(|z|^a\,u - b\arctan u\right)}{\left(1+u^2\right)^{\frac{b}{2}}}du$$
$$+\frac{z}{|z|^{a^2}}\frac{e^{|z|^a}\,|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{2\cos\left(|z|^a\,u - b\arctan u\right)\cos\left(a\arctan u\right) - \cos\left(|z|^a\,u + (a-b)\arctan u\right)}{\left(1+u^2\right)^{\frac{b+a}{2}}}du$$
$$+O\left(\left(\frac{z}{|z|^{a^2}}\right)^2\frac{e^{|z|^a}\,|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{\cos\left(|z|^a\,u - b\arctan u\right)}{\left(1+u^2\right)^{\frac{b}{2}}}du\right)$$

With

$$\begin{aligned} H &= 2\cos\left(|z|^a\,u - b\arctan u\right)\cos\left(a\arctan u\right) - \cos\left(|z|^a\,u + (a-b)\arctan u\right) \\ &= \cos\left(|z|^a\,u - (a+b)\arctan u\right) \end{aligned}$$

we simplify to

$$\begin{aligned} E_{a,b}(z) &= \frac{e^{|z|^a}\,|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{\cos\left(|z|^a\,u - b\arctan u\right)}{\left(1+u^2\right)^{\frac{b}{2}}}du \\ &+\frac{z}{|z|^{a^2}}\frac{e^{|z|^a}\,|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{\cos\left(|z|^a\,u - (a+b)\arctan u\right)}{\left(1+u^2\right)^{\frac{b+a}{2}}}du \\ &+O\left(\left(\frac{z}{|z|^{a^2}}\right)^2\frac{e^{|z|^a}\,|z|^{a(1-b)}}{\pi}\int_0^\infty \frac{\cos\left(|z|^a\,u - b\arctan u\right)}{\left(1+u^2\right)^{\frac{b}{2}}}du\right) \end{aligned}$$

Invoking (73) with $c=|z^a|$ shows that $\frac{e^{-|z^a|}|z|^{a(b-1)}}{\Gamma(b)} = \frac{1}{\pi}\int_0^\infty \frac{\cos(|z^a|u-b\arctan u)}{(1+u^2)^{\frac{b}{2}}}du$ and leads, for large a and $z>1$, to

$$E_{a,b}(z) = \frac{1}{\Gamma(b)}+\frac{z}{\Gamma(a+b)}+O\left(\left(\frac{z}{|z|^{a^2}}\right)^2\right)$$

which implies, compared to the Taylor series (1), $E_{a,b}(z) = \frac{1}{\Gamma(b)}+\frac{z}{\Gamma(a+b)}+\sum_{k=2}^\infty \frac{z^k}{\Gamma(ak+b)}$, that $\sum_{k=2}^\infty \frac{z^k}{\Gamma(ak+b)} = O\left(|z|^{2-2a^2}\right)$.

6.2 Complex integral for $E_{a,b}(z)$ deduced from Cauchy's residue theorem

33. *Deductions from Cauchy's residue theorem.* If $f(z)$ is an entire function and $\lim_{r\to\infty}\left|\frac{f\left(re^{i\theta}\right)}{\sin\pi re^{i\theta}}\right|\to 0$ in a semicircle with either $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$, then it follows directly from Cauchy's residue theorem [47] for $0<c<1$ that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi w}f(w)\,dw = \begin{cases} -\sum_{k=1}^\infty (-1)^k f(k) & \text{if } -\frac{\pi}{2}<\theta<\frac{\pi}{2} \\ \sum_{k=0}^\infty (-1)^k f(-k) & \text{if } \frac{\pi}{2}<\theta<\frac{3\pi}{2} \end{cases} \tag{76}$$

where if $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, the contour is closed over the positive $\mathrm{Re}(w)$-plane, else over the negative $\mathrm{Re}(w)$-plane. Equation (76) is similar to Plana's summation formula [44, p. 438]. The definition (1) of $E_{a,b}(z)=\sum_{k=0}^\infty \frac{z^k}{\Gamma(b+ak)}$ contains $f(w)=\frac{z^w}{\Gamma(b+aw)}$, which is an entire function in w. The asymptotic behavior (176) of $\frac{1}{\left|\Gamma\left(b+are^{i\theta}\right)\right|}=\frac{(ar)^{\frac{1}{2}-b}}{\sqrt{2\pi}}e^{-ar(\ln(ar)-1)\cos\theta+\theta ar\sin\theta}\left(1+O\left(\frac{1}{r}\right)\right)$ in **art**. 69 shows that the above contour can be closed over the positive $\operatorname{Re}(w)$-plane, resulting in[11]

$$E_{a,b}(z)=-\frac{1}{2\pi i}\int_{c+\infty e^{-i\theta}}^{c+\infty e^{i\theta}}\frac{\pi}{\sin\pi w}\frac{(-z)^w}{\Gamma(b+aw)}dw \qquad \text{for } -1<c<0 \tag{78}$$

where the path above and below the real w-axis follow the lines $c+re^{\pm i\theta}$, where $0<\theta<\frac{\pi}{2}$. Thus, the line of integration cannot be parallel with the imaginary axis, unless $a<2$. If $a=b=1$, the reflection formula (161) leads to Mellin's integral $\frac{1}{2\pi i}\int_{c+\infty}^{c+\infty}\Gamma(w)(-z)^w dw=e^z$.

Let us consider the contour $\mathcal{C}$, consisting of the line $c+re^{i\theta}$ with $0\leq r\leq T$, the line parallel to the real axis from $c+Te^{i\theta}$ to the left at $c-m+Te^{i\theta}$, the line back to the real axis at $c-m$ (and the reflection of this parallelogram around the real axis). The path parallel to the real axis

$$\int_{c+Te^{-i\theta}}^{c-m+Te^{i\theta}}\frac{\pi}{\sin\pi w}\frac{(-z)^w}{\Gamma(b+aw)}dw=\int_{c}^{c-m}\frac{\pi}{\sin\pi\left(x+Te^{-i\theta}\right)}\frac{(-z)^{x+Te^{-i\theta}}}{\Gamma\left(b+ax+aTe^{-i\theta}\right)}dx$$

vanishes by (176) for $T\to\infty$ provided $0<\theta<\frac{\pi}{2}$. The contour $\mathcal{C}$ encloses the poles $\frac{\pi}{\sin\pi w}$ at $w=-k$ from $k=1$ to m with residue $(-1)^k$. Hence, by shifting the lines $c+re^{\pm i\theta}$ to $c-m+re^{\pm i\theta}$, maintaining the angle $0<\theta<\frac{\pi}{2}$, we deform the integral (78) into

$$E_{a,b}(z)=\sum_{k=1}^{m}\frac{1}{\Gamma(b-ka)}\frac{1}{z^k}-\frac{1}{2\pi i}\int_{c'+\infty e^{-i\theta}}^{c'+\infty e^{i\theta}}\frac{\pi}{\sin\pi w}\frac{(-z)^w}{\Gamma(b+aw)}dw \qquad \text{for } -1-m<c'<m$$

For complex z and for any a, it is generally complicated to bound the integral for large $|z|$ to deduce an asymptotic expansion.

34. We assume here complex z and b, but a is real and positive. If we choose $\theta=\frac{\pi}{2}$ in (78), then we must restrict $0<a<2$ due to $\frac{1}{|\Gamma(b+air)|}=\frac{(ar)^{\frac{1}{2}-b}}{\sqrt{2\pi}}e^{\frac{\pi}{2}ar}\left(1+O\left(\frac{1}{r}\right)\right)$ in (177). In that case, we evaluate the contour in (78) along the line $w=c+it$,

$$E_{a,b}(-z)=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\sin\pi(c+it)}\frac{z^{c+it}}{\Gamma(b+ac+iat)}dt \qquad \text{for } -1<c<0$$

Since $c<0$ and $0<a<2$, the reflection formula (161) yields

$$\frac{1}{\Gamma(b-a|c|+ait)}=-\Gamma(1-b+a|c|-ait)\frac{\sin\pi(a|c|-b-ait)}{\pi}$$

and

$$E_{a,b}(-z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi(a|c|-b-ait)}{\sin\pi(-|c|+it)}\Gamma(a|c|+1-b-ait)z^{-|c|+it}dt$$

We change the sign of c and obtain

$$E_{a,b}(-z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi(ac-b-ait)}{\sin\pi(c-it)}\frac{\Gamma(ac+1-b-ait)}{z^{c-it}}dt \qquad \text{for } 0<c<1$$

[11] The integral (78) is rewritten with the reflection formula of the Gamma function as a Barnes-Mellin type integral

$$E_{a,b}(-z)=-\frac{1}{2\pi i}\int_{c+\infty e^{-i\theta}}^{c+\infty e^{i\theta}}\frac{\Gamma(1-w)\Gamma(w)}{\Gamma(b+aw)}z^w dw \qquad \text{for } -1<c<0 \text{ and } 0<\theta<\frac{\pi}{2} \tag{77}$$

In order to replace $\Gamma(ac+1-b-ait)$ by Euler's integral (147), we must require that $\mathrm{Re}\,(ac+1-b) > 0$ or $ac+1 > \mathrm{Re}\,(b)$

$$E_{a,b}(-z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \frac{\sin\pi(a(c-it)-b)}{\sin\pi(c-it)}\frac{1}{z^{c-it}}\int_0^\infty u^{a(c-it)-b}e^{-u}du \qquad \text{for } 0<c<1$$

Since $z = re^{i\theta}$, then $z^{c-it} = \left(re^{i\theta}\right)^{c-it} = r^{c-it}e^{i\theta(c-it)} = r^{c-it}e^{i\theta c}e^{\theta t}$ and the integral becomes

$$E_{a,b}\left(-re^{i\theta}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \frac{\sin\pi(a(c-it)-b)}{\sin\pi(c-it)}\frac{1}{r^{c-it}e^{i\theta c}e^{\theta t}}\int_0^\infty u^{a(c-it)-b}e^{-u}du \qquad \text{for } 0<c<1$$

We can interchange the integrals by absolute convergence, provided that $0<a<1$ and that $|\theta|<\pi$,

$$E_{a,b}(-z) = -\int_0^\infty u^{-b}e^{-u}du\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi(a(c-it)-b)}{\sin\pi(c-it)}\left(\frac{u^a}{z}\right)^{c-it}dt\right) \qquad \text{for } 0<c<1 \quad (79)$$

The integral between brackets in (79) can be recasted with $y=\frac{u^a}{z}$ as

$$Q_{a,b}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi(a(c-it)-b)}{\sin\pi(c-it)}y^{c-it}dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\sin\pi(aw-b)}{\sin\pi w}y^w dw \qquad \text{for } 0<c<1 \quad (80)$$

and (79) becomes

$$E_{a,b}(-z) = -\int_0^\infty u^{-b}e^{-u}Q_{a,b}\left(\frac{u^a}{z}\right)du \quad (81)$$

In **art**. 36 below, we prove for complex y with $|\arg y|<\pi$ and $0<a<1$ that

$$Q_{a,b}(y) = \frac{1}{\pi}\left(\frac{y\sin\pi(a-b) - y^2\sin\pi b}{1+2y\cos\pi a + y^2}\right) \quad (82)$$

Substitution of (82) into (79) yields, for $0<a\le 1$, $a+1>\mathrm{Re}\,(b)$ and complex z with $|\arg z|<\pi$,

$$E_{a,b}(-z) = \frac{1}{\pi}\int_0^\infty u^{-b}e^{-u}\left(\frac{-\frac{u^a}{z}\sin\pi(a-b) + \left(\frac{u^a}{z}\right)^2\sin\pi b}{1+2\frac{u^a}{z}\cos\pi a + \left(\frac{u^a}{z}\right)^2}\right)du \quad (83)$$

We present several reformulations of the integral (83). First, we split (83) into two parts, for $0<a\le 1$, $a+1>\mathrm{Re}\,(b)$ and complex z with $|\arg z|<\pi$,

$$E_{a,b}(-z) = \frac{-z\sin\pi(a-b)}{\pi}\int_0^\infty \frac{u^{a-b}e^{-u}du}{z^2+2zu^a\cos\pi a + u^{2a}} + \frac{\sin\pi b}{\pi}\int_0^\infty\frac{u^{2a-b}e^{-u}du}{z^2+2zu^a\cos\pi a+u^{2a}} \quad (84)$$

which equals (69) without residue sum. A second rewriting of (83)

$$E_{a,b}(-z) = \frac{1}{\pi}\int_0^\infty u^{a-b}e^{-u}\left(\frac{-z\sin\pi(a-b)+u^a\sin\pi b}{(z\sin\pi a)^2+(z\cos\pi a+u^a)^2}\right)du$$

illustrates that the denominator is always positive for real z. Third, let $u = z^{\frac{1}{a}}t$ in (83), then we must additionally require that $\mathrm{Re}\left(z^{\frac{1}{a}}\right) > 0$, thus $\mathrm{Re}\left(r^{\frac{1}{a}}e^{i\frac{\theta}{a}}\right) = r^{\frac{1}{a}}\cos\left(\frac{\theta}{a}\right) > 0$. We obtain, for $\mathrm{Re}\left(z^{\frac{1}{a}}\right) > 0$, $0<a\le 1$ and $a+1>\mathrm{Re}\,(b)$,

$$E_{a,b}(-z) = \frac{z^{\frac{1-b}{a}}}{\pi}\int_0^\infty t^{-b}e^{-z^{\frac{1}{a}}t}\left(\frac{-t^a\sin\pi(a-b)+t^{2a}\sin\pi b}{1+2t^a\cos\pi a + t^{2a}}\right)dt \quad (85)$$

35. *The integral* $Q_{a,b}(y)$. The contour of the integral $Q_{a,b}(y)$ in (80) for complex $y=|y|e^{i\theta}$ with $|\theta|<\pi$ can be closed over either half-plane, because

$$\lim_{w\to\infty}\frac{\sin\pi(aw-b)}{\sin\pi w}y^w=\lim_{|w|\to\infty}e^{\{\pi(a-1)-\theta\}|w|\sin\varphi}\,|y|^{|w|\cos\varphi}$$

vanishes if $|y|<1$, $\cos\varphi>0$, $a<1$ and $|\theta|<\pi$. If $|y|<1$, then we close the contour in (80) over $\operatorname{Re}(w)>0$-plane and obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dt\frac{\sin\pi(aw-b)}{\sin\pi w}y^w&=-\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\sin\pi(ak-b)\,y^k\\&=\frac{e^{i\pi(a-b)}y}{2\pi i}\sum_{k=0}^{\infty}\left(-e^{i\pi a}y\right)^k-\frac{e^{-i(\pi a-b)}y}{2\pi i}\sum_{k=0}^{\infty}\left(-e^{-i\pi a}y\right)^k\\&=\frac{y}{2\pi i}\left(\frac{e^{i\pi(a-b)}}{1+e^{i\pi a}y}-\frac{e^{-i\pi(a-b)}}{1+e^{-i\pi a}y}\right)\\&=\frac{1}{\pi}\left(\frac{y\sin\pi(a-b)-y^2\sin\pi b}{1+2y\cos\pi a+y^2}\right)\end{aligned}$$

The derivation also shows, for real a and b, that $Q_{a,b}(|y|)=\frac{|y|}{\pi}\operatorname{Im}\left(\frac{e^{i\pi(a-b)}}{1+e^{i\pi a}|y|}\right)$ and $|Q_{a,b}(y)|\leq\frac{1}{\pi}\sum_{k=1}^{\infty}|y|^k=\frac{1}{\pi}\frac{|y|}{1-|y|}$ for $|y|<1$.

If $|y|>1$, then we close the contour over the $\operatorname{Re}(w)<0$-plane,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dt\frac{\sin\pi(aw-b)}{\sin\pi w}(u^a)^w&=\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\sin\pi(-ak-b)\,y^{-k}\\&=\frac{1}{\pi}\left(\frac{y\sin\pi(a-b)-y^2\sin\pi b}{1+2y\cos\pi a+y^2}\right)\end{aligned}$$

Both the case $|y|<1$ and $|y|>1$ lead to the same result and $Q_{a,b}(1)=\frac{1}{2\pi}\left(\frac{\sin\pi(a-b)-\sin\pi b}{1+\cos\pi a}\right)=\frac{1}{2\pi}\frac{\sin\pi\left(\frac{a}{2}-b\right)}{\cos\frac{\pi a}{2}}$.

In summary, for any complex y with $|\arg y|<\pi$ and for $0<a<1$, we have proved (82). Moreover, $|Q_{a,b}(y)|\leq\frac{1}{\pi}\sum_{k=0}^{\infty}|y|^{-k}=\frac{1}{\pi}\frac{1}{1-|y|^{-1}}=\frac{1}{\pi}\frac{|y|}{|y|-1}$ for $|y|>1$. Hence, for any y, it holds that $|Q_{a,b}(y)|\leq\frac{1}{\pi}\frac{|y|}{||y|-1|}$.

36. *The integral* $Q_{a,b}(u)$ *for real* u. The contour in (80) for real $y=u$ can be rewritten as

$$Q_{a,b}(u)=\frac{u^{\frac{a}{2}}}{\pi}\left\{\sin\pi\left(\frac{a}{2}-b\right)\int_0^\infty\frac{\cosh\pi at}{\cosh\pi t}\cos(at\log u)\,dt-\cos\pi\left(\frac{a}{2}-b\right)\int_0^\infty\frac{\sinh\pi at}{\cosh\pi t}\sin(at\log u)\,dt\right\}$$

Since $Q_{a,1}(u)=Q_{a,1}\left(\frac{1}{u}\right)$, as follows from (82), it holds that

$$\int_0^\infty\frac{\sinh\pi at}{\cosh\pi t}\sin(at\log u)\,dt=\frac{u^a-1}{u^a+1}\tan\left(\frac{\pi a}{2}\right)\int_0^\infty\frac{\cosh\pi at}{\cosh\pi t}\cos(at\log u)\,dt$$

so that

$$Q_{a,b}(u)=\frac{u^{\frac{a}{2}}}{\pi}\sin\pi\left(\frac{a}{2}-b\right)\left\{1-\frac{u^a-1}{u^a+1}\cot\pi\left(\frac{a}{2}-b\right)\tan\left(\frac{\pi a}{2}\right)\right\}\int_0^\infty\frac{\cosh\pi at}{\cosh\pi t}\cos(at\log u)\,dt$$

It follows then from (82) that

$$\int_0^\infty \frac{\cosh \pi at}{\cosh \pi t} \cos\left(at \log u\right) dt = \frac{u^{\frac{a}{2}} \left(\sin \pi \left(a-b\right) - u^a \sin \pi b\right)}{\left(1+2u^a \cos \pi a + u^{2a}\right) \left\{\sin \pi \left(\frac{a}{2}-b\right) + \frac{1-u^a}{1+u^a} \cos \pi \left(\frac{a}{2}-b\right) \tan\left(\frac{\pi a}{2}\right)\right\}}$$

whose right-hand is independent of b, so that b can be chosen at will. The simplest choice[12] is $b = \frac{a}{2}$, then

$$\int_0^\infty \frac{\cosh \pi at}{\cosh \pi t} \cos\left(at \log u\right) dt = \frac{u^{\frac{a}{2}} \left(1+u^a\right) \cos \frac{\pi a}{2}}{1+2u^a \cos \pi a + u^{2a}}$$

Suppose that we ignore the restriction that the k-sum in

$$Q_{a,b}\left(t^a\right) = \frac{1}{\pi} \left(\frac{t^a \sin \pi \left(a-b\right) - t^{2a} \sin \pi b}{1+2t^a \cos \pi a + t^{2a}}\right) = \frac{1}{\pi} \sum_{k=1}^{\infty} \left(-1\right)^{k-1} \sin \pi \left(ak-b\right) t^{ak}$$

only converges for $t<1$ and that we substitute the series formally back in (85) and change the order of integration and summation. Then, we find

$$E_{a,b}\left(-z\right) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin \pi \left(ak-b\right) \Gamma\left(ak-b+1\right)}{\left(-z\right)^k} = -\sum_{k=1}^{\infty} \frac{1}{\left(-z\right)^k \Gamma\left(b-ak\right)} = \frac{1}{\Gamma\left(b\right)} - E_{-a,b}\left(-\frac{1}{z}\right)$$

which is precisely equal to (68), in spite of the divergence of the series!

6.3 Integral for $E_a\left(z\right)$ with $0<a<1$

If $b=1$, then the integral (85) simplifies to

$$E_a\left(-z\right) = \frac{\sin \pi a}{\pi} \int_0^\infty \frac{t^{a-1} e^{-z^{\frac{1}{a}} t}}{1+2t^a \cos \pi a + t^{2a}} dt \tag{86}$$

which is listed in [4, eq. (34)] for real $z=x$ and deducible from (70). We present several variations of the integral (86) for $b=1$ and $0<a<1$.

37. *Deductions from the integral (86).* Berberan-Santos [4, eq. (35)] mentions[13] that partial integration of (86) results in

$$E_a\left(-x\right) = 1 - \frac{1}{2a} + \frac{x^{\frac{1}{a}}}{\pi a} \int_0^\infty \arctan\left(\frac{u^a + \cos\left(\pi a\right)}{\sin\left(\pi a\right)}\right) e^{-x^{\frac{1}{a}} u} du \tag{87}$$

Indeed, let $y=u^a$ in (86), then

$$E_a\left(-x\right) = \frac{\sin \pi a}{\pi a} \int_0^\infty \frac{e^{-x^{\frac{1}{a}} y^{\frac{1}{a}}}}{1+2y \cos \pi a + y^2} dy$$

[12]It is readily verified that

$$\frac{\sin \pi \left(a-b\right) - u^a \sin \pi b}{\sin \pi \left(\frac{a}{2}-b\right) + \frac{1-u^a}{1+u^a} \cos \pi \left(\frac{a}{2}-b\right) \tan\left(\frac{\pi a}{2}\right)} = \cos\left(\frac{\pi a}{2}\right)\left(1+u^a\right)$$

[13]The integral in Berberan-Santos [4, eq. (35)], $E_a\left(-x\right) = 1 - \frac{1}{2a} + \frac{x^{\frac{1}{a}}}{\pi} \int_0^\infty \arctan\left(\frac{u^a + \cos(\pi a)}{\sin(\pi a)}\right) e^{-x^{\frac{1}{a}} u} du$, misses a factor of $\frac{1}{a}$ before the integral.

With $\frac{1}{1+2y\cos\pi a+y^2}=\frac{1}{2i\sin\pi a}\left(\frac{1}{e^{-i\pi a}+y}-\frac{1}{e^{i\pi a}+y}\right)$, we have

$$E_a(-x)=\frac{1}{2\pi i a}\int_0^\infty\left(\frac{1}{e^{-i\pi a}+y}-\frac{1}{e^{i\pi a}+y}\right)e^{-x^{\frac{1}{a}}y^{\frac{1}{a}}}dy \tag{88}$$

as well as

$$\begin{aligned}\int\frac{dy}{1+2y\cos\pi a+y^2}&=\frac{1}{2i\sin\pi a}\left(\log\frac{e^{-i\pi a}+y}{e^{i\pi a}+y}\right)=\frac{1}{2i\sin\pi a}\left(\log\frac{y+\cos\pi a-i\sin\pi a}{y+\cos\pi a+i\sin\pi a}\right)\\&=\frac{1}{2i\sin\pi a}\left(\log\frac{1-i\frac{\sin\pi a}{y+\cos\pi a}}{1+i\frac{\sin\pi a}{y+\cos\pi a}}\right)=-\frac{1}{\sin\pi a}\arctan\left(\frac{\sin\pi a}{y+\cos\pi a}\right)\end{aligned}$$

because $\arctan z=\frac{i}{2}\log\frac{1-iz}{1+iz}$. Hence, we find the indefinite integral with a constant K,

$$\int\frac{u^{a-1}du}{1+2u^a\cos\pi a+u^{2a}}=-\frac{1}{a\sin\pi a}\arctan\left(\frac{\sin\pi a}{u^a+\cos\pi a}\right)+K \tag{89}$$

With this preparation, partial integration of (86) yields

$$\begin{aligned}E_a(-x)&=\frac{1}{a\pi}\arctan\left(\frac{\sin\pi a}{\cos\pi a}\right)-\frac{x^{\frac{1}{a}}}{a\pi}\int_0^\infty\arctan\left(\frac{\sin\pi a}{u^a+\cos\pi a}\right)e^{-x^{\frac{1}{a}}u}du\\&=1-\frac{x^{\frac{1}{a}}}{a\pi}\int_0^\infty\arctan\left(\frac{\sin\pi a}{u^a+\cos\pi a}\right)e^{-x^{\frac{1}{a}}u}du\end{aligned}$$

After invoking $\arctan x=\frac{\pi}{2}-\arctan\frac{1}{x}$ for $x>0$, we arrive at (87).

Further, using the integral for $\arctan z=\int_0^z\frac{dt}{1+t^2}$,

$$1-E_a(-x)=\frac{x^{\frac{1}{a}}}{a\pi}\int_0^\infty du\int_0^{\frac{\sin\pi a}{u^a+\cos\pi a}}dt\frac{e^{-x^{\frac{1}{a}}u}}{1+t^2}$$

and reverse the integrals, provided that $0<a\le\frac{1}{2}$,

$$1-E_a(-x)=\frac{x^{\frac{1}{a}}}{a\pi}\int_0^{\tan\pi a}\frac{\int_0^{\left(\frac{\sin\pi a}{t}-\cos\pi a\right)^{\frac{1}{a}}}e^{-x^{\frac{1}{a}}u}du}{1+t^2}dt$$

Hence, we obtain, for $0<a\le\frac{1}{2}$,

$$E_a(-x)=\frac{1}{a\pi}\int_0^{\tan\pi a}\frac{e^{-\left(x\left(\frac{\sin\pi a}{t}-\cos\pi a\right)\right)^{\frac{1}{a}}}}{1+t^2}dt \tag{90}$$

After letting $u^a=\frac{\sin\pi a}{t}-\cos\pi a$ in (90), we retrieve (86) again.

38. *Bounds from the integral (86).* We split the integration interval in (86) into two parts,

$$E_a(-x)=\frac{\sin\pi a}{\pi}\int_1^\infty\frac{u^{a-1}\left(e^{-x^{\frac{1}{a}}u^{-1}}+e^{-x^{\frac{1}{a}}u}\right)}{1+2u^a\cos\pi a+u^{2a}}du$$

First,

$$\begin{aligned}\int_1^\infty\frac{u^{a-1}e^{-x^{\frac{1}{a}}u^{-1}}}{1+2u^a\cos\pi a+u^{2a}}du&=e^{-x^{\frac{1}{a}}}\int_1^\infty\frac{u^{a-1}e^{x^{\frac{1}{a}}\left(1-u^{-1}\right)}}{1+2u^a\cos\pi a+u^{2a}}du\\&\ge e^{-x^{\frac{1}{a}}}\int_1^\infty\frac{u^{a-1}}{1+2u^a\cos\pi a+u^{2a}}du\end{aligned}$$

Using (89) and $\frac{\sin \pi a}{1+\cos \pi a} = \tan \frac{\pi a}{2}$, we arrive at the lower bound,

$$\frac{\sin \pi a}{\pi} \int_1^\infty \frac{u^{a-1} e^{-x^{\frac{1}{a}} u^{-1}}}{1+2u^a \cos \pi a + u^{2a}} du \geq \frac{1}{\pi a} e^{-x^{\frac{1}{a}}} \arctan\left(\tan \frac{\pi a}{2}\right) = \frac{1}{2} e^{-x^{\frac{1}{a}}}$$

Second, after partial integration and again using (89), we obtain

$$\begin{aligned}\frac{\sin \pi a}{\pi} \int_1^\infty \frac{u^{a-1} e^{-x^{\frac{1}{a}} u}}{1+2u^a \cos \pi a + u^{2a}} du &= \frac{1}{2} e^{-x^{\frac{1}{a}}} - \frac{x^{\frac{1}{a}}}{\pi a} \int_1^\infty e^{-x^{\frac{1}{a}} u} \arctan\left(\frac{\sin \pi a}{u^a + \cos \pi a}\right) du \\ &= \frac{1}{2} e^{-x^{\frac{1}{a}}} - \frac{x^{\frac{1}{a}} e^{-x^{\frac{1}{a}}}}{\pi a} \int_1^\infty e^{-x^{\frac{1}{a}}(u-1)} \arctan\left(\frac{\sin \pi a}{u^a + \cos \pi a}\right) du\end{aligned}$$

so that

$$E_a(-x) \geq e^{-x^{\frac{1}{a}}} - \frac{x^{\frac{1}{a}} e^{-x^{\frac{1}{a}}}}{\pi a} \int_1^\infty e^{-x^{\frac{1}{a}}(u-1)} \arctan\left(\frac{\sin \pi a}{u^a + \cos \pi a}\right) du$$

Further, we may bound the latter integral,

$$\begin{aligned}\int_1^\infty e^{-x^{\frac{1}{a}}(u-1)} \arctan\left(\frac{\sin \pi a}{u^a + \cos \pi a}\right) du &< \int_1^\infty e^{-x^{\frac{1}{a}}(u-1)} \arctan\left(\frac{\sin \pi a}{1 + \cos \pi a}\right) du \\ &= \frac{\pi a}{2} \int_1^\infty e^{-x^{\frac{1}{a}}(u-1)} du = \frac{\pi a}{2x^{\frac{1}{a}}}\end{aligned}$$

which leads to the lower bound, for $0 < a < 1$,

$$E_a(-x) \geq \frac{1}{2} e^{-x^{\frac{1}{a}}}$$

It follows directly from (86) that

$$\begin{aligned}E_a(-x) &= \frac{\sin \pi a}{\pi} \int_0^\infty \left(1 - \frac{2u^a \cos \pi a + u^{2a}}{1+2u^a \cos \pi a + u^{2a}}\right) u^{a-1} e^{-x^{\frac{1}{a}} u} du \\ &= \frac{\sin \pi a}{\pi} \frac{\Gamma(a)}{x} - \frac{\sin \pi a}{\pi} \int_0^\infty \frac{(2\cos \pi a + u^a)}{1+2u^a \cos \pi a + u^{2a}} u^{2a-1} e^{-x^{\frac{1}{a}} u} du\end{aligned}$$

Further, for $0 < a < \frac{1}{2}$ (because then $\cos \pi a \geq 0$),

$$\begin{aligned}E_a(-x) &\leq \frac{\sin \pi a}{\pi} \frac{\Gamma(a)}{x} - \frac{\sin \pi a}{\pi} \int_0^1 \frac{(2\cos \pi a + u^a)}{1+2u^a \cos \pi a + u^{2a}} u^{2a-1} e^{-x^{\frac{1}{a}} u} du \\ &\leq \frac{\sin \pi a}{\pi} \frac{\Gamma(a)}{x} - \frac{\sin \pi a}{\pi} \frac{2\cos \pi a}{2+2\cos \pi a} e^{-x^{\frac{1}{a}}} \int_0^1 u^{2a-1} du\end{aligned}$$

and

$$E_a(-x) \leq \frac{\sin \pi a}{\pi} \frac{\Gamma(a)}{x} - \frac{e^{-x^{\frac{1}{a}}}}{4a\pi} \frac{\sin 2\pi a}{1+\cos \pi a}$$

which illustrates, for $0 < a < \frac{1}{2}$, that $E_a(-x)$ is bounded by

$$\frac{\sin \pi a}{\pi} \frac{\Gamma(a)}{x} - \frac{e^{-x^{\frac{1}{a}}}}{4a\pi} \frac{\sin 2\pi a}{1+\cos \pi a} \geq E_a(-x) \geq \frac{1}{2} e^{-x^{\frac{1}{a}}} \tag{91}$$

7 The Mittag-Leffler function in probability theory

39. *Monotonicity of* $E_{a,b}(-x)$ *for* $0 \leq a \leq 1$. Widder [56, Chapter IV] devotes an entire chapter to absolutely and completely monotone functions. The Hausdorff–Bernstein–Widder[14] theorem [56, p. 161, Theorem 12a] states that a necessary and sufficient condition that a function $\varphi(x)$ on $[0,\infty)$ should be completely monotonic is that there exists a bounded and non-decreasing function $f(u)$ such that the integral

$$\varphi(x) = \int_0^\infty e^{-xu} df(u)$$

converges for all real $x \geq 0$. In other words, a function $\varphi(x)$ is completely monotonic on $[0,\infty)$ if and only if $\varphi(x)$ is a Laplace transform of a bounded and non-decreasing measure $f(u)$. The fact that $(-1)^n \frac{d^n \varphi(x)}{dx^n} = \int_0^\infty x^n e^{-xu} df(u) \geq 0$ for all non-negative integers n is a direct consequence of the Hausdorff–Bernstein–Widder theorem, but the condition $(-1)^n \frac{d^n \varphi(x)}{dx^n} \geq 0$ for all non-negative integers n is also sufficient [6, p. 56-59] and thus an equivalent statement for complete monotonicity of a function $\varphi(x)$.

Since the integrand in (86) is positive for $0 \leq a \leq 1$ because

$$(1-u^a)^2 \leq 1 + 2u^a \cos \pi a + u^{2a} \leq (1+u^a)^2$$

(86) shows that $E_a(-x) > 0$ for $0 < a \leq 1$. Hence, the integral (86) directly demonstrates that $E_a(-x) > 0$ for $0 < a \leq 1$ is completely monotonic. Also, the case for $b = a$ in (85) reduces to

$$x^{1-\frac{1}{a}} E_{a,a}(-x) = \frac{\sin \pi a}{\pi} \int_0^\infty \frac{u^a}{1 + 2u^a \cos \pi a + u^{2a}} e^{-x^{\frac{1}{a}} u} du \tag{92}$$

illustrating that also $x^{1-\frac{1}{a}} E_{a,a}(-x)$ is complete monotonic. The monotonicity of $E_a(-x) > 0$ for $0 < a \leq 1$ was first conjectured by Feller [14, Section 7] and later proved by Pollard [39].

Pollard[15] [39] introduces $\frac{1}{t+z} = \int_0^\infty e^{-s(t+z)} ds$ in Mittag-Leffler's integral (71) and obtains a Laplace transform,

$$E_{a,b}(-z) = \int_0^\infty e^{-sz} \left(\frac{1}{2\pi i a} \int_{C_{a\phi}} t^{\frac{1-b}{a}} e^{-st + t^{\frac{1}{a}}} dt \right) ds$$

from which he proves[16] that $E_a(-x)$ is completely monotonic for real $x \geq 0$ and $0 \leq a \leq 1$, in the sense that

$$E_a(-x) = \int_0^\infty e^{-xt} dF_a(t)$$

where $F_a(t)$ is nondecreasing, bounded and a probability distribution. In other words, $E_a(z)$ for $0 \leq a \leq 1$ has no zeros on the negative real axis. Pollard [39] also explicitly determined the non-negative function $F_a'(t)$. However, his proof is not easy and, therefore, omitted, but replaced by our derivation in **art**. 41.

40. *Monotonicity of* $E_{a,b}(-x)$ *for* $0 < a \leq 1$ *and* $b \geq a$. Schneider [45] extended the range of the parameter b by proving that $E_{a,b}(-x) > 0$ for $0 \leq a \leq 1$ and $b > a$. Schneider's proof is involved,

[14] Widder [56, p. 144] mentions that Hausdorff, Bernstein and himself have independently proved the theorem.

[15] Pollard's PhD advisor was D. V. Widder.

[16] Berberan-Santos [4] claims that monotonicity, defined by $(-1)^n \frac{d^n}{dx^n} E_a(-x) \geq 0$ for all n, follows from $g(t) > 0$ in **art**. 22 for $0 \leq a \leq 1$, but I find his argument circular.

based on Fox functions, and Schneider also derives the corresponding probability measure. Miller and Samko [32] presented a simple proof, which we include here.

The integral (57) with $\lambda = -\frac{z}{x^a}$, $b = a$ and $w = b - a$ becomes

$$E_{a,b}(-z) = \frac{x^{-a}}{\Gamma(b-a)} \int_0^x \left(1 - \frac{u}{x}\right)^{b-a-1} u^{a-1} E_{a,a}\left(-z\left(\frac{u}{x}\right)^a\right) du$$

After substitution $t = \left(\frac{u}{x}\right)^a$ or $u = xt^{\frac{1}{a}}$, we find, for $b > a > 0$,

$$E_{a,b}(-z) = \frac{1}{a\Gamma(b-a)} \int_0^1 \left(1 - t^{\frac{1}{a}}\right)^{b-a-1} E_{a,a}(-zt)\, dt \tag{93}$$

It follows from the differentiation formula (17) for $b = 1$ that $az\frac{d}{dz}E_{a,1}(z) = E_{a,0}(z)$. Since $E_{a,0}(z) = zE_{a,a}(z)$ by (5), it holds that

$$a\left.\frac{d}{du}E_{a,1}(u)\right|_{u=z} = E_{a,a}(z) \tag{94}$$

With the chain rule, $\frac{d}{dt}E_{a,1}(-zt) = \frac{d}{du}E_{a,1}(u)\left.\frac{du}{dt}\right|_{u=-zt} = -\frac{z}{a}E_{a,a}(-zt)$, (94) indicates that $E_{a,a}(-z) = -a\frac{d}{dz}E_{a,1}(-z)$. **Art.** 39 shows that $E_{a,1}(-z) = E_a(-z)$ is completely monotonous for $0 < a \leq 1$ satisfying $(-1)^n \frac{d^n E_a(-z)}{dz^n} \geq 0$ for all non-negative integers n, so that $E_{a,a}(-z) > 0$ is completely monotonous as well. Since the integrand in (93) is non-negative, we conclude that $E_{a,b}(-z) > 0$ for $0 < a \leq 1$ and $b \geq a$.

41. $E_a(-x)$ *with* $0 < a < 1$ *in probability theory.* We construct two probability density functions from the Mittag-Leffler function $E_a(-x)$ with $0 < a < 1$ and show that $E_a(-s^a)$ is both a probability generating function and a probability distribution.

a. After replacing x by s^a in (86), the Laplace transform

$$E_a(-s^a) = \frac{\sin \pi a}{\pi} \int_0^\infty \frac{t^{a-1}}{1 + 2t^a \cos \pi a + t^{2a}} e^{-st} dt$$

indicates that

$$f_a(t) = \frac{\sin \pi a}{\pi} \frac{t^{a-1}}{1 + 2t^a \cos \pi a + t^{2a}} \tag{95}$$

is a probability density function (pdf) for $t > 0$ and for $0 < a \leq 1$. Indeed, the Laplace transform (192) of the non-negative function $f_a(t)$ is [50] a probability generating function (pgf) $\varphi_X(z) = E\left[e^{-zX}\right]$ of a random variable X, provided $\varphi_X(0) = 1$. Moreover, with $f_a(1) = \frac{1}{2\pi}\tan\frac{\pi a}{2}$, the pdf (95) obeys the functional equation – recall that $Q_{a,1}(u) = Q_{a,1}\left(\frac{1}{u}\right)$ in (82) –

$$tf_a(t) = \frac{1}{t} f_a\left(\frac{1}{t}\right)$$

Its companion, that follows similarly from (92) as

$$s^{a-1}E_{a,a}(-s^a) = \frac{\sin \pi a}{\pi} \int_0^\infty \frac{t^a}{1 + 2t^a \cos \pi a + t^{2a}} e^{-st} dt$$

with non-negative function

$$g_a(t) = \frac{\sin \pi a}{\pi} \frac{t^a}{1 + 2t^a \cos \pi a + t^{2a}} = tf_a(t) = g_a\left(\frac{1}{t}\right)$$

is, in contrast to $E_a(-s^a)$, not[17] a probability generating function, because $\lim_{s\to0} s^{a-1}E_{a,a}(-s^a) = \infty$ for $a<1$. We verify from Laplace transform theory that

$$-\frac{d}{ds}E_a(-s^a) = s^{a-1}E_{a,a}(-s^a) \tag{96}$$

in agreement with (19) taken into account (5).

b. The Laplace transform (54) with $b=1$ and $x=-1$,

$$\int_0^\infty e^{-st}E_a(-t^a)\,dt = \frac{s^{a-1}}{s^a+1}$$

and with $b=a$ and $x=-1$

$$\int_0^\infty e^{-st}t^{a-1}E_{a,a}(-t^a)\,dt = \frac{1}{s^a+1} \tag{97}$$

are only valid for $s^a>1$. Hence[18], we cannot conclude from $\lim_{s\to0}\frac{1}{s^a+1}=1$ for $a>0$ that (97) is a pgf $\varphi_M(s)$, while $\int_0^\infty e^{-st}E_a(-t^a)\,dt = \frac{s^{a-1}}{s^a+1}$ is not. However, integration of (96) yields $\int_p^q t^{a-1}E_{a,a}(-t^a)\,dt = E_a(-p^a)-E_a(-q^a)$. If $p=0$, then $E_{a,1}(0)=\frac{1}{\Gamma(1)}=1$, while $\lim_{q\to\infty}E_a(-q^a)=0$, as follows from the integral $E_a(-s^a)=\frac{\sin\pi a}{\pi}\int_0^\infty \frac{t^{a-1}}{1+2t^a\cos\pi a+t^{2a}}e^{-st}dt$. Hence, it holds that

$$\int_0^\infty t^{a-1}E_{a,a}(-t^a)\,dt = 1$$

illustrating that $t^{a-1}E_{a,a}(-t^a)$ can be regarded as a probability density function and that the integral in (97) exists for $\operatorname{Re}(s)\geq0$ and $a>0$. Thus, by analytic continuation, the pgf $\varphi_M(s)=E\left[e^{-sM}\right]$ in (97) is valid for $\operatorname{Re}(s)\geq0$ with corresponding pdf

$$f_M(t) = t^{a-1}E_{a,a}(-t^a) \qquad \text{for } 0<a<1 \tag{98}$$

of a random variable M and (96) demonstrates that the corresponding Mittag-Leffler distribution for $0<a<1$ is

$$F_M(t) = \Pr[M\leq t] = \int_0^t f_M(u)\,du = 1-E_a(-t^a) \tag{99}$$

with mean $E[M] = -\varphi_M'(0) = \lim_{s\to0}\frac{as^{a-1}}{(s^a+1)^2}=\infty$. In fact, for $0<a<1$, the pgf (97) is not analytic at $s=0$, implying that the Taylor series around $s=0$ does not exist, nor any derivative. Hence, the Mittag-Leffler random variable M, defined by the pgf (97) and pdf (98) for $0<a<1$, does not possess any finite moment $E\left[M^k\right]$. In the limit $a\to1$, the Mittag-Leffler random variable M becomes an exponential random variable with mean 1.

42. *Probabilistic properties of the Mittag-Leffler random variable.* The sum $S_n=\sum_{j=1}^n M_j$ of n i.i.d Mittag-Leffler random variables $M_1, M_2, \ldots, M_n$, each with the same Mittag-Leffler distribution (99), has the pgf [50, p. 30]

$$\varphi_{S_n}(z) = E\left[e^{-z\sum_{j=1}^n M_j}\right] = \varphi_M^n(z) = (1+z^a)^{-n}$$

[17] However, insertion into (97) below leads, for $\operatorname{Re}(s)>0$, to

$$\int_0^\infty \frac{u^a}{1+2u^a\cos\pi a+u^{2a}}\frac{du}{s+u} = \frac{\pi}{\sin\pi a}\frac{1}{s^a+1}$$

[18] This observation was communicated to me by Rui Ferreira.

If we choose the scaling parameter β in $\varphi_{S_n}(\beta z) = (1 + \beta^a z^a)^{-n}$ equal to $\beta^a = \frac{1}{n}$, then

$$\lim_{n\to\infty} \varphi_{S_n}\left(\frac{z}{n^{\frac{1}{a}}}\right) = \lim_{n\to\infty}\left(1 + \frac{z^a}{n}\right)^{-n} = e^{-z^a}$$

Hence, the scaled sum $\beta S_n = \sum_{j=1}^n n^{-\frac{1}{a}} M_j$ tends for $n \to \infty$ to a random variable R with pgf equal to

$$\varphi_R(z) = E\left[e^{-zR}\right] = e^{-z^a} \tag{100}$$

whose form belongs to the class of stable distributions[19]. If $a = 1$, then $R = 1$ and not random. Since e^{-z^a} with $0 < 1 < a$ is only analytic for $\operatorname{Re}(z) > 0$, inverse Laplace transform (193) provides us with the pdf

$$\begin{aligned} f_R(t) &= \frac{1}{2\pi i}\int_{c-\infty e^{-i\phi}}^{c+\infty e^{i\phi}} e^{-z^a} e^{zt} dz \qquad c > 0 \text{ and } \phi > \frac{\pi}{2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{2\pi i}\int_{c-\infty e^{-i\phi}}^{c+\infty e^{i\phi}} z^{ka} e^{zt} dz \end{aligned}$$

Introducing Hankel's integral (182)

$$f_R(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{-ak-1}}{\Gamma(-ka)}$$

and the reflection formula (161) results in

$$f_R(t) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \frac{\Gamma(ka+1)\sin \pi ka}{t^{ak+1}} \qquad \text{with } 0 < 1 < a \tag{101}$$

which is derived in another, more complicated way by Pollard [38]. Integration leads to the distribution

$$1 - F_R(t) = \Pr[R > t] = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \frac{\Gamma(ka)\sin \pi ka}{t^{ak}}$$

with the interesting result that $\lim_{t\to 0} \Pr[R \geq t] = 1$, while $\lim_{t\to 0} f_R(t) = \infty$.

The sum of n i.i.d. random variables $R_1, R_2, \ldots, R_n$ with same distribution $f_R(t)$ in (101) equals $\sum_{j=1}^n R_j = n^{\frac{1}{a}} R$, because

$$E\left[e^{-z\sum_{j=1}^n R_j}\right] = \varphi_R^n(z) = e^{-nz^a} = e^{-\left(n^{\frac{1}{a}} z\right)^a} = E\left[e^{-zn^{\frac{1}{a}} R}\right]$$

Thus, $\sum_{j=1}^n R_j = n^{\frac{1}{a}} R$ expresses self-similarity: a sum of random variables maintains the same distribution upon scaling, which is an alternative description to a "stable" distribution.

[19] Let X_1 and X_2 be i.i.d random variables, similarly distributed as a random variable X. The random variable X is stable if for any constants $a > 0$, $b > 0$, $c > 0$ and d, the random variable $aX_1 + bX_2$ has the same distribution as $cX + d$, denoted as $aX_1 + bX_2 \overset{d}{=} cX + d$. Another definition [15, p. 170] states that X is stable if and only if $\sum_{j=1}^n X_j \overset{d}{=} c_n X + d_n$ for any integer $n > 1$ and where the constant $c_n > 0$ and $d_n \in \mathbb{R}$.

Gorenflo and Mainardi [20] discuss fractional diffusion processes and their relation to Levy stable distributions.

43. *The Mittag-Leffler and Weibull random variable.* We consider the random variable $X = RW$, where W is independent of the stable random variable R and will be chosen later. The pgf of X is computed by invoking conditional expectations [50, p. 32],

$$\varphi_X(z) = E\left[e^{-zRW}\right] = E_W\left[E_R\left[e^{-zRW}\middle| W\right]\right]$$

With the pgf (100) of the stable random variable R, the inner conditional expectation is a random variable equal to

$$E_R\left[e^{-zRW}\middle| W\right] = e^{-z^a W^a}$$

so that

$$\varphi_X(z) = E_W\left[e^{-z^a W^a}\right]$$

Let us now define the random variable $Y = W^a$, then the expectation $E_W\left[e^{-z^a W^a}\right]$ becomes

$$\varphi_Y(z) = E_Y\left[e^{-z^a Y}\right] = \varphi_Y(z^a)$$

If Y is an exponential random variable with mean μ, then $\varphi_Y(z) = E\left[e^{-zY}\right] = \frac{1}{z+\mu}$. Hence, choosing the mean equal $\mu = 1$, then shows that

$$\varphi_X(z) = \frac{1}{z^a + 1}$$

and the pgf (97) demonstrates that $X = M$ has a Mittag-Leffler distribution. The random variable $W = Y^{\frac{1}{a}}$ has the distribution $\Pr[W \le x] = \Pr\left[Y^{\frac{1}{a}} \le x\right] = \Pr[Y \le x^a] = \left(1 - e^{-x^a}\right)$ with the pdf [50, p. 18]

$$f_W(x) = \frac{d\Pr[W \le x]}{dx} = a x^{a-1} e^{-x^a}$$

which illustrates that W is a Weibull random variable [50, p. 59] with $E\left[W^b\right] = \Gamma\left(\frac{b}{a} + 1\right)$ for any real $b > -a$. Hence, all moments $E\left[W^k\right]$ for non-negative integer k exist. The Weibull distribution is one of the three possible limit extremal distributions of a sequence of i.i.d. random variables [5, pp. 65-69] and reduces for $a = 1$ to the exponential distribution, just like the Mittag-Leffler distribution (99). Here, we have shown for the parameter $0 < a < 1$ that the Weibull random variable $W = \frac{M}{R}$ is the quotient of the Mittag-Leffler M and stable R random variable, whose moments do not exist.

44. *The scaled random variable $x^{\frac{1}{a}}R$.* Based on Laplace transforms, Feller [15, p. 453] shows that the distribution $\Pr\left[R > \frac{t}{x^{1/a}}\right] = 1 - F_R\left(\frac{t}{x^{\frac{1}{a}}}\right)$ has a Laplace transform equal to $E_a(-st^a) = \sum_{k=0}^{\infty} \frac{(-s)^k t^{ak}}{\Gamma(1+ka)}$. We present a direct computation. Partial integration of the pgf $\varphi_X(z) = \int_0^\infty e^{-zt} f_X(t)\, dt$,

$$\int_0^\infty e^{-zt}(1 - F_X(t))\, dt = \frac{1 - \varphi_X(z)}{z}$$

is transformed, after letting $z \to \beta z$ for $\beta > 0$, and substituting $u = \beta t$, into

$$\int_0^\infty e^{-zu}\left(1 - F_X\left(\frac{u}{\beta}\right)\right) du = \frac{1}{z}(1 - \varphi_X(\beta z))$$

Applied to the stable random variable R with pgf $\varphi_R(z) = e^{-z^a}$,

$$\int_0^\infty e^{-zu}\left(1 - F_R\left(\frac{u}{\beta}\right)\right) du = \frac{1}{z}\left(1 - e^{-\beta^a z^a}\right)$$

and choosing $x = \beta^a > 0$ yields

$$\int_0^\infty e^{-zu}\left(1 - F_R\left(\frac{u}{x^{\frac{1}{\alpha}}}\right)\right) du = \frac{1}{z}\left(1 - e^{-xz^a}\right)$$

Taking the Laplace transform of both sides with respect to x,

$$\begin{aligned}\int_0^\infty dx e^{-sx} \int_0^\infty du e^{-zu}\left(1 - F_R\left(\frac{u}{x^{\frac{1}{\alpha}}}\right)\right) &= \frac{1}{z}\int_0^\infty dx e^{-sx}\left(1 - e^{-xz^a}\right) \\ &= \frac{1}{z}\left(\frac{1}{s} - \frac{1}{s+z^a}\right) = \frac{1}{s}\left(\frac{z^{a-1}}{s+z^a}\right)\end{aligned}$$

Introducing the Laplace transform (54) with $b = 1$ and $x = -s$

$$\int_0^\infty e^{-zt} E_a\left(-st^a\right) dt = \frac{z^{a-1}}{z^a + s}$$

and interchanging the integrals on the left-hand side, allowed by absolute convergence,

$$\int_0^\infty dt e^{-zt} \int_0^\infty dx e^{-sx}\left(1 - F_R\left(\frac{t}{x^{\frac{1}{\alpha}}}\right)\right) = \frac{1}{s}\int_0^\infty e^{-zt} E_a\left(-st^a\right) dt$$

finally leads to

$$\int_0^\infty e^{-sx}\left(1 - F_R\left(\frac{t}{x^{\frac{1}{\alpha}}}\right)\right) dx = \frac{1}{s} E_a\left(-st^a\right)$$

which is, however, a factor $\frac{1}{s}$ different from Feller's [15, p. 453] result above[20].

8 Miscellanea

45. *A Taylor series approach with Fermi-Dirac integrals.* We introduce the Taylor series of the entire function $\frac{1}{\Gamma(b+ak)}$ around q into the definition (1) of $E_{a,b}(z)$,

$$E_{a,b}(z) = \frac{1}{\Gamma(b)} + \sum_{k=1}^\infty \frac{z^k}{\Gamma(b+ak)} = \frac{1}{\Gamma(b)} + \sum_{k=1}^\infty z^k \sum_{m=0}^\infty \frac{1}{m!}\frac{d^m}{du^m}\frac{1}{\Gamma(u)}\bigg|_{u=q} (ak+b-q)^m$$

and

$$E_{a,b}(z) = \frac{1}{\Gamma(b)} + \frac{1}{\Gamma(q)}\frac{z}{1-z} + \sum_{m=1}^\infty \frac{1}{m!}\frac{d^m}{du^m}\frac{1}{\Gamma(u)}\bigg|_{u=q} a^m \sum_{k=1}^\infty z^k\left(k + \frac{b-q}{a}\right)^m$$

The reversal in the k- and m-sum leads to a confinement of $|z| < 1$. We will now choose $q = b$ and evaluate the series $\sum_{k=1}^\infty k^m z^k$, that converges for $|z| < 1$.

The Fermi-Dirac integral of order p is defined as

$$F_p(y) = \frac{1}{\Gamma(p+1)}\int_0^\infty \frac{x^p}{1+e^{x-y}}\, dx \tag{102}$$

The value of the zero argument in y is immediately written in terms of the Eta function,

$$F_p(0) = \eta(p+1) \tag{103}$$

[20] In the limit $s \to 0$, the right-hand side diverges and the left-hand side is $\int_0^\infty \left(1 - F_R\left(\frac{t}{x^{\frac{1}{\alpha}}}\right)\right) dx = at^a \int_0^\infty (1 - F_R(u))\, u^{-a-1} du = at^a \int_0^\infty \Pr[R > u]\, u^{-a-1} du$ illustrating that the integrand at the origin is $O\left(u^{-a-1}\right)$, leading to a diverging integral. Hence, the factor $\frac{1}{s}$ is essential.

where the Eta function $\eta(s)$ is related to the Riemann Zeta function $\zeta(s)$ as

$$\eta(s) = (1 - 2^{1-s})\zeta(s) \tag{104}$$

By expanding $\frac{1}{1+e^{x-y}} = \frac{e^{-x+y}}{1+e^{-x+y}} = \sum_{k=1}^{\infty}(-1)^{k-1}e^{-k(x-y)}$ for $\operatorname{Re}(y) < 0$ in (102), the Dirichlet series for all complex p is readily deduced as

$$F_p(y) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(e^y)^k}{k^{p+1}} \tag{105}$$

In particular, $F_{-1}(y) = \frac{1}{1+e^{-y}}$. Hence, we can write $\sum_{k=1}^{\infty} k^m (-z)^k = -F_{-m-1}(\log z)$ for $|z| < 1$ and

$$E_{a,b}(-z) = \frac{1}{\Gamma(b)} - \sum_{m=0}^{\infty}\frac{1}{m!}\frac{d^m}{du^m}\frac{1}{\Gamma(u)}\bigg|_{u=b} F_{-m-1}(\log z)a^m \tag{106}$$

For integer negative order and $k > 0$, it can be shown [48] that

$$F_{-k}(y) = \frac{d^{k-1}}{dy^{k-1}}\left(\frac{1}{1+e^{-y}}\right) = \sum_{m=1}^{k}(m-1)!(-1)^{m-1}\mathcal{S}_k^{(m)}\left(\frac{1}{1+e^{-y}}\right)^m$$

where $\mathcal{S}_k^{(m)}$ is the Stirling Number of the Second Kind [1, 24.1.4]. Since $\frac{1}{1+e^{-y}} = 1 - \frac{1}{1+e^{y}}$, which is equivalent to $F_{-1}(y) = 1 - F_{-1}(-y)$, the k-th derivative shows that, for $k > 1$,

$$F_{-k}(y) = (-1)^k F_{-k}(-y) \tag{107}$$

and, thus extending the above for $|z| < 1$ to,

$$E_{a,b}(-z) = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{d^m}{du^m}\frac{1}{\Gamma(u)}\bigg|_{u=b} F_{-m-1}(-\log z)(-a)^m \tag{108}$$

Stretching the convergence constraint in (108) to $z = 1$ and using (103) results in

$$E_{a,b}(-1) = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{d^m}{dz^m}\frac{1}{\Gamma(z)}\bigg|_{z=b}\eta(-m)(-a)^m$$

Further by (104), it holds that $\eta(-m) = (1-2^{1+m})\zeta(-m) = (1-2^{1+m})\frac{(-1)^m}{m+1}B_{m+1}$, because $\zeta(-n) = \frac{(-1)^n}{n+1}B_{n+1}$ and $\zeta(-2n) = 0$ for $n > 0$. Taking into account that the odd Bernoulli numbers $B_{2m+1} = 0$ for $m > 0$, we find

$$E_{a,b}(-1) = -\frac{1}{2\Gamma(b)} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\frac{d^{2k-1}}{dz^{2k-1}}\frac{1}{\Gamma(z)}\bigg|_{z=b}(2^{2k}-1)a^{2k-1}$$

which converges fast for small a. Since $E_{a,b}(-1)$ in (1) is an alternating series with decreasing coefficients for $a > 0$ and $b > 1.462$, it holds that $\frac{1}{\Gamma(b)} < E_{a,b}(-1) < \frac{1}{\Gamma(a+b)}$.

The major interest of the expansion (108) lies in its fast convergence for small a, whereas the definition (1) is converging slower for small a. Moreover, rewriting (9) as

$$E_{a,b}(z) = \frac{1}{n}\sum_{r=0}^{n-1}E_{\frac{a}{n},b}\left(z^{\frac{1}{n}}e^{i\frac{2\pi r}{n}}\right)$$

illustrates that any real a can be transformed to a value smaller than 1 by choosing $n = [a] + 1$, where $[a]$ is the largest integer smaller or equal to a. Indeed, for $|z| < 1$, (108) becomes

$$E_{a,b}(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{du^m} \frac{1}{\Gamma(u)}\bigg|_{u=b} \left\{ \frac{\sum_{r=0}^{[a]} F_{-m-1}\left(-\frac{\log z}{[a]+1} - i\pi\left(\frac{2r}{[a]+1} + 1\right)\right)}{[a]+1} \right\} \left(-\frac{a}{[a]+1}\right)^m$$

46. *A Taylor series approach based on the inverse of the Gamma function.* We present a related approach as in **art**. 45 based on the modified Taylor series, tuneable in the complex parameter p and derived in [48],

$$\frac{1}{\Gamma(z)} = \frac{e^{-pz}}{2\pi i} \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_0^{\infty} e^{-u} \left\{ (p - \log(u) + i\pi)^k - (p - \log(u) - i\pi)^k \right\} du \tag{109}$$

where the Taylor coefficient $c_k(p) = \frac{1}{k!} \frac{d^k}{du^k} \frac{e^{pu}}{\Gamma(u)}\Big|_{u=0}$ of the Taylor series of the entire function $\frac{e^{pz}}{\Gamma(z)} = \sum_{k=0}^{\infty} c_k(p) z^k$ around $z_0 = 0$ is

$$c_k(p) = \frac{1}{2\pi i} \frac{1}{k!} \int_0^{\infty} e^{-u} \left\{ (p - \log(u) + i\pi)^k - (p - \log(u) - i\pi)^k \right\} du \tag{110}$$

We apply (109) to the Mittag-Leffler function $E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}$,

$$E_{a,b}(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} (an+b)^k e^{-p(an+b)} c_k(p) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} (-1)^k \left. \frac{d^k e^{-t(an+b)}}{dt^k} \right|_{t=p} c_k(p)$$

We assume that a reversal of the summations is allowed,

$$\begin{aligned}
E_{a,b}(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{d^k}{dt^k} \left(\sum_{n=0}^{\infty} z^n e^{-t(an+b)} \right)\bigg|_{t=p} c_k(p) \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left(\frac{e^{-tb}}{1 - ze^{-ta}} \right)\bigg|_{t=p} \int_0^{\infty} e^{-u} \left\{ (p - \log(u) + i\pi)^k - (p - \log(u) - i\pi)^k \right\} du
\end{aligned} \tag{111}$$

where $|ze^{-ap}| < 1$ or $p > \frac{\log|z|}{a}$. First, a verification is given. Thereafter, we proceed with the above series (111).

Verification: Assuming that the k-sum and integral in (111) can be reversed, yields

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_0^{\infty} e^{-u} \left\{ \begin{array}{c} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left(\frac{e^{-tb}}{1-ze^{-ta}} \right)\Big|_{t=p} (p - \log(u) + i\pi)^k \\ - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left(\frac{e^{-tb}}{1-ze^{-ta}} \right)\Big|_{t=p} (p - \log(u) - i\pi)^k \end{array} \right\} du$$

where the Taylor series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left(\frac{e^{-tb}}{1 - ze^{-ta}} \right)\bigg|_{t=p} (p - \log(u) - i\pi)^k = \frac{e^{-tb}}{1 - ze^{-ta}}\bigg|_{t=\log u + i\pi} = \frac{u^{-b} e^{-bi\pi}}{1 - zu^{-a} e^{-ai\pi}}$$

leads to

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_0^{\infty} e^{-u} u^{-b} \left\{ \frac{e^{bi\pi}}{1 - zu^{-a} e^{ai\pi}} - \frac{e^{-bi\pi}}{1 - zu^{-a} e^{-ai\pi}} \right\} du$$

Since $\frac{e^{bi\pi}}{1-zu^{-a}e^{ai\pi}} - \frac{e^{-bi\pi}}{1-zu^{-a}e^{-ai\pi}} = 2i\left(\frac{\sin\pi b+zu^{-a}\sin\pi(a-b)}{1-2zu^{-a}\cos a\pi+z^2u^{-2a}}\right)$, we arrive at

$$E_{a,b}(z) = \frac{1}{\pi}\int_0^\infty e^{-u}u^{a-b}\left\{\frac{z\sin\pi(a-b)+u^a\sin\pi b}{u^{2a}-2zu^a\cos a\pi+z^2}\right\}du$$

which equals (84), but, as shown in **art**. 29, which is only correct if $0<a\leq 1$ and $a-b>-1$. In other words, the reversal of operators has limited the scope of the parameters $a\in(0,1]$ and $b<a+1$.

Returning to the series (111). We invoke Leibniz' rule to $\frac{d^k}{dt^k}\left(\frac{e^{-tb}}{1-ze^{-ta}}\right) = \frac{1}{z}\frac{d^k}{dt^k}\left(\frac{e^{t(a-b)}}{e^{ta-\log z}-1}\right)$,

$$\frac{d^k}{dt^k}\left(\frac{e^{t(a-b)}}{e^{ta-\log z}-1}\right) = \sum_{n=0}^{k}\binom{k}{n}\frac{d^{k-n}}{dt^{k-n}}\left(e^{t(a-b)}\right)\frac{d^n}{dt^n}\left(\frac{1}{e^{ta-\log z}-1}\right)$$

The last derivative can be exactly executed [48] as

$$\frac{d^n}{dt^n}\left(\frac{1}{e^{ta-\log z}-1}\right)\Bigg|_{t=p} = a^n\frac{d^n}{dw^n}\left(\frac{1}{e^{w-\log z}-1}\right)\Bigg|_{w=ap} = a^n\sum_{j=0}^{n}\frac{(-1)^j j! e^{j(ap-\log z)}}{\left(e^{ap-\log z}-1\right)^{j+1}}\mathcal{S}_n^{(j)}$$

so that

$$\frac{d^k}{dt^k}\left(\frac{e^{t(a-b)}}{e^{ta}-z}\right)\Bigg|_{t=p} = e^{p(a-b)}\sum_{n=0}^{k}\binom{k}{n}(a-b)^{k-n}a^n\sum_{j=0}^{n}\frac{(-1)^j j! e^{jap}}{\left(e^{ap}-z\right)^{j+1}}\mathcal{S}_n^{(j)}$$

Introducing the k-th derivative into the series (111) leads to

$$E_{a,b}(z) = e^{p(a-b)}\sum_{k=0}^{\infty}(-1)^k c_k(p)\sum_{n=0}^{k}\binom{k}{n}(a-b)^{k-n}a^n\sum_{j=0}^{n}\frac{(-1)^j j! e^{jap}}{\left(e^{ap}-z\right)^{j+1}}\mathcal{S}_n^{(j)} \tag{112}$$

For $z=0$, (112) simplifies to $E_{a,b}(0) = e^{-pb}\sum_{k=0}^\infty(-1)^k c_k(p)\sum_{n=0}^k\binom{k}{n}(a-b)^{k-n}a^n\sum_{j=0}^n(-1)^j j!\mathcal{S}_n^{(j)}$. With a generating function [37, 26.8.10] of the Stirling numbers of the second kind, $x^n = \sum_{j=0}^n j!\,\mathcal{S}_n^{(j)}\binom{x}{j}$, and $\binom{-z}{j} = (-1)^j\binom{z-1+j}{j}$ so that $\binom{-1}{j} = (-1)^j$, we obtain

$$\begin{aligned}E_{a,b}(0) &= e^{-pb}\sum_{k=0}^{\infty}(-1)^k c_k(p)\sum_{n=0}^{k}\binom{k}{n}(a-b)^{k-n}(-a)^n\\ &= e^{-pb}\sum_{k=0}^{\infty}c_k(p)(-1)^k(a-b-a)^k = e^{-pb}\sum_{k=0}^{\infty}c_k(p)\,b^k\end{aligned}$$

Since $\frac{e^{pz}}{\Gamma(z)} = \sum_{k=0}^\infty c_k(p)z^k$, we arrive indeed at $E_{a,b}(0) = e^{-pb}\frac{e^{pb}}{\Gamma(b)} = \frac{1}{\Gamma(b)}$. If $p=0$, then (112) reduces, for $|z|<1$, to

$$E_{a,b}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^k}{du^k}\frac{1}{\Gamma(u)}\Bigg|_{u=0}\sum_{n=0}^{k}\binom{k}{n}(a-b)^{k-n}a^n\sum_{j=0}^{n}\frac{(-1)^j j!}{(1-z)^{j+1}}\mathcal{S}_n^{(j)}$$

Further, since $p>\frac{\log|z|}{a}$, we choose $p=\frac{\log|z|}{a}+t$ with $t>0$ and (112) becomes

$$E_{a,b}(z) = |z|^{-\frac{b}{a}}e^{t(a-b)}\sum_{k=0}^{\infty}(-1)^k c_k\left(\frac{\log|z|}{a}+t\right)\sum_{n=0}^{k}\binom{k}{n}(a-b)^{k-n}a^n\sum_{j=0}^{n}\frac{(-1)^j j! e^{jta}}{\left(e^{ta}-\frac{z}{|z|}\right)^{j+1}}\mathcal{S}_n^{(j)}$$

Unfortunately, the series (112) and its modification above are found to be numerically very inaccurate.

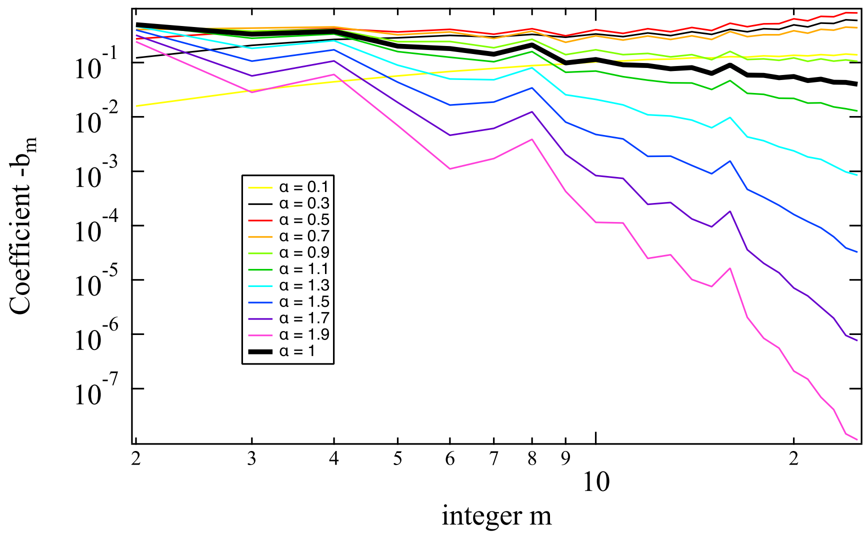

Figure 1: The coefficients $-b_m$ versus m up to $m = 25$ on a log-log plot (ignoring the only positive coefficient $b_1 = \frac{\Gamma(b)}{\Gamma(a+b)}$) for various values of $a = \alpha$ with $b = 1$ of the Mittag-Leffler function $E_{a,b}(z)$. The thick line represents the exponential function e^z.

47. *A product form for* $E_{a,b}(z)$. Inspired by Euler's recipe [12, art. 283, p. 237-238] that led him to the generating function of the prime numbers, the famous Euler product $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ of the Riemann Zeta function $\zeta(z)$ for $\mathrm{Re}(z) > 1$, we have applied Euler's idea to the Taylor series $f(z) = \sum_{k=0}^{\infty} f_k(z_0)(z - z_0)^k$ of a complex function, which results into the product form [48]

$$f(z) = \frac{f_0}{\prod_{m=1}^{\infty} \left(1 - b_m(z_0)(z - z_0)^m\right)} \tag{113}$$

The coefficients $b_m(z_0)$ satisfy a recursion,

$$b_m(z_0) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k\ f_0^k(z_0)}\, s[k,m](z_0) - \sum_{n|m;n<m} \frac{n}{m} \left(b_n(z_0)\right)^{\frac{m}{n}} \tag{114}$$

with starting value $b_1(z_0) = \frac{f_1(z_0)}{f_0(z_0)}$. The first sum in the recursion (114) is precisely equal to the Taylor coefficient of the Taylor series of $\log f(z)$ in (37). Fig. 1 shows the computation of the coefficients $b_m = b_m(0)$, via the recursion (114), for Taylor coefficients $f_k = \frac{1}{\Gamma(ak+b)}$ of the Mittag-Leffler function $E_{a,b}(z)$ around $z_0 = 0$. Fig. 1 suggests that $(b_m)_{E_\alpha(z)} \sim m^{-\alpha}$ for sufficiently large m. Since the computation of the recursion (114) is expensive for $E_\alpha(z)$, only the first 25 coefficients have been computed. However, for the exponential function e^z, drawn in black thick line in Fig. 1, the recursion (114) considerably simplifies as the first sum with characteristic coefficients vanishes as follows from $\log f(z)$ in (37), because $\log e^z = z$. This observation demonstrates that the coefficients $(b_m)_{e^z}$ are rational numbers that do not dependent upon the Taylor coefficients of e^z, apart from $f_1 = f_0 = 1$, leading to $(b_1)_{e^z} = 1$. The recursion (114) simplifies to

$$(b_m)_{e^z} = -\frac{1}{m} \sum_{n|m;n<m} n\ \left((b_n)_{e^z}\right)^{\frac{m}{n}} = -\frac{1}{m}\left(1 + \sum_{n|m;1<n<m} n\ \left((b_n)_{e^z}\right)^{\frac{m}{n}}\right)$$

and contains the sum of the divisors of m scaled by integer powers of previous divisor sums $(b_n)_{e^z}$. In particular, if $m = p$ is a prime p, then $(b_p)_{e^z} = -\frac{1}{p} b_1^m = -\frac{1}{p}$ and only primes satisfy $(b_m)_{e^z} = -\frac{1}{m}$, although, for large m, all $(b_m)_{e^z} \sim \frac{1}{m}$ as shown in Fig. 2. Hence, apart from relatively small

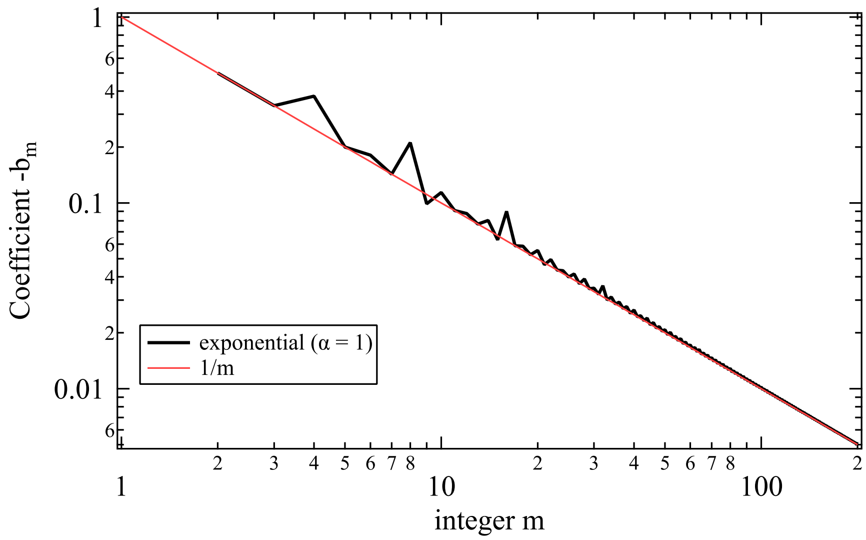

Figure 2: The coefficients $(b_m)_{e^z}$ on a log-log plot.

fluctuations around the asymptotic $(b_m)_{e^z} \sim m^{-1}$ due to the irregular behavior of the number of divisors [24] of an integer m, the scaling law $(b_m)_{E_\alpha(z)} \sim m^{-\alpha}$ is numerically demonstrated for $\alpha = 1$.

The convergence of the product (113) around z_0 is rather difficult to determine in general. For the exponential function e^z, the product (113) around $z_0 = 0$ converges for $|z - z_0| = |z| < 1$. For the Mittag-Leffler function $E_\alpha(z)$, the product (113) around $z_0 = 0$ converge for $|z|$ around 1; slightly larger than one for $\alpha > 1$ and slightly smaller than one for $\alpha < 1$. The small convergence radius clearly limits the practical use of the product (113).

48. *Derivation of $E_{a,b}(z)$ with respect to the parameters a and b.* From the definition (1), partial differentiating yields

$$\begin{aligned}
\frac{\partial}{\partial a} E_{a,b}(z) &= \frac{\partial}{\partial a}\left(\frac{1}{\Gamma(b)} + \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(b+ak)}\right) \\
&= \sum_{k=1}^{\infty} \frac{d}{dy} \frac{1}{\Gamma(y)}\bigg|_{y=b+ak} \frac{dy}{da} z^k = -\sum_{k=1}^{\infty} \frac{\psi(y)}{\Gamma(y)}\bigg|_{y=b+ak} k z^k
\end{aligned}$$

while, similarly but containing index $k = 0$,

$$\frac{\partial}{\partial b} E_{a,b}(z) = -\sum_{k=0}^{\infty} \frac{\psi(y)}{\Gamma(y)}\bigg|_{y=b+ak} z^k$$

Hence, we observe that

$$\frac{\partial}{\partial a} E_{a,b}(z) = z \frac{\partial^2}{\partial z \partial b} E_{a,b}(z)$$

Partial differentiating m-times gives

$$\frac{\partial^m}{\partial a^m} E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{d^m}{dy^m} \frac{1}{\Gamma(y)}\bigg|_{y=b+ak} k^m z^k$$

which suggest to let $z = ye^w$ so that

$$\frac{\partial^m}{\partial a^m} E_{a,b}(ye^w) = \sum_{k=0}^{\infty} \frac{d^m}{dy^m} \frac{1}{\Gamma(y)}\bigg|_{y=b+ak} k^m y^k e^{kw} = \frac{\partial^m}{\partial w^m} \sum_{k=0}^{\infty} \frac{d^m}{dy^m} \frac{1}{\Gamma(y)}\bigg|_{y=b+ak} y^k e^{kw}$$

leading to the partial differentiation equation for any integer $m \geq 0$ and any y (independent of a, b and w),

$$\frac{\partial^m}{\partial a^m} E_{a,b}(ye^w) = \frac{\partial^{2m}}{\partial w^m \partial b^m} E_{a,b}(ye^w) \tag{115}$$

49. *Möbius inversion.* The first Möbius inversion pair is

$$g(x) = \sum_{n=1}^{\infty} f(nx) \qquad \Longleftrightarrow \qquad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx) \tag{116}$$

where $\mu(n)$ is the Möbius function. The Möbius function satisfies the functional equation

$$\sum_{k|n} \mu(k) = \delta_{n,1} \tag{117}$$

and shows that $\mu(1) = 1$. Since $k = 1$ is always a divisor, we obtain from (117) the recursion

$$\mu(m) = -1 - \sum_{k|m;1<k<m} \mu(k)$$

from which $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$ and so on. The Möbius function is an important function in number theory and in the theory of the Riemann Zeta function [24], where it is shown that $\mu(n) = (-1)^k$ if the prime factorization of n contains k different primes, else $\mu(n) = 0$. Hence, $|\mu(n)| = 1$.

Let $f(x) = \frac{z^x}{\Gamma(b+x)}$ in (116), then

$$g(x) = \sum_{n=1}^{\infty} f(nx) = \sum_{n=1}^{\infty} \frac{z^{nx}}{\Gamma(b+nx)} = E_{x,b}(z^x) - \frac{1}{\Gamma(b)}$$

and Möbius inversion $f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx)$ in (116) yields, for $x \neq 0$,

$$\frac{z^x}{\Gamma(b+x)} = \sum_{n=1}^{\infty} \mu(n) \left(E_{nx,b}(z^{nx}) - \frac{1}{\Gamma(b)} \right) \tag{118}$$

With $y = z^x$, (118) simplifies to $\frac{y}{\Gamma(b+x)} = \sum_{n=1}^{\infty} \mu(n) \left(E_{nx,b}(y^n) - \frac{1}{\Gamma(b)} \right)$. After multiplying both sides in (118) by z^{b-1}, differentiating with respect to z and invoking (19) again leads to (118) with b replaced by $b-1$.

Invoking $E_{2a,b}\left(z^2\right)=\frac{E_{a,b}(z)+E_{a,b}(-z)}{2}$ in **art**. 2 in (118)

$$\begin{aligned}\frac{z^x}{\Gamma(b+x)}&=\sum_{n=1}^{\infty}\mu(n)\left(2E_{2nx,b}\left(z^{2nx}\right)-E_{nx,b}\left(-z^{nx}\right)-\frac{1}{\Gamma(b)}\right)\\&=2\sum_{n=1}^{\infty}\mu(n)\left(E_{2nx,b}\left(z^{2nx}\right)-\frac{1}{\Gamma(b)}\right)-\sum_{n=1}^{\infty}\mu(n)\left(E_{nx,b}\left(-z^{nx}\right)-\frac{1}{\Gamma(b)}\right)\end{aligned}$$

and using (118) leads to

$$\sum_{n=1}^{\infty}\mu(n)\left(E_{nx,b}\left(-z^{nx}\right)-\frac{1}{\Gamma(b)}\right)=\frac{2z^{2x}}{\Gamma(b+2x)}-\frac{z^x}{\Gamma(b+x)}\tag{119}$$

which is an instance of the general Möbius function identity, proved in [48],

$$\sum_{j=1}^{n}\mu(j)\sum_{q=1}^{\left[\frac{n}{j}\right]}(-1)^q f(qj)=2f(2)-f(1)\tag{120}$$

holds for any function f and any $n>1$.

50. *Mertens function.* Applying Abel summation using the Mertens[21] function $\gamma_{-1}(k)=\sum_{l=1}^{k}\mu(l)$, we obtain

$$f(x)=\sum_{k=1}^{\infty}\gamma_{-1}(k)\ \left(g(kx)-g\left((k+1)\,x\right)\right)+\lim_{N\to\infty}g(Nx)\gamma_{-1}(N)$$

Hence,

$$\frac{z^x}{\Gamma(b+x)}=\sum_{k=1}^{\infty}\gamma_{-1}(k)\ \left(E_{kx,b}\left(z^{kx}\right)-E_{kx+x,b}\left(z^{kx+x}\right)\right)+\lim_{N\to\infty}\left(E_{Nx,b}\left(z^{Nx}\right)-\frac{1}{\Gamma(b)}\right)\gamma_{-1}(N)$$

and the limit vanishes if $x>0$, resulting in

$$\frac{z^x}{\Gamma(b+x)}=\sum_{k=1}^{\infty}\gamma_{-1}(k)\ \left(E_{kx,b}\left(z^{kx}\right)-E_{kx+x,b}\left(z^{kx+x}\right)\right)\qquad\text{for } x>0\tag{121}$$

With $\int_{kx}^{(k+1)x}\frac{d}{da}E_{a,b}\left(z^a\right)da=E_{kx+x,b}\left(z^{kx+x}\right)-E_{kx,b}\left(z^{kx}\right)$, the corresponding integral representation of (121) is

$$\begin{aligned}\frac{z^x}{\Gamma(b+x)}&=-\sum_{k=1}^{\infty}\gamma_{-1}(k)\int_{kx}^{(k+1)x}\frac{d}{da}E_{a,b}\left(z^a\right)da\\&=-\sum_{k=1}^{\infty}\int_{kx}^{(k+1)x}\gamma_{-1}\left(\left[\frac{a}{x}\right]\right)\frac{d}{da}E_{a,b}\left(z^a\right)da\end{aligned}$$

and

$$\frac{z^x}{\Gamma(b+x)}=-\int_{x}^{\infty}\gamma_{-1}\left(\left[\frac{a}{x}\right]\right)\frac{d}{da}E_{a,b}\left(z^a\right)da\qquad\text{for } x>0\tag{122}$$

[21] A sufficient condition to prove the Riemann Hypothesis is to demonstate that the Mertens function behaves as $\gamma_{-1}(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for large x.

Similarly for (119), it holds that

$$\frac{2z^{2x}}{\Gamma(b+2x)} - \frac{z^{x}}{\Gamma(b+x)} = -\int_{x}^{\infty} \gamma_{-1}\left(\left[\frac{a}{x}\right]\right) \frac{d}{da} E_{a,b}\left(-z^{a}\right) da \qquad \text{for } x > 0 \tag{123}$$

Although (122) and (123) are remarkable, further progress to estimate the order of $\gamma_{-1}([x])$ requires a study of $\frac{d}{da}E_{a,b}(yz^{a})$.

51. *Apelblat series.* Inspired by series in [2], we call

$$g_{a,b}(t) = \sum_{k=0}^{\infty} f_k t^{b+k-1} E_{a,b+k}\left(xt^{a}\right)$$

an Apelblat series, where the Taylor series of the function $f(z) = \sum_{k=0}^{\infty} f_k z^k$ around $z_0 = 0$ converges for $|z| \leq R$. Evidently, if $f_k = 0$ for $k > n$, then $f(z)$ is a polynomial of order n in z and $f(z)$ is an entire function. We generalize the method of Apelblat [2]. We take the Laplace transform $\mathcal{L}[.]$ of both sides and use (54),

$$\mathcal{L}\left[g_{a,b}(t)\right] = \sum_{k=0}^{\infty} f_k \mathcal{L}\left[t^{b+k-1}E_{a,b+k}\left(xt^{a}\right)\right] = \sum_{k=0}^{\infty} f_k \frac{s^{a-b-k}}{s^{a}-x} = \frac{s^{a-b+\beta}}{s^{a}-x}\sum_{k=0}^{\infty} f_k s^{-k-\beta}$$

and obtain a product of Laplace transformed functions

$$\mathcal{L}\left[g_{a,b}(t)\right] = \mathcal{L}\left[t^{b-\beta-1}E_{a,b-\beta}\left(xt^{a}\right)\right]\frac{1}{s^{\beta}} f\left(\frac{1}{s}\right) \tag{124}$$

The convolution theorem for the Laplace transform suggests us to find the inverse Laplace transform $\mathcal{L}^{-1}\left[\frac{1}{s^{\beta}}f\left(\frac{1}{s}\right)\right]$ of $\frac{1}{s^{\beta}}f\left(\frac{1}{s}\right)$. Apelblat [2] observes and demonstrates that elegant series follow if a closed form for $\mathcal{L}^{-1}\left[\frac{1}{s^{\beta}}f\left(\frac{1}{s}\right)\right]$ exist, else we can proceed with (193) and Hankel's integral (182)

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^{\beta}}f\left(\frac{1}{s}\right)\right] &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s^{\beta}}f\left(\frac{1}{s}\right)e^{st}ds = \frac{1}{2\pi i}\sum_{k=0}^{\infty} f_k \int_{c-i\infty}^{c+i\infty} s^{-k-\beta}e^{st}ds \\
&= t^{\beta-1}\sum_{k=0}^{\infty} f_k \frac{t^{k}}{\Gamma(\beta+k)}
\end{aligned}$$

After taking the inverse Laplace transform of both sides in (124), we formally arrive at the Apelblat series, for the free parameter $\operatorname{Re}(\beta) > 0$,

$$\sum_{k=0}^{\infty} f_k t^{b+k-1}E_{a,b+k}\left(xt^{a}\right) = \int_{0}^{t}(t-u)^{b-\beta-1}E_{a,b-\beta}\left(x(t-u)^{a}\right)u^{\beta-1}\sum_{k=0}^{\infty} f_k\frac{u^{k}}{\Gamma(\beta+k)}du \tag{125}$$

which directly follows from (57) for $w = \beta + k$ and $b \to b - \beta$. Hence, the property (125) of the Apelblat series is a consequence of the generalized integration property in **art**. 19.

Examples a. The Taylor series of $(1+cz)^{p} = \sum_{k=0}^{\infty}\binom{p}{k}c^{k}z^{k}$ converges for all complex p provided $|z| < 1$. The right-hand series in (125) becomes with $f_k = \binom{p}{k}c^{k} = \frac{\Gamma(p+1)}{k!\Gamma(p-k+1)}c^{k} = \frac{\Gamma(-p+k)}{k!\Gamma(-p)}(-c)^{k}$

$$\sum_{k=0}^{\infty} f_k\frac{u^{k}}{\Gamma(\beta+k)} = \frac{1}{\Gamma(-p)}\sum_{k=0}^{\infty}\frac{\Gamma(-p+k)}{k!\Gamma(\beta+k)}(-cu)^{k}$$

which reduces if we choose $\beta=-p$ to $\sum_{k=0}^{\infty} f_k \frac{u^k}{\Gamma(\beta+k)} = \frac{1}{\Gamma(-p)} e^{-cu}$. The Apelblat series (125) then becomes, for $q=-p$ and $0<\operatorname{Re}(q)<\operatorname{Re}(b)$,

$$\sum_{k=0}^{\infty} \binom{-q}{k} c^k t^{b+k-1} E_{a,b+k}(xt^a) = \frac{1}{\Gamma(q)} \int_0^t u^{b-q-1} E_{a,b-q}(xu^a)(t-u)^{q-1} e^{-c(t-u)} du$$

For $q=1$, $b=2$, $c=x=1$, we retrieve the series [2, eq. (64)] and for $q=2$, $b=2$, $c=x=1$, the series [2, eq. (68)].

b. The Taylor series of the Bessel function $\left(\frac{z}{2}\right)^{-p} J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+1+k)} \left(\frac{z}{2}\right)^{2k}$ in [1, 9.1.10] has only even Taylor coefficients $j_{2k} = \frac{\left(-\frac{1}{4}\right)^k}{k!\Gamma(p+1+k)}$ and the odd $j_{2k+1}=0$. If $f_{2k+1}=0$, then

$$\sum_{k=0}^{\infty} f_k \frac{u^k}{\Gamma(\beta+k)} = \sum_{k=0}^{\infty} f_{2k} \frac{u^{2k}}{\Gamma(\beta+2k)} = \frac{\sqrt{\pi}}{2^{\beta-1}} \sum_{k=0}^{\infty} f_{2k} \frac{\left(\frac{u}{2}\right)^{2k}}{\Gamma\left(\frac{\beta}{2}+k\right)\Gamma\left(\frac{\beta+1}{2}+k\right)}$$

where the duplication formula of Gamma function is used. If $f_{2k} = \binom{p}{k} c^{2k} = \frac{\Gamma(-p+k)}{k!\Gamma(-p)} \left(-c^2\right)^k$ then

$$\sum_{k=0}^{\infty} f_k \frac{u^k}{\Gamma(\beta+k)} = \frac{\sqrt{\pi}}{2^{\beta-1}\Gamma(-p)} \sum_{k=0}^{\infty} \frac{\Gamma(-p+k)(-1)^k \left(\frac{cu}{2}\right)^{2k}}{k!\Gamma\left(\frac{\beta}{2}+k\right)\Gamma\left(\frac{\beta+1}{2}+k\right)}$$

Choosing $-p=\frac{\beta}{2}$ yields

$$\sum_{k=0}^{\infty} f_k \frac{u^k}{\Gamma(\beta+k)} = \frac{\sqrt{\pi}}{2^{\beta-1}\Gamma\left(\frac{\beta}{2}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{cu}{2}\right)^{2k}}{k!\Gamma\left(\frac{\beta-1}{2}+1+k\right)} = \frac{\sqrt{\pi}\left(\frac{cu}{2}\right)^{-\frac{\beta-1}{2}}}{2^{\beta-1}\Gamma\left(\frac{\beta}{2}\right)} J_{\frac{\beta-1}{2}}(cu)$$

and

$$\sum_{k=0}^{\infty} \binom{-\frac{\beta}{2}}{k} c^{2k} t^{b+2k-1} E_{a,b+2k}(xt^a) = \frac{\sqrt{\pi}\left(\frac{2}{c}\right)^{\frac{\beta-1}{2}}}{2^{\beta-1}\Gamma\left(\frac{\beta}{2}\right)} \int_0^t (t-u)^{b-\beta-1} E_{a,b-\beta}\left(x(t-u)^a\right) u^{\frac{\beta-1}{2}} J_{\frac{\beta-1}{2}}(cu)\, du$$

while choosing $-p=\frac{\beta+1}{2}$ yields

$$\sum_{k=0}^{\infty} f_k \frac{u^k}{\Gamma(\beta+k)} = \frac{\sqrt{\pi}}{2^{\beta-1}\Gamma\left(\frac{\beta+1}{2}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{cu}{2}\right)^{2k}}{k!\Gamma\left(\frac{\beta}{2}+k\right)} = \frac{\sqrt{\pi}\left(\frac{cu}{2}\right)^{-\frac{\beta-2}{2}}}{2^{\beta-1}\Gamma\left(\frac{\beta+1}{2}\right)} J_{\frac{\beta-2}{2}}(cu)$$

and the Apelblat series (125) then becomes

$$\sum_{k=0}^{\infty} \binom{-\frac{\beta+1}{2}}{k} c^{2k} t^{b+2k-1} E_{a,b+2k}(xt^a) = \frac{\sqrt{\pi}\left(\frac{c}{2}\right)^{1-\frac{\beta}{2}}}{2^{\beta-1}\Gamma\left(\frac{\beta+1}{2}\right)} \int_0^t (t-u)^{b-\beta-1} E_{a,b-\beta}\left(x(t-u)^a\right) u^{\frac{\beta}{2}} J_{\frac{\beta}{2}-1}(cu)\, du$$

For $\beta=0$, $b=1$, $c=x=1$, we retrieve the series [2, eq. (74)],

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} t^{2k} E_{a,1+2k}(t^a) = -\int_0^t E_a(u^a) J_1(t-u)\, du$$

c. Let us now consider the hypergeometric function [1, 15.1] with Taylor series around the origin,

$$F(p,q;r;z) = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)} \sum_{k=0}^{\infty} \frac{\Gamma(p+k)\Gamma(q+k)}{\Gamma(r+k)\,k!} z^k \tag{126}$$

With (126), the Apelblat series (125) at the right-hand side becomes

$$\sum_{k=0}^{\infty} f_k \frac{c^k u^k}{\Gamma(\beta+k)} = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)} \sum_{k=0}^{\infty} \frac{\Gamma(p+k)\Gamma(q+k)}{\Gamma(r+k)\Gamma(\beta+k)} \frac{c^k u^k}{k!}$$

and, if we choose q equal to β, then

$$\sum_{k=0}^{\infty} f_k \frac{c^k u^k}{\Gamma(\beta+k)} = \frac{\Gamma(r)}{\Gamma(p)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(p+k)}{\Gamma(r+k)} \frac{(cu)^k}{k!} = \frac{1}{\Gamma(\beta)} M(p,r,cu)$$

where $M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{u^k}{k!}$ is Kummer's confluent hypergeometric function [1, 13.1.2]. The Apelblat series (125) thus becomes

$$\frac{\Gamma(r)}{\Gamma(p)} \sum_{k=0}^{\infty} \frac{\Gamma(p+k)\Gamma(\beta+k)}{\Gamma(r+k)\,k!} c^k t^{b+k-1} E_{a,b+k}(xt^a) = \int_0^t (t-u)^{b-\beta-1} E_{a,b-\beta}(x(t-u)^a)\, u^{\beta-1} M(p,r,cu)\, du$$

In order to use the property [1, 13.3.2] of the Kummer function,

$$\lim_{a\to\infty} M\left(a,b,-\frac{z}{a}\right) = \Gamma(b)\, z^{\frac{1-b}{2}} J_{b-1}\left(2\sqrt{z}\right)$$

we first choose $c = -\frac{1}{p}$ and then take the limit $p \to \infty$ of both sides becomes, with $\lim_{p\to\infty} \frac{\Gamma(p+k)}{\Gamma(p)p^k} = 1$ (see [1, 6.1.47]),

$$\sum_{k=0}^{\infty} \frac{\Gamma(\beta+k)}{\Gamma(r+k)\,k!} (-1)^k t^{b+k-1} E_{a,b+k}(xt^a) = \int_0^t (t-u)^{b-\beta-1} E_{a,b-\beta}(x(t-u)^a)\, u^{\beta+\frac{1-r}{2}-1} J_{r-1}\left(2\sqrt{u}\right) du$$

Let $\beta = r$, then $r = b-1$, we have

$$\sum_{k=0}^{\infty} (-1)^k \frac{t^{b+k-1}}{k!} E_{a,b+k}(xt^a) = \int_0^t E_a(x(t-u)^a)\, u^{\frac{b-2}{2}} J_{b-2}\left(2\sqrt{u}\right) du$$

which simplified for $b=2$ to

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^k}{(k-1)!} E_{a,k+1}(xt^a) = \int_0^t E_a(xu^a)\, J_0\left(2\sqrt{t-u}\right) du$$

52. *First limit for* $E_{a,b}(z)$. The Gauss product $\Pi(n,z)$ in (150) indicates that

$$\frac{1}{\Pi(n,z)} = (1+z)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\cdots\left(1+\frac{z}{n}\right) e^{-z\log n}$$

and defines the Gamma function $\Gamma(z+1) = \lim_{n\to\infty} \Pi(n,z)$ as a limit. Thus, $\Gamma(z) = \frac{1}{z}\lim_{n\to\infty} \Pi(n,z)$ and

$$\begin{aligned} E_{a,b}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)} = \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{(b+ak)\,z^k}{\Pi(n,b+ak)} \\ &= \sum_{k=0}^{\infty} \lim_{n\to\infty} (b+ak)\, z^k \prod_{j=1}^{n} \left(1+\frac{b+ak}{j}\right) e^{-(b+ak)\log n} \end{aligned}$$

With the generating function [37, Chapter 26] of the Stirling Numbers of the First Kind for $k>0$,

$$\prod_{j=1}^{n}\left(1+\frac{z}{j}\right)=\frac{1}{n!}\prod_{j=1}^{n}(z+j)=\frac{1}{n!}\sum_{j=0}^{n}S_{n+1}^{(j+1)}\left(-1\right)^{n-j}z^{j}$$

we obtain

$$E_{a,b}\left(z\right)=\sum_{k=0}^{\infty}\lim_{n\to\infty}\frac{1}{n!}\sum_{j=0}^{n}S_{n+1}^{(j+1)}\left(-1\right)^{n-j}z^{k}\left(b+ak\right)^{j+1}e^{-(b+ak)\log n}$$

and arrive at the first limit

$$E_{a,b}\left(z\right)=\lim_{n\to\infty}\frac{1}{n!}\sum_{j=0}^{n}S_{n+1}^{(j+1)}\left(-1\right)^{n-j}\left(\sum_{k=0}^{\infty}\frac{z^{k}\left(b+ak\right)^{j+1}}{n^{b+ak}}\right) \tag{127}$$

53. We will derive a finite series for the infinite series in brackets in (127), which we define as

$$T_{j}\left(z;n\right)=\sum_{k=0}^{\infty}\frac{z^{k}\left(b+ak\right)^{j}}{n^{b+ak}} \tag{128}$$

Since n in (127) grows unboundedly, we may assume that $\left|\frac{z}{n^{a}}\right|<1$ and we find

$$T_{0}\left(z;n\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{n^{b+ak}}=\frac{1}{n^{b}}\sum_{k=0}^{\infty}\left(\frac{z}{n^{a}}\right)^{k}=\frac{1}{n^{b}}\frac{1}{1-\frac{z}{n^{a}}}=\frac{n^{a-b}}{n^{a}-z}$$

Let $n=e^{x}$, then

$$T_{j}\left(z;e^{x}\right)=\sum_{k=0}^{\infty}z^{k}\left(b+ak\right)^{j}e^{-(b+ak)x}$$

and $T_{0}\left(z;e^{x}\right)=\frac{e^{(a-b)x}}{e^{ax}-z}=\frac{e^{-bx}}{1-e^{\log z-ax}}$. Differentiation shows that $\frac{d}{dx}T_{j-1}\left(z;e^{x}\right)=-T_{j}\left(z;e^{x}\right)$ and iteration leads us to

$$T_{j}\left(z;e^{x}\right)=(-1)^{j}\frac{d^{j}}{dx^{j}}T_{0}\left(z;e^{x}\right)=(-1)^{j}\frac{d^{j}}{dx^{j}}\left(\frac{e^{(a-b)x}}{e^{ax}-z}\right)$$

Invoking Leibniz' rule,

$$\begin{aligned}(-1)^{j}T_{j}\left(z;e^{x}\right)&=\sum_{m=0}^{j}\binom{j}{m}\frac{d^{j-m}}{dx^{j-m}}e^{(a-b)x}\frac{d^{m}}{dx^{m}}\left(\frac{1}{e^{ax}-z}\right)\\&=\sum_{m=0}^{j}\binom{j}{m}\left(a-b\right)^{j-m}e^{(a-b)x}\frac{d^{m}}{dx^{m}}\left(\frac{1}{e^{ax}-z}\right)\end{aligned}$$

Now,

$$\frac{d^{m}}{dx^{m}}\left(\frac{1}{e^{ax}-z}\right)=\frac{d^{m}}{dy^{m}}\left(\frac{1}{e^{y}-z}\right)\frac{d^{m}y}{dx^{m}}=a^{m}\frac{d^{m}}{dy^{m}}\left(\frac{1}{e^{y}-z}\right)$$

Provided that $b\neq 0$ and recalling that $\mathcal{S}_{m}^{(0)}=\delta_{m0}$ for the Stirling Numbers of the Second Kind $\mathcal{S}_{m}^{(k)}$, the Taylor series, derived in [48],

$$\frac{1}{e^{z+b}-1}=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\frac{(-1)^{k}k!e^{kb}}{(e^{b}-1)^{k+1}}\mathcal{S}_{m}^{(k)}\right)\frac{z^{m}}{m!} \tag{129}$$

indicates that

$$\frac{d^m}{dz^m}\left(\frac{1}{e^{z+b}-1}\right)\Bigg|_{z=0} = \sum_{k=0}^{m}\frac{(-1)^k k! e^{kb}}{(e^b-1)^{k+1}}\mathcal{S}_m^{(k)} = \sum_{k=0}^{m}\frac{(-1)^k k! e^{k(b+z)}}{(e^{b+z}-1)^{k+1}}\mathcal{S}_m^{(k)}\Bigg|_{z=0}$$

Thus, $\frac{d^m}{dz^m}\left(\frac{1}{e^{z+b}-1}\right) = \sum_{k=0}^{m}\frac{(-1)^k k! e^{k(b+z)}}{(e^{b+z}-1)^{k+1}}\mathcal{S}_m^{(k)}$ and with $\frac{1}{e^y-z} = \frac{1}{z}\frac{1}{e^{y-\log z}-1}$, we find

$$\frac{d^m}{dy^m}\left(\frac{1}{e^y-z}\right) = \frac{1}{z}\frac{d^m}{dy^m}\left(\frac{1}{e^{y-\log z}-1}\right) = \frac{1}{z}\sum_{k=1}^{m}\frac{(-1)^k k! e^{k(y-\log z)}}{(e^{y-\log z}-1)^{k+1}}\mathcal{S}_m^{(k)}$$

and

$$\begin{aligned}\frac{d^m}{dx^m}\left(\frac{1}{e^{ax}-z}\right) &= a^m\frac{d^m}{dy^m}\left(\frac{1}{e^y-z}\right) = \frac{a^m}{z}\sum_{k=1}^{m}\frac{(-1)^k k! e^{k(ax-\log z)}}{(e^{ax-\log z}-1)^{k+1}}\mathcal{S}_m^{(k)}\\ &= a^m\sum_{k=1}^{m}\frac{(-1)^k k! e^{kax}}{(e^{ax}-z)^{k+1}}\mathcal{S}_m^{(k)}\end{aligned}$$

Combining all,

$$\begin{aligned}(-1)^j T_j(z;e^x) &= \sum_{m=0}^{j}\binom{j}{m}(a-b)^{j-m}e^{(a-b)x}\frac{d^m}{dx^m}\left(\frac{1}{e^{ax}-z}\right)\\ &= (a-b)^j e^{(a-b)x}\sum_{m=0}^{j}\binom{j}{m}\left(\frac{a}{a-b}\right)^m\sum_{k=0}^{m}\frac{(-1)^k k! e^{kax}}{(e^{ax}-z)^{k+1}}\mathcal{S}_m^{(k)}\end{aligned}$$

In summary, provided that $e^x > |z|$, we arrive at

$$T_j(z;e^x) = \sum_{k=0}^{\infty} z^k (b+ak)^j e^{-(b+ak)x} = (b-a)^j e^{(a-b)x}\sum_{m=0}^{j}\binom{j}{m}\left(\frac{a}{a-b}\right)^m\sum_{k=0}^{m}\frac{(-1)^k k! e^{kax}}{(e^{ax}-z)^{k+1}}\mathcal{S}_m^{(k)} \tag{130}$$

54. *Second limit for* $E_{a,b}(z)$. We transform $n = e^x$ back in (130),

$$T_j(z;n) = \sum_{k=0}^{\infty} z^k\frac{(b+ak)^j}{n^{(b+ak)}} = (b-a)^j n^{(a-b)}\left(\sum_{m=0}^{j}\binom{j}{m}\left(\frac{a}{a-b}\right)^m\sum_{k=0}^{m}\frac{(-1)^k k! n^{ka}}{(n^a-z)^{k+1}}\mathcal{S}_m^{(k)}\right)$$

Substitution into the first limit (127) yields

$$\begin{aligned}E_{a,b}(z) &= \lim_{n\to\infty}\frac{1}{n!}\sum_{j=0}^{n} S_{n+1}^{(j+1)}(-1)^{n-j}\left(\sum_{k=0}^{\infty}\frac{z^k(b+ak)^{j+1}}{n^{b+ak}}\right)\\ &= \lim_{n\to\infty}\frac{(-1)^{n-1}n^{(a-b)}}{n!}\sum_{j=1}^{n+1}S_{n+1}^{(j)}(a-b)^j n^{(a-b)}\left(\sum_{m=0}^{j}\binom{j}{m}\left(\frac{a}{a-b}\right)^m\sum_{k=0}^{m}\frac{(-1)^k k! n^{ka}}{(n^a-z)^{k+1}}\mathcal{S}_m^{(k)}\right)\end{aligned}$$

After using $S_m^{(0)} = \delta_{0m}$ to produce the same zero lower bound[22] in the summations, we arrive at the second limit

$$E_{a,b}(z) = \lim_{n\to\infty}\frac{(-1)^{n-1}n^{(a-b)}}{n!}\sum_{j=0}^{n+1}S_{n+1}^{(j)}(a-b)^j\sum_{m=0}^{j}\binom{j}{m}\left(\frac{a}{a-b}\right)^m\sum_{k=0}^{m}\frac{(-1)^k k! n^{ka}}{(n^a-z)^{k+1}}\mathcal{S}_m^{(k)} \tag{131}$$

[22] Expression (131) also holds for all lower bounds equal to 1 instead of 0.

We concentrate on the finite triple sum in (131)

$$U_n = \sum_{j=0}^{n+1} S_{n+1}^{(j)} (a-b)^j \sum_{m=0}^{j} \binom{j}{m} \left(\frac{a}{a-b}\right)^m \sum_{k=0}^{m} \frac{(-1)^k k! n^{ka}}{(n^a - z)^{k+1}} \mathcal{S}_m^{(k)}$$

and will present several alternatives for U_n and thus slightly different limits of (131). We reverse the m-sum and k-sum,

$$U_n = \sum_{j=0}^{n+1} S_{n+1}^{(j)} (a-b)^j \sum_{k=0}^{j} \frac{(-1)^k k! n^{ka}}{(n^a - z)^{k+1}} \sum_{m=k}^{j} \binom{j}{m} \left(\frac{a}{a-b}\right)^m \mathcal{S}_m^{(k)}$$

Reversing additionally the j-sum and k-sum,

$$U_n = \frac{1}{n^a - z} \sum_{k=0}^{n+1} (-1)^k k! \left(\frac{n^a}{n^a - z}\right)^k \sum_{j=k}^{n+1} S_{n+1}^{(j)} (a-b)^j \sum_{m=k}^{j} \binom{j}{m} \left(\frac{a}{a-b}\right)^m \mathcal{S}_m^{(k)}$$

Finally, we reverse the j- and m-sum,

$$U_n = \frac{1}{n^a - z} \sum_{k=0}^{n+1} (-1)^k k! \left(\frac{n^a}{n^a - z}\right)^k \sum_{m=k}^{n+1} \left(\frac{a}{a-b}\right)^m \mathcal{S}_m^{(k)} \sum_{j=m}^{n+1} \binom{j}{m} S_{n+1}^{(j)} (a-b)^j$$

The sum

$$Q = \sum_{j=m}^{n+1} \binom{j}{m} S_{n+1}^{(j)} (a-b)^j = \frac{1}{m!} \sum_{j=m}^{n+1} \frac{j!}{(j-m)!} S_{n+1}^{(j)} (a-b)^j$$

is computed from $\frac{\Gamma(x+1)}{\Gamma(x+1-m)} = \sum_{k=0}^{m} S_m^{(k)} x^k$ and $\frac{d^n}{dx^n} \frac{\Gamma(x+1)}{\Gamma(x+1-m)} = \frac{d^n}{dx^n} \prod_{k=0}^{m-1} (x-k) = \sum_{k=0}^{m} S_m^{(k)} \frac{d^n}{dx^n} x^k = \sum_{j=0}^{m} S_m^{(j)} \frac{j!}{(j-n)!} x^{j-n}$. Hence,

$$\begin{aligned} Q &= \frac{1}{m!} \sum_{j=m}^{n+1} \frac{j!}{(j-m)!} S_{n+1}^{(j)} (a-b)^j = \frac{(a-b)^m}{m!} \sum_{j=m}^{n+1} \frac{j!}{(j-m)!} S_{n+1}^{(j)} (a-b)^{j-m} \\ &= \frac{(a-b)^m}{m!} \left. \frac{d^m}{dx^m} \frac{\Gamma(x+1)}{\Gamma(x-n)} \right|_{x=a-b} = \frac{(a-b)^m}{m!} \left. \frac{d^m}{dx^m} \prod_{k=0}^{n} (x-k) \right|_{x=a-b} \end{aligned}$$

We return to the sum,

$$\begin{aligned} U_n &= \frac{1}{n^a - z} \sum_{k=0}^{n+1} (-1)^k k! \left(\frac{n^a}{n^a - z}\right)^k \sum_{m=k}^{n+1} \left(\frac{a}{a-b}\right)^m \mathcal{S}_m^{(k)} \frac{(a-b)^m}{m!} \left. \frac{d^m}{dx^m} \frac{\Gamma(x+1)}{\Gamma(x-n)} \right|_{x=a-b} \\ &= \frac{1}{n^a - z} \sum_{k=0}^{n+1} (-1)^k k! \left(\frac{n^a}{n^a - z}\right)^k \sum_{m=k}^{n+1} a^m \frac{\mathcal{S}_m^{(k)}}{m!} \left. \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)}\right) \right|_{x=a-b} \end{aligned}$$

Reversing the k- and m-sum yields

$$U_n = \frac{1}{n^a - z} \sum_{m=0}^{n+1} \frac{a^m}{m!} \left. \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)}\right) \right|_{x=a-b} \sum_{k=0}^{m} (-1)^k k! \mathcal{S}_m^{(k)} \left(\frac{1}{1 - n^{-a} z}\right)^k$$

55. *From the limit (131) back to the Taylor series (1).* From the variants of U_n in **art**. 54, we now verify the correctness of (131). A simplification of $T_j(z; e^x)$ in (130)

$$\sum_{k=1}^{\infty} k^j \left(e^{-b}\right)^k = \frac{(-1)^j}{e^b - 1} \sum_{k=1}^{j} (-1)^k k! \mathcal{S}_j^{(k)} \left(\frac{1}{1 - e^{-b}}\right)^k \tag{132}$$

shows, with $n^{-a}z = e^{-b}$, that

$$\sum_{k=1}^{m} (-1)^k \, k! \mathcal{S}_m^{(k)} \left(\frac{1}{1 - n^{-a}z} \right)^k = (-1)^m \left(\frac{n^a - z}{z} \right) \sum_{k=1}^{\infty} k^m \left(n^{-a}z \right)^k$$

Then,

$$\begin{aligned} U_n &= \frac{1}{n^a - z} \sum_{m=0}^{n+1} \frac{a^m}{m!} \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)} \right) \Bigg|_{x=a-b} (-1)^m \left(\frac{n^a - z}{z} \right) \sum_{k=1}^{\infty} \frac{k^m z^k}{n^{ak}} \\ &= \frac{1}{z} \sum_{m=0}^{n+1} \frac{(-a)^m}{m!} \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)} \right) \Bigg|_{x=a-b} \sum_{k=1}^{\infty} \frac{k^m z^k}{n^{ak}} \\ &= \frac{1}{z} \sum_{k=1}^{\infty} \left(\frac{z}{n^a} \right)^k \sum_{m=0}^{n+1} \frac{(-ak)^m}{m!} \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)} \right) \Bigg|_{x=a-b} \end{aligned}$$

If n is large, then general Taylor expansion demonstrates that

$$\begin{aligned} \lim_{n\to\infty} \sum_{m=0}^{n+1} \frac{(-ak)^m}{m!} \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)} \right) \Bigg|_{x=a-b} &= \lim_{n\to\infty} \sum_{m=0}^{\infty} \frac{(-ak)^m}{m!} \frac{d^m}{dx^m} \left(\frac{\Gamma(x+1)}{\Gamma(x-n)} \right) \Bigg|_{x=a-b} \\ &= \lim_{n\to\infty} \frac{\Gamma(-a(k-1)-b+1)}{\Gamma(-a(k-1)-b-n)} \end{aligned}$$

Since $E_{a,b}(z) = \lim_{n\to\infty} \frac{(-1)^{n-1} n^{(a-b)}}{n!} U_n$, we obtain

$$E_{a,b}(z) = \lim_{n\to\infty} \frac{(-1)^{n-1} \, n^{(a-b)}}{n!} \frac{1}{z} \sum_{k=1}^{\infty} \left(\frac{z}{n^a} \right)^k \frac{\Gamma(-a(k-1)-b+1)}{\Gamma(-a(k-1)-b-n)}$$

The reflection formula (161) indicates that $\frac{\Gamma(-a(k-1)-b+1)}{\Gamma(-a(k-1)-b-n)} = (-1)^{n-1} \frac{\Gamma(1+a(k-1)+b+n)}{\Gamma(a(k-1)+b)}$ and

$$\begin{aligned} E_{a,b}(z) &= \lim_{n\to\infty} \frac{n^{(a-b)}}{n!} \frac{1}{z} \sum_{k=1}^{\infty} \left(\frac{z}{n^a} \right)^k \frac{\Gamma(1 + a(k-1) + b + n)}{\Gamma(a(k-1)+b)} \\ &= \lim_{n\to\infty} n^{(a-b)} \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z}{n^a} \right)^{k+1} \frac{\Gamma(n+1+ak+b)}{\Gamma(n+1)\,\Gamma(ak+b)} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)} \lim_{n\to\infty} \frac{n^{(a-b)}\Gamma(n+1+ak+b)}{n^{a(k+1)}\Gamma(n+1)} \end{aligned}$$

Finally, $\lim_{n\to\infty} \frac{n^{(a-b)}\Gamma(n+1+ak+b)}{n^{a(k+1)}\Gamma(n+1)} = \lim_{n\to\infty} \frac{n^{(a-b)}}{n^{a(k+1)}} n^{ak+b} = 1$ and we arrive again at the defining infinite series (1).

9 The integral $I_{a,b}(z)$

We will study properties of the integral $I_{a,b}(z) = \int_0^\infty \frac{z^u}{\Gamma(b+au)} du$ in (2), which is the "continuous-sum" variant of the Mittag-Leffler function $E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)}$.

56. *Complex argument* z. For $z = re^{i\theta}$, the integral in (2),

$$I_{a,b}\left(re^{i\theta}\right) = \int_0^\infty \frac{r^u e^{i\theta u}}{\Gamma(b+au)} du$$

is split up in a real and imaginary part, assuming that a and b are real, as

$$I_{a,b}\left(re^{i\theta}\right) = \int_0^\infty \frac{r^u \cos\theta u}{\Gamma(b+au)}\,du + i\int_0^\infty \frac{r^u \sin\theta u}{\Gamma(b+au)}\,du$$

which illustrates that $I_{a,b}\left(re^{i\theta}\right)$ is only real if $\theta = 0$, i.e. only when z is a real non-negative number. In contrast, the definition (1) of the Mittag-Leffler function $E_{a,b}(z)$,

$$E_{a,b}\left(re^{i\theta}\right) = \sum_{k=0}^{\infty} \frac{r^k e^{ik\theta}}{\Gamma(b+ak)} = \sum_{k=0}^{\infty} \frac{r^k \cos k\theta}{\Gamma(b+ak)} + i\sum_{k=0}^{\infty} \frac{r^k \sin k\theta}{\Gamma(b+ak)}$$

demonstrates that $E_{a,b}\left(re^{i\theta}\right)$ is real along the entire real axis, because $\sin\theta = 0$ for $\theta = m\pi$ for $m \in \mathbb{Z}$. The point $z = 0$ is a singularity as shown in **art**. 62 below. For $r > 0$, both integrals exists and $I_{a,b}(z)$ is thus defined along the negative real z-axis, where z^u has a branch cut. Both integrals decrease in θ for positive a and b and demonstrate that $\left|I_{a,b}\left(re^{i\theta}\right)\right|$ decreases with $\theta \geq 0$. Hence, we find that $\left|I_{a,b}\left(re^{i\theta}\right)\right| \leq I_{a,b}(r)$ and that $\lim_{\theta\to\infty} I_{a,b}\left(re^{i\theta}\right) = 0$, and that $I_{a.b}(z)$ is multi-valued function, which necessitates, just as for the logarithm, to limit the argument θ of z to the usual range $[0, 2\pi]$ or $[-\pi, \pi]$.

Moreover, after substitution of $t = \theta u$ for $\theta \neq 0$,

$$I_{a,b}\left(re^{i\theta}\right) = \frac{1}{\theta}\int_0^\infty \frac{\left(r^{\frac{1}{\theta}}\right)^t e^{it}}{\Gamma\left(b + \frac{a}{\theta}t\right)}\,dt$$

shows that

$$I_{a,b}\left(re^{i\theta}\right) = \frac{1}{\theta} I_{\frac{a}{\theta},b}\left(r^{\frac{1}{\theta}} e^{i}\right)$$

Thus, the integral at any complex number $z = re^{i\theta}$ with for $\theta \neq 0$ can be mapped to an evaluation along the straight line with angle equal to 1 radius.

57. *Functional relations of the integral* $I_{a,b}(z)$. In contrast to the Mittag-Leffler function $E_{a,b}(z)$, we can scale the integral $I_{a,b}(z)$ easily by considering various (real) substitutions in (2). We consider a linear transformation $b + au = \beta + \alpha v$ with $a > 0$ and $\alpha > 0$. Thus, let $u = \frac{\beta - b}{a} + \frac{\alpha}{a}v$ in (2), then

$$\begin{aligned} I_{a,b}(z) &= \int_0^\infty \frac{z^u}{\Gamma(b+au)}\,du = \frac{\alpha}{a} z^{\frac{\beta-b}{a}} \int_{\frac{b-\beta}{\alpha}}^\infty \frac{z^{\frac{\alpha}{a}v}}{\Gamma(\beta+\alpha v)}\,dv \\ &= \frac{\alpha}{a} z^{\frac{\beta-b}{a}} \left(\int_0^\infty \frac{\left(z^{\frac{\alpha}{a}}\right)^v}{\Gamma(\beta+\alpha v)}\,dv - \int_0^{\frac{b-\beta}{\alpha}} \frac{z^{\frac{\alpha}{a}v}}{\Gamma(\beta+\alpha v)}\,dv \right) \end{aligned}$$

Using the definition (2) leads to[23]

$$I_{a,b}(z) = \frac{\alpha}{a} z^{\frac{\beta-b}{a}} \left(I_{\alpha,\beta}\left(z^{\frac{\alpha}{a}}\right) - \int_0^{\frac{b-\beta}{\alpha}} \frac{z^{\frac{\alpha}{a}v}}{\Gamma(\beta+\alpha v)}\,dv \right) \tag{133}$$

The simplest form of (133) arises for $\alpha = 1$ and $\beta = 0$,

$$I_{a,b}(z) = \frac{z^{-b}}{a}\int_b^\infty \frac{\left(z^{\frac{1}{a}}\right)^v}{\Gamma(v)}\,dv \tag{134}$$

[23] Differentiating $z^{-\frac{\beta-b}{a}} I_{a,b}(z) = \frac{\alpha}{a}\int_{\frac{b-\beta}{\alpha}}^\infty \frac{z^{\frac{\alpha}{a}v}}{\Gamma(\beta+\alpha v)}\,dv$ with respect to b yields $\frac{d}{db}\left(z^{\frac{b-\beta}{a}} I_{a,b}(z)\right) = -\frac{1}{a}\frac{z^{\frac{b-\beta}{a}}}{\Gamma(b)}$, which is independent of α and leads to the linear differential equation in b, $\frac{dI_{a,b}(z)}{db} + \frac{\log z}{a} I_{a,b}(z) = -\frac{1}{a\Gamma(b)}$.

which, similarly to the Mittag-Leffler function $E_{a,b}(z)$ and exhibited in the differential recursion (19) in **art.** 6, illustrates the appearance of the natural $w = z^a$ map, suggesting to consider $I_{a,b}(z^a)$ rather than $I_{a,b}(z)$. Indeed, the integral in (134) further looses the parameter a,

$$I_{a,b}(z^a) = \int_0^\infty \frac{z^{au} du}{\Gamma(b+au)} = \frac{z^{-ab}}{a} \int_b^\infty \frac{e^{v \log z}}{\Gamma(v)} dv$$

If $b > \beta > 0$ and $z = x$ is real and positive, then the integral in (133) is non-negative and

$$I_{a,b}(x) < \frac{\alpha}{a} x^{\frac{\beta-b}{a}} I_{\alpha,\beta}\left(x^{\frac{\alpha}{a}}\right)$$

which bears resemblance to the last expression for $E_{m,b}(z) < \frac{z^{\frac{1-b}{m}}}{m} e^{z^{\frac{1}{m}}}$ shown in **art.** 7. The situation simplifies considerably when we choose $\beta = b$ in (133),

$$I_{a,b}(z) = \frac{\alpha}{a} I_{\alpha,b}\left(z^{\frac{\alpha}{a}}\right)$$

In particular, for $a > 0$ and $\alpha = \frac{1}{a}$, the map

$$I_{\frac{1}{a},b}(z) = a^2 I_{a,b}\left(z^{a^2}\right) \tag{135}$$

suggests a similar relation between $E_{\frac{1}{a},b}(z)$ and $E_{a,b}(z)$ based on (64) and Bieberbach's deductions in **art.** 31.

58. *Bounding $I_{a,b}(x)$ for positive real a, b and x.* The integral in (134)

$$az^b I_{a,b}(z) = \int_b^\infty e^{\left(\frac{\log z}{a}\right)v - \log\Gamma(v)} dv = \int_b^\infty e^{\left(\frac{\log r}{a}\right)v - \log\Gamma(v)} e^{i\frac{\theta}{a}v} dv$$

can be bounded for real $z = x$ using (172), resulting in

$$\frac{e^{-\frac{1}{12b}}}{\sqrt{2\pi}} \int_b^\infty v^{\frac{1}{2}} e^{\left(\frac{\log x}{a}+1\right)v - v\log v} dv \le ax^b I_{a,b}(x) \le \frac{1}{\sqrt{2\pi}} \int_b^\infty v^{\frac{1}{2}} e^{\left(\frac{\log x}{a}+1\right)v - v\log v} dv$$

If a function $f(x)$ is positive and increasing for all $x \in [a, b]$, the integral is bounded by the lower and upper Riemann sum,

$$f(k) < \int_k^{k+1} f(x)\, dx < f(k+1)$$

such that for integers $n, N \in [a, b]$, we obtain the inequalities

$$\sum_{k=n}^{N} f(k) < \int_n^{N+1} f(x)\, dx < \sum_{k=n+1}^{N+1} f(k) \tag{136}$$

with the opposite inequality signs if $f(x)$ is positive and decreasing. The integrand $f(u) = \frac{z^u}{\Gamma(b+au)}$ of $I_{a,b}(z) = \int_0^\infty \frac{z^u}{\Gamma(b+au)} du$ attains a maximum around $u_{\max} \approx \frac{z^{\frac{1}{a}} - b}{a}$ for $b > 0$. Indeed, the derivative $f'(u) = \frac{z^u}{\Gamma(b+au)} (\log z - a\psi(b+au))$ vanishes when $\log z = a\psi(b+au)$ and the expression (175) for the digamma function indicates, for large z, that $\psi(z) \approx \log z$. For negative real b, on the other hand, there may exist more than one extremum. Applying (136) with $N = \lfloor u_{\max} \rfloor$ yields, for positive a, b and x

$$\sum_{k=0}^{\lfloor u_{\max} \rfloor} \frac{x^k}{\Gamma(b+ak)} < \int_0^{\lfloor u_{\max} \rfloor + 1} \frac{x^u}{\Gamma(b+au)} du < \sum_{k=1}^{\lfloor u_{\max} \rfloor + 1} \frac{x^k}{\Gamma(b+ak)}$$

and

$$\sum_{k=\lfloor u_{\max}\rfloor+1}^{\infty}\frac{x^k}{\Gamma(b+ak)} > \int_{\lfloor u_{\max}\rfloor+1}^{\infty}\frac{x^u}{\Gamma(b+au)}du > \sum_{k=\lfloor u_{\max}\rfloor+2}^{\infty}\frac{x^k}{\Gamma(b+ak)}$$

We rewrite the latter inequalities as,

$$\begin{aligned}E_{a,b}(x) &> I_{a,b}(x) + \sum_{k=0}^{\lfloor u_{\max}\rfloor}\frac{x^k}{\Gamma(b+ak)} - \int_0^{\lfloor u_{\max}\rfloor+1}\frac{x^u}{\Gamma(b+au)}du \\ &= I_{a,b}(x) + \sum_{k=0}^{\lfloor u_{\max}\rfloor+1}\frac{x^k}{\Gamma(b+ak)} - \int_0^{\lfloor u_{\max}\rfloor+1}\frac{x^u}{\Gamma(b+au)}du - \frac{x^{\lfloor u_{\max}\rfloor+1}}{\Gamma(b+a(\lfloor u_{\max}\rfloor+1))}\end{aligned}$$

and use the former inequalities,

$$E_{a,b}(x) > I_{a,b}(x) - \frac{x^{\lfloor u_{\max}\rfloor+1}}{\Gamma(b+a(\lfloor u_{\max}\rfloor+1))}$$

Analogously, we find

$$E_{a,b}(x) < I_{a,b}(x) + \frac{x^{\lfloor u_{\max}\rfloor+1}}{\Gamma(b+a(\lfloor u_{\max}\rfloor+1))}$$

59. *Differential recursion.* Differentiating the integral $I_{a,b}(z)$ in (2) and using the functional equation (146) of the Gamma function,

$$\frac{d}{dz}z^{b-1}I_{a,b}(z^a) = \int_0^\infty \frac{(au+b-1)\,z^{au+b-2}}{\Gamma(b+au)}\,du = \int_0^\infty \frac{z^{au+b-2}}{\Gamma(b-1+au)}\,du$$

leads to a recursion equation in

$$\frac{d}{dz}\left\{z^{b-1}I_{a,b}(z^a)\right\} = z^{b-2}I_{a,b-1}(z^a) \tag{137}$$

which is precisely the same as for $E_{a,b}(.)$ in (19).

60. *A complex integral representation for $I_{a,b}(z)$.* We start by concentrating on the integral

$$z^{b-1}I_{a,b}(z^a) = \int_0^\infty \frac{z^{b-1+au}}{\Gamma(b+au)}\,du$$

whose Laplace transform is

$$L_{a,b}(s) = \int_0^\infty \left\{z^{b-1}I_{a,b}(z^a)\right\}e^{-zs}dz = \int_0^\infty e^{-zs}\int_0^\infty \frac{z^{b-1+au}}{\Gamma(b+au)}du\,dz$$

The reversal of the integrals is justified by absolute convergence,

$$\begin{aligned}L_{a,b}(s) &= \int_0^\infty \frac{du}{\Gamma(b+au)}\int_0^\infty z^{b+au-1}e^{-zs}dz = \int_0^\infty \frac{du}{\Gamma(b+au)}\frac{\Gamma(b+au)}{s^{b+au}} \\ &= s^{-b}\int_0^\infty e^{-au\log s}du = \frac{s^{-b}}{a\log s}\end{aligned}$$

The inverse Laplace transform (193) returns a complex integral,

$$z^{b-1}I_{a,b}(z^a) = \frac{1}{2\pi ia}\int_{c-i\infty}^{c+i\infty}\frac{e^{zs}}{s^b\log s}ds \tag{138}$$

where $c > 1$ because $L_{a,b}(s)$ is only analytic for $\text{Re}(s) > 1$ due to the pole at $s = 1$. After replacing z^a by z in (138) and combining with the definition (2), we obtain

$$I_{a,b}(z) = \int_0^\infty \frac{e^{u \log z}}{\Gamma(b+au)} du = \frac{z^{\frac{1-b}{a}}}{2\pi i a} \int_{c-i\infty}^{c+i\infty} \frac{e^{z^{\frac{1}{a}} s}}{s^b \log s} ds \qquad c > 1 \tag{139}$$

a. For $\text{Re}\left(z^{\frac{1}{a}}\right) < 0$, the contour in (139) can be closed over the positive $\text{Re}(s)$-plane, in which the integrand is analytic and[24] $I_{a,b}(z) = 0$. Let $z = re^{i\theta}$ with $\theta = \arg z \in [-\pi, \pi]$ and recalling that $a > 0$, then $\text{Re}\left(z^{\frac{1}{a}}\right) = r^{\frac{1}{a}} \cos \frac{\theta}{a}$ and $\text{Re}\left(z^{\frac{1}{a}}\right) < 0$ requires that $\cos \frac{\theta}{a} < 0$, which is equivalent to $\frac{\pi}{2} < \frac{\pm\theta}{a} < \pi$ or $\frac{\pi a}{2} < |\arg z| < \pi a$. The latter condition, combined with $0 < |\arg z| < \pi$ is only possible if $0 < a < 2$. If $1 \leq a < 2$, then the combined condition means that $\frac{\pi}{2} < |\arg z| < \pi$ or that $\text{Re}(z) < 0$. Only if $0 < a < \frac{1}{2}$, then the combined condition means that $\frac{\pi}{2} < |\arg z| < \frac{\pi a}{2}$ or that $\text{Re}(z) > 0$, while for $\frac{1}{2} \leq a \leq 1$, $\text{Re}(z)$ can be either sign.

b. For $\text{Re}\left(z^{\frac{1}{a}}\right) > 0$, we close the contour in (139) over the negative $\text{Re}(s)$-plane around the branch cut of $s^b \ln(s)$, which is the negative real axis. Thus, we consider the contour C that consists of the line at $c > 1$, the quarter of a circle with infinite radius from $\frac{\pi}{2}$ to $\pi - \varepsilon$, the line segment above the real negative axis from minus infinity to $s = 0$, the circle around the origin $s = 0$ from $\pi - \varepsilon$ back to $-\pi - \varepsilon$ with radius δ, the line segment below the real negative axis from $s = 0$ towards minus infinity, the quarter circle with infinite radius back to close the contour C. This contour encloses the pole at $s = 1$, whose residue is $\lim_{s\to 1} \frac{e^{z^{\frac{1}{a}} s}(s-1)}{s^b \ln s} = e^{z^{\frac{1}{a}}}$. Cauchy's Residue Theorem [47] results in

$$\frac{1}{2\pi i} \int_C \frac{e^{z^{\frac{1}{a}} s}}{s^b \ln s} ds = e^{z^{\frac{1}{a}}}$$

while the evaluation of the contour C yields

$$\frac{1}{2\pi i} \int_C \frac{e^{z^{\frac{1}{a}} s}}{s^b \ln s} ds = az^{\frac{b-1}{a}} I_{a,b}(z) + \frac{1}{2\pi i} \int_\infty^0 \frac{e^{-z^{\frac{1}{a}} x} d\left(e^{i(\pi-\varepsilon)} x\right)}{x^b e^{ib(\pi-\varepsilon)} \ln\left(xe^{i(\pi-\varepsilon)}\right)} + \frac{1}{2\pi i} \int_0^\infty \frac{e^{-z^{\frac{1}{a}} x} d\left(e^{i(-\pi-\varepsilon)} x\right)}{x^b e^{ib(-\pi-\varepsilon)} \ln\left(xe^{i(-\pi-\varepsilon)}\right)}$$

since the parts of C along the circles vanish for $\text{Re}(z) > 0$, but for $\delta \to 0$ only provided $\text{Re}(b) \leq 1$. Hence, we obtain

$$az^{\frac{b-1}{a}} I_{a,b}(z) = e^{z^{\frac{1}{a}}} + \frac{1}{2\pi i} \int_0^\infty \frac{e^{-z^{\frac{1}{a}} x}}{x^b} \left(\frac{e^{ib\pi}}{\ln x - i\pi} - \frac{e^{-ib\pi}}{\ln x + i\pi} \right) dx$$

Finally, with $\frac{e^{ib\pi}}{\ln x - i\pi} - \frac{e^{-ib\pi}}{\ln x + i\pi} = 2\pi i \frac{\frac{\sin b\pi}{\pi} \ln x + \cos b\pi}{\left(\pi^2 + (\ln x)^2\right)}$ and the definition (2), we arrive, for $\text{Re}(b) \leq 1$ and $\text{Re}\left(z^{\frac{1}{a}}\right) > 0$, at

$$I_{a,b}(z) = \int_0^\infty \frac{z^u}{\Gamma(b+au)} du = \frac{z^{\frac{1-b}{a}}}{a} \left\{ e^{z^{\frac{1}{a}}} + \int_0^\infty \frac{e^{-z^{\frac{1}{a}} x}}{x^b} \left(\frac{\frac{\sin b\pi}{\pi} \ln x + \cos b\pi}{\pi^2 + (\ln x)^2} \right) dx \right\} \tag{140}$$

[24] Hence, for $\text{Re}\left(z^{\frac{1}{a}}\right) = \text{Re}\left(e^{\frac{1}{a}\log z}\right) < 0$, it holds that

$$0 = \int_0^\infty \frac{e^{u \log r} e^{iu\theta}}{\Gamma(b+au)} du = \frac{r^{-\frac{b}{a}}}{a} \int_b^\infty \frac{e^{w \frac{\log r}{a}} e^{i(w-b)\frac{\theta}{a}}}{\Gamma(w)} dw = \frac{r^{-\frac{b}{a}}}{a} \int_b^\infty \frac{e^{w \frac{\log r}{a}} \left\{\cos\left(\frac{\theta}{a}(w-b)\right) + i \sin\left(\frac{\theta}{a}(w-b)\right)\right\}}{\Gamma(w)} dw$$

implying that

$$\int_b^\infty \frac{e^{w \frac{\log r}{a}} \cos\left(\frac{\theta}{a}(w-b)\right)}{\Gamma(w)} dw = \int_b^\infty \frac{e^{w \frac{\log r}{a}} \sin\left(\frac{\theta}{a}(w-b)\right)}{\Gamma(w)} dw = 0$$

while, for $\operatorname{Re}\left(z^{\frac{1}{a}}\right) < 0$, it holds that $I_{a,b}(z) = 0$. Expression (140) is obviously related to Bierberbach's integral in **art**. 31, written as

$$E_{a,b}(z) = \frac{z^{\frac{1-b}{a}}}{a}\left\{e^{z^{\frac{1}{a}}} + \frac{1}{2\pi i}\frac{1}{z}\int_{C'_{a\phi}} \frac{\left(\frac{t}{z}\right)^{\frac{1-b}{a}} e^{t^{\frac{1}{a}}}}{\left(\frac{t}{z}\right) - 1} dt\right\} \tag{141}$$

For $b = 1$, (140) simplifies to

$$I_{a,1}(z) = \frac{1}{a}\left\{e^{z^{\frac{1}{a}}} - \int_0^\infty \frac{e^{-z^{\frac{1}{a}}x}}{x\left(\pi^2 + (\ln x)^2\right)} dx\right\}$$

where $\int_0^\infty \frac{e^{-\lambda x}}{x\left(\pi^2+(\ln x)^2\right)}dx$ is increasing[25] in $\operatorname{Re}(\lambda) > 0$ from 0 to 1. Hence, for $\operatorname{Re}\left(z^{\frac{1}{a}}\right) > 0$, the following lower and upper bound hold,

$$\frac{1}{a}e^{z^{\frac{1}{a}}} - \frac{1}{a} < I_{a,1}(z) < \frac{1}{a}e^{z^{\frac{1}{a}}}$$

With $\alpha = \beta = 1$ in (133) and $I_{1,1}\left(z^{\frac{1}{a}}\right)$ from (140), we find for $\operatorname{Re}\left(z^{\frac{1}{a}}\right) > 0$,

$$I_{a,b}(z) = \frac{z^{\frac{1-b}{a}}}{a}\left(e^{z^{\frac{1}{a}}} - \int_0^\infty \frac{e^{-z^{\frac{1}{a}}x}}{x\left(\pi^2 + (\ln x)^2\right)} dx - \int_0^{b-1} \frac{\left(z^{\frac{1}{a}}\right)^x}{x\Gamma(v)} dx\right)$$

Comparison with (140) indicates, for $b \le 1$ and $\operatorname{Re}(z) > 0$, that[26]

$$\int_0^\infty \frac{e^{-zx}}{x^b}\left(\frac{\frac{\sin b\pi}{\pi}\ln x + \cos b\pi}{\pi^2 + (\ln x)^2}\right) dx = -\int_0^\infty \frac{e^{-zx}}{x\left(\pi^2 + (\ln x)^2\right)} dx - \int_0^{b-1} \frac{z^v}{\Gamma(v+1)} dv$$

61. *Another complex integral representation for* $I_{a,b}(z)$. Another complex integral follows directly from Hankel's contour (178) as

$$I_{a,b}(z) = \frac{1}{2\pi i}\int_C w^{-b}e^w dw \int_0^\infty e^{-u(a\log w - \log z)} du$$

Only if $\operatorname{Re}(a\log w - \log z) > 0$, which is equivalent to $\operatorname{Re}\left(\log\frac{w^a}{z}\right) = \log\left|\frac{w^a}{z}\right| > 0$ and $\left|\frac{w^a}{z}\right| > 1$, then

$$I_{a,b}(z) = \frac{1}{2\pi i}\int_C \frac{w^{-b}e^w}{a\log w - \log z} dw \qquad \text{with } |w| > |z|^{\frac{1}{a}} \tag{142}$$

where the constraint $|w| > |z|^{\frac{1}{a}}$ requires to deform the contour C (as explained in **art**. 30). The contour integral (142) bears a resemblance to the basic complex integral (67) for $E_{a,b}(z)$, whereas the

[25] Indeed (see [50, p.73]),

$$\int_0^\infty \frac{e^{-\lambda x}}{x\left(\pi^2 + (\ln x)^2\right)} dx = \int_{-\infty}^\infty \frac{e^{-\lambda e^t}}{\pi^2 + t^2} dt \le \int_{-\infty}^\infty \frac{dt}{\pi^2 + t^2} = 1$$

[26] Differentiating with respect to b, $-\frac{z^{b-1}}{\Gamma(b)} = \int_0^\infty \frac{e^{-zx}}{\pi^2+(\ln x)^2}\frac{d}{db}\left(x^{-b}\left(\frac{\sin b\pi}{\pi}\ln x + \cos b\pi\right)\right) dx$ leads, using the reflection formula (161), to an identity $\frac{z^{b-1}}{\Gamma(b)} = \frac{\sin b\pi}{\pi}\int_0^\infty e^{-zx}x^{-b}dx = \frac{\sin b\pi}{\pi}\frac{\Gamma(1-b)}{z^{1-b}}$.

above integral (139) is closer to the Mittag-Leffler integral (71), although a formal substitution $s = \frac{w^a}{z}$ in (142) leads to

$$I_{a,b}(z) = \frac{z^{\frac{1-b}{a}}}{2\pi i a} \int_{C'} \frac{s^{\frac{1-b}{a}-1} e^{(zs)^{\frac{1}{a}}}}{\log s} ds$$

62. *A series for* $I_{a,b}(z)$. The Taylor series of $I_{a,b}(z)$ around $z = \zeta$ equals

$$I_{a,b}(z) = I_{a,b}(\zeta) + \sum_{k=0}^{\infty} \frac{1}{k!} \left. \frac{d^k I_{a,b}(z)}{dz^k} \right|_{z=\zeta} (z-\zeta)^k$$

where the derivative for real $a > 0$

$$\left. \frac{d^k I_{a,b}(z)}{dz^k} \right|_{z=\zeta} = \int_0^{\infty} \frac{\left. \frac{d^k}{dz^k} z^u \right|_{z=\zeta}}{\Gamma(b+au)} du = \frac{1}{\zeta^k} \int_0^{\infty} \frac{u(u-1)\dots(u-k+1)\zeta^u}{\Gamma(b+au)} du$$
$$= \frac{1}{\zeta^k} \int_0^{\infty} \frac{\Gamma(u+1)\zeta^u}{\Gamma(u-k+1)\Gamma(b+au)} du$$

converges for all k, except when $\zeta = 0$. Hence, in contrast to $E_{a,b}(z)$, the function $I_{a,b}(z)$ is not entire and has an essential singularity at $z = 0$, where none of the derivatives exists. Since $\left. \frac{dI_{a,b}(z)}{dz} \right|_{z=x} > 0$ for positive real x and a, b, the function $I_{a,b}(x)$ increases for all $x > 0$.

We expand the integrand of the integral $I_{a,b}(z)$, defined in (2), in a Taylor series around b,

$$\frac{1}{\Gamma(b+au)} = \sum_{j=0}^{\infty} \frac{1}{j!} \left. \frac{d^j}{dy^j} \frac{1}{\Gamma(y)} \right|_{y=b} (au)^j$$

which converges for all u and b, and obtain

$$I_{a,b}(z) = \int_0^{\infty} \frac{z^u}{\Gamma(b+au)} du = \sum_{j=0}^{\infty} \frac{a^j}{j!} \left. \frac{d^j}{dy^j} \frac{1}{\Gamma(y)} \right|_{y=b} \int_0^{\infty} u^j e^{-u(-\log z)} du$$

Only if $\mathrm{Re}(\log z) < 0$ or $0 \leq |z| < 1$, then we arrive at

$$I_{a,b}(z) = -\frac{1}{\log z} \sum_{j=0}^{\infty} \left. \frac{d^j}{dy^j} \frac{1}{\Gamma(y)} \right|_{y=b} \left(-\frac{a}{\log z} \right)^j \tag{143}$$

but this series only converges when $\left| \frac{a}{\log z} \right| < 1$. Indeed, alternatively, after p-times repeated partial integration, we obtain

$$I_{a,b}(z) = -\frac{1}{\log z} \sum_{j=0}^{p-1} \left. \frac{d^j}{dy^j} \frac{1}{\Gamma(y)} \right|_{y=b} \left(-\frac{a}{\log z} \right)^j + \left(-\frac{a}{\log z} \right)^p \int_0^{\infty} \left. \frac{d^p}{dy^p} \frac{1}{\Gamma(y)} \right|_{y=b+au} z^u \, du$$

where the last integral exists for all p and z. Thus, if $p \to \infty$ and $\left| \frac{a}{\log z} \right| < 1$, then repeated partial integration again produces (143). The series (143) indicates that $I_{a,b}(0) = 0$.

10 Epilogue

Driven by the surge in fractional analysis [30], the Mittag-Leffler function $E_{a,b}(z)$, called by Mainardi [29] the "Queen Function of the Fractional Calculus", gains increasing interest. I will end this work by enumerating some open problems. Although results [58] exist, the determination of the zeros of $E_{a,b}(z)$ in the complex plane still stands on the agenda. Indeed, Weierstrass's entire function theory [47] shows that any entire function can be represented as a product form that contains all the zeros and such a product form for $E_{a,b}(z)$ has not been found yet. A general Lagrange series for $G\left(f^{-1}(z)\right)$, thus a function $G(z)$ of the inverse function $f^{-1}(z)$ of the function $f(z)$, in terms of characteristic coefficients (**art**. 9), is available (see e.g. [53, art. 342], [49, Appendix A]). Thus, the zero most close to a point z_0 in the complex plane can be approximated accurately, if the Taylor coefficients of $f(z)$ around z_0 are known or easily computable. Related to this root-locus problem is the study of the inverse function, which is the solution $z = E_{a,b}^{-1}(w)$ of $w = E_{a,b}(z)$. Mainly numerical computations of the inverse Mittag-Leffler function exist (see e.g. [26]), but few analytic results.

63. *The Garrappa-Popolizio conjecture*[27]. For any complex number z, the Garrappa-Popolizio conjecture claims the truth of the two inequalities

$$|E_{a,b}(z)| \leq E_{a,b}(\operatorname{Re} z) \qquad \text{for } 0 < a < 1 \text{ and } a \leq b \tag{144}$$

and

$$|E_{a,b}(z)| \geq E_{a,b}(\operatorname{Re} z) \qquad \text{for } a \geq 1 \text{ and } a \geq b \tag{145}$$

The inequality (145) is true for $a = b = 1$, namely equality in (145) holds because $|e^z| = \left|e^{x+iy}\right| = e^x = e^{\operatorname{Re} z}$. Roberto Garrappa has informed[28] me that inequalities (144) and (145) were verified by a huge number of computations for a wide range of the parameters a and b, but a proof of (144) and (145) is still missing.

Acknowledgement I am grateful to Professor F. Mainardi and Professor A. Apelblat for their very stimulating discussions. After the submission of the first version on arXiv:2005.13330, they have contacted me and pointed me to earlier work. At the time of writing, Professor F. Mainardi continues to stimulate and to express his deep interest in the Mittag-Leffler function.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, Inc., New York, 1968.

[2] A. Apelblat. Differentiation of the Mittag-Leffler functions with respect to parameters in the Laplace transform approach. *Mathematics*, 8(657), doi:10.3390/math8050657 2020.

[3] M. Berberan-Santos. Analytical inversion of the Laplace transform without contour integration: application to luminescence decay laws and other relaxation functions. *Journal of Mathematical Chemistry*, 38(2):165–173, August 2005.

[4] M. Berberan-Santos. Properties of the Mittag-Leffler relaxation function. *Journal of Mathematical Chemistry*, 38(4):629–635, November 2005.

[5] M. A. Berger. *An Introduction to Probabiliy and Stochastic Processes*. Springer-Verlag, New York, 1993.

[27] Francesco Mainardi has emailed me this conjecture on 3 December 2023.
[28] email on 14 December 2023.

[6] S. Bernstein. Sur les fonctions absolument monotones. *Acta Mathematica*, 52:1–66, 1929.

[7] L. Bieberbach. *Lehrbuch der Funktionentheorie*, volume II. Teubner in Leipzig, 1934; reprinted by Chelsea Publishing Company, New York, 1945.

[8] I. V. Blagouchine. Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to π^{-1}. *arXiv1408.3902v9*, May 2016.

[9] H. M. Edwards. *Riemann's Zeta Function.* Academic Press, New York, 1974.

[10] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions*, volume 3 of *California Institute of Technology Bateman Manuscript Project.* McGraw-Hill Book Company, New York, 1953.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions*, volume 1 of *California Institute of Technology Bateman Manuscript Project.* McGraw-Hill Book Company, New York, 1953.

[12] L. Euler. *Introductio in analysin infinitorum, I.* German translation by H. Maser in 1885. Julius Springer, Berlin, 1748.

[13] M. A. Evgrafov. *Analytic Functions.* W. B. Saunders Company, 1966; Reprinted by Dover Publications, Inc., New York, dover 2019 edition, 2019.

[14] W. Feller. Fluctuation theory of recurrent events. *Transactions of the American Mathematical Society*, 67:98–119, 1949.

[15] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 2. John Wiley & Sons, New York, 2nd edition, 1971.

[16] C. F. Gauss. Disquisitiones generales circa seriem infinitam $1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(1+\alpha)\beta(1+\beta)}{1.2\gamma(\gamma+1)}xx+\frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{1.2.3\gamma(\gamma+1)(2+\gamma)}x^3$ etc., pars prior. *Commentationes societatis regiae scientiarum Gottingensis recentiores vol. II* in Gauss Werke, Band III, pages 125–162, 1813.

[17] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications.* Springer, 2014.

[18] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications.* Springer, second edition, 2020.

[19] R. Gorenflo and F. Mainardi. *Fractional Calculus: Integral and Differential Equations of Fractional Order*, volume Fractals and Fractional Calculus in Continuum, Mechanics, A. Carpinteri and F. Mainardi (Editors) of *Cism Courses and Lectures, No. 378.* Springer Verlag, Wien and New York, 1997.

[20] R. Gorenflo and F. Mainardi. Parametric subordination in fractional diffusion processes. *arXiv:1210.8414*, October 2012.

[21] B. Gross. Note on the inversion of the Laplace transform. *Philosophical Magazine A*, 41(317):534–544, 1950.

[22] J. Hadamard. *Essai sur l'étude des functions données par leur développement de Taylor & Étude sur les propriétés des functions entières et en particulier d'une function considérée par Riemann*, volume VIII and IX of *Extraits du Journal de Mathématiques pures et appliquées, 4^e Série.* Gauthier-Villars et fils, Paris, 1892-1893.

[23] G. H. Hardy. *A Course of Pure Mathematics.* Cambridge University Press, 10th edition, 2006.

[24] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford University Press, London, 6th edition, 2008.

[25] H. J. Haubold, A. M. Mathai, and R. K. Saxena. Mittag-Leffler functions and their applications. *Journal of Applied Mathematics*, 2011(Article ID 298628), 2011.

[26] R. Hilfer and H. J. Seybold. Computation of the generalized Mittag-Leffler function and its inverse in the complex plane. *Integral Transforms and Special Functions*, 17(9):637–652, 2006.

[27] F. Klein. *Vorlesungen über die Hypergeometrische Funktion.* Verlag von Julius Springer, Berlin, 1933.

[28] F. Mainardi. On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t>0$ with $0<\alpha<1$. *Discrete and Continuous Dynamical Systems Series B*, 19(7):2267–2278, September 2014.

[29] F. Mainardi. Why the Mittag-Leffler function can be considered the Queen function of the Fractional Calculus? *Entropy (MDPI)*, 22(12):1359/1–29, 2020.

[30] F. Mainardi. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models.* World Scientific Publishing, London, U.K., second edition, 2021.

[31] A. I. Markushevich. *Theory of Functions of a Complex Variable*, volume I – III. Chelsea Publishing Company, New York, 1985.

[32] K. S. Miller and S. G. Samko. A note on the complete monotonicity of the generalized Mittag-Leffler function. *Real Analysis Exchange*, 32(2):753–755, 1997/8.

[33] M. G. Mittag-Leffler. Sur la nouvelle function $E_\alpha(x)$. *Comptes Rendus hebdomadaires de Séances de l'Academié des Sciences, Paris*, 137:554–558, 12 October 1903.

[34] M. G. Mittag-Leffler. Une généralisation de l'intégrale de Laplace-Abel. *Comptes Rendus hebdomadaires de Séances de l'Academié des Sciences, Paris*, 136:537–539, 2 Mars 1903.

[35] M. G. Mittag-Leffler. Sur la représentation analytique d'une branche uniforme d'une function monogène (cinquième note). *Acta Mathematica*, 29:101–181, 1905.

[36] N. Nielsen. *Die Gammafunktion: Band I. Handbuch der Theorie der Gammafunktion und Band II. Theorie des Integrallogarithmus und verwandter Transzendenten.* B. G. Teubner, Leipzig 1906; republished by Chelsea, New York, 1956.

[37] F. W. J. Olver, D. W Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions.* Cambridge University Press, New York, 2010.

[38] H. Pollard. The representation of e^{-x^λ} as a Laplace integral. *Bulletin of the American Mathematical Society*, 52:908–910, 1946.

[39] H. Pollard. The completely monotonic character of the Mittag-Leffler function $e_a(-x)$. *Bulletin of the American Mathematical Society*, 54:1115–1116, 1948.

[40] A. Y. Popov and A. M. Sedletskii. Distribution of roots of Mittag-Leffler functions. *Journal of Mathematical Sciences*, 190(2):209–409, April 2013.

[41] H. Rademacher. *Topics in Analytic Number Theory.* Springer-Verlag, Berlin, 1973.

[42] J. Riordan. *Combinatorial Identities.* John Wiley & Sons, New York, 1968.

[43] T. Sandev and Z. Tomovski. *Fractional Equations and Models, Theory and Applications*, volume 61 of *Developments in Mathematics*. Springer, Switzerland, 2019.

[44] G. Sansone and J. Gerretsen. *Lectures on the Theory of Functions of a Complex Variable*, volume 1 and 2. P. Noordhoff, Groningen, 1960.

[45] W. R. Schneider. Completely monotone generalized Mittag-Leffler functions. *Expositiones Mathematicae*, 14:3–16, 1996.

[46] E. C. Titchmarsh. *Introduction to the Theory of Fourier Integrals.* Oxford University Press, Ely House, London W. I, 2nd edition, 1948.

[47] E. C. Titchmarsh. *The Theory of Functions.* Oxford University Press, Amen House, London, 1964.

[48] P. Van Mieghem. *Characteristic Coefficients of a Complex Function.* unpublished, 1993-2020.

[49] P. Van Mieghem. The asymptotic behaviour of queueing systems: Large deviations theory and dominant pole approximation. *Queueing Systems*, 23:27–55, 1996.

[50] P. Van Mieghem. *Performance Analysis of Complex Networks and Systems.* Cambridge University Press, Cambridge, U.K., 2014.

[51] P. Van Mieghem. The Mittag-Leffler function. *arXiv:2005.13330*, 2020.

[52] P. Van Mieghem. The origin of the fractional derivative and fractional non-Markovian continuous time processes. *Physical Review Research*, 4(2):023242, 2022.

[53] P. Van Mieghem. *Graph Spectra for Complex Networks.* Cambridge University Press, Cambridge, U.K., second edition, 2023.

[54] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, U.K., cambridge mathematical library edition, 1995.

[55] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press, Cambridge, U.K., cambridge mathematical library edition, 1996.

[56] D. V. Widder. *The Laplace transform*. Princeton University Press, Princeton, 1946.

[57] A. Wiman. Über den Fundamentalsatz in der Teorie der Funktionen $e_a(x)$. *Acta mathematica*, 29:191–205, 1905.

[58] A. Wiman. Über die Nullstellen der Funktionen $e_a(x)$. *Acta mathematica*, 29:217–234, 1905.

A The Gamma function $\Gamma(z)$

We review properties of the Gamma function. Besides the basic functions [23, Chapter IX and X] like the exponential, logarithm, circular or trigonometric functions (as sinus, cosinus, tangens, etc.), the Gamma function is the next important complex function. Nearly all books on complex function theory [11, 13, 31, 44, 47, 55] treat the Gamma function.

The Gamma function $\Gamma(z)$ is an extension of the factorial $n! = 1.2.3\ldots n$ in the integers $n \geq 1$ to complex numbers z. The factorial obeys

$$n! = n(n-1)!$$

which directly generalizes to the functional equation $\Pi(z) = z\Pi(z-1)$ with $n! = \Pi(n)$ and $\Pi(0) = 1$, in the notation of Gauss in his truly impressive manuscript [16, p. 146]. Later in 1814, Legendre defined the Gamma function by its current notation $\Gamma(z) = \Pi(z-1)$, with $\Gamma(1) = 1$, and the functional equation $\Pi(z) = z\Pi(z-1)$ translates to the Gamma function $\Gamma(z)$ as

$$\Gamma(z+1) = z\Gamma(z) \tag{146}$$

The first step in the theory of the Gamma function consists of finding a solution of the functional equation (146). Euler has proposed his famous integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad \text{for } \operatorname{Re}(z) > 0 \tag{147}$$

Partial integration of (147) shows that Euler's integral (147) obeys the functional equation (146) and $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$. However, since Euler's integral is only valid for $\operatorname{Re}(z) > 0$, other ingenious methods have been devised that are valid for all complex numbers z.

In his beautiful book on the Gamma function [36], Nielsen cites the historic achievements and reviews most contributions before 1906. Nielsen [36] starts his book with the functional equation (146) of the Gamma function and immediately remarks that any solution can be multiplied by a periodic function $\omega(z) = \omega(z+1)$ with period 1. Next, Nielsen [36] concentrates on the digamma function, which is the logarithmic derivative $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ and the functional equation (146) tells us that

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{148}$$

After n iterations,

$$\psi(z) = \psi(z+n) - \sum_{k=0}^{n-1} \frac{1}{z+k}$$

and taking the limit $n \to \infty$, we formally obtain $\psi(z) = \lim_{n\to\infty} \psi(z+n) - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{z+k}$. However, the latter series does not converge[29], implying that $\lim_{n\to\infty} \psi(z+n) = \infty$. Nielsen then applies Weierstrass's factorization theory for entire functions [47] and deduces Weierstrass's product,

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n} \tag{149}$$

which we will derive from Gauss's product (156) in **art.** 64. Weierstrass created his magnificent theory for entire functions, a pearl of complex function theory, inspired by Gauss's product (156) and Gauss's remark on factorization in [16, p. 146].

A.1 Gauss's approach

Iterating the functional equation (146) n-times gives $\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)}$, but purely iterating (146) for non-integer values is not successful. Therefore, Gauss [16, p. 144] proposes to consider the more general form

$$\Pi(k,z) = \frac{\Pi(k)\,\Pi(z)}{\Pi(k+z)} k^z = \frac{1}{(z+1)} \frac{2}{(z+2)} \frac{3}{(z+3)} \cdots \frac{k}{(z+k)} k^z \tag{150}$$

which satisfies

$$\Pi(k,z+1) = \Pi(k,z) \frac{(z+1)}{\left(1+\frac{z+1}{k}\right)} \tag{151}$$

as well as

$$\Pi(k+1,z) = \Pi(k,z) \frac{\left(1+\frac{1}{k}\right)^{z+1}}{\left(1+\frac{z+1}{k}\right)} \tag{152}$$

Iterating (152), with $\Pi(1,z) = \frac{1}{z+1}$, results in

$$\Pi(k,z) = \frac{1}{z+1} \frac{2^{z+1}}{(2+z)} \frac{3^{z+1}}{2^z(3+z)} \cdots \frac{k^{z+1}}{(k-1)^z(k+z)} = \frac{1}{z+1} \prod_{n=2}^{k} \frac{n^{z+1}}{(n-1)^z(n+z)} \tag{153}$$

The interesting observation from $\lim_{k\to\infty} \frac{\left(1+\frac{1}{k}\right)^{z+1}}{\left(1+\frac{z+1}{k}\right)} = 1$ in (152) is that $\lim_{k\to\infty} \Pi(k,z)$ exist for all z, which Gauss demonstrates after taking the logarithm of both sides, while the first functional equation (151) indicates that $\lim_{k\to\infty} \Pi(k,z)$ satisfies the functional equation $\Pi(z+1) = (z+1)\Pi(z)$. Combining both, Gauss is led to

$$\Pi(z) = \lim_{k\to\infty} \Pi(k,z) = \lim_{k\to\infty} \frac{\Pi(k)\,\Pi(z)}{\Pi(k+z)} k^z \tag{154}$$

which is equivalent with $\Gamma(z+1) = \Pi(z)$ to

$$\Gamma(z) = \lim_{k\to\infty} \frac{k!\Gamma(z)}{\Gamma(k+1+z)} k^z = \lim_{k\to\infty} \frac{k^z k!}{z(z+1)\ldots(z+k)}$$

[29] Indeed,

$$\sum_{k=0}^{n} \frac{1}{z+k} = \frac{1}{z} + \sum_{k=1}^{n_0} \frac{1}{k\left(1+\frac{z}{k}\right)} + \sum_{k=1+n_0}^{n} \frac{1}{k\left(1+\frac{z}{k}\right)}$$

We can choose $n_0 > |z|$, so that $0 < \left|1+\frac{z}{k}\right| < 2$ and $\left|\sum_{k=1+n_0}^{n} \frac{1}{k\left(1+\frac{z}{k}\right)}\right| > \frac{1}{2}\sum_{k=1+n_0}^{n} \frac{1}{k} \to \infty$, because the harmonic series $\sum_{k=1}^{n} \frac{1}{k}$ diverges for $n \to \infty$.

Rewriting Gauss's definition in (154) as $\Gamma(z) = \lim_{k\to\infty} \Pi(k, z-1)$ and introducing (153) yields

$$\Gamma(z) = \lim_{k\to\infty} \frac{1}{z} \prod_{n=2}^{k} \frac{n^z}{(n-1)^{z-1}(n+z-1)} = \lim_{k\to\infty} \frac{1}{z} \prod_{n=1}^{k-1} \frac{(n+1)^z}{n^{z-1}(n+z)}$$

Finally[30], we arrive at Gauss's infinite product[31] for the Gamma function

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^z \left(1+\frac{z}{n}\right)^{-1} \tag{156}$$

which converges for all complex z, except for the integers at $z = 0, -1, -2, \ldots$, at which $\Gamma(z)$ has simples poles. The inverse of the product (156) shows that $\frac{1}{\Gamma(z)}$ is an entire function and that $\Gamma(z)$ has no zeros in the finite complex plane.

Of course, the proposal of (150) by Gauss was crucial towards his elegant product (156). Gauss posited (150) without providing intuition. Perhaps the most convincing argument for Gauss's starting point (150) is given by Klein [27, p. 71], who gives three definitions of the Gamma function, of which the third is also discussed by Gauss himself [16, p. 151]. Klein [27, p. 74] starts from the Beta-integral, studied by Euler and valid for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(q) > 0$,

$$B(z,q) = \frac{\Gamma(z)\Gamma(q)}{\Gamma(z+q)} = \int_0^1 u^{z-1}(1-u)^{q-1}\,du \tag{157}$$

for $q = k+1$ and makes the substitution $u = \frac{v}{k}$,

$$B(z,k+1) = \int_0^1 u^{z-1}(1-u)^k\,du = k^{-z}\int_0^k v^{z-1}\left(1-\frac{v}{k}\right)^k dv$$

Using $\lim_{k\to\infty}\left(1-\frac{v}{k}\right)^k = e^{-v}$ and Euler's integral (147), Klein [27, p. 74] arrives for $\mathrm{Re}(z) > 0$ at

$$\Gamma(z) = \int_0^\infty v^{z-1}e^{-v}dv = \lim_{k\to\infty}\left(k^z B(z,k+1)\right) = \lim_{k\to\infty}\left(k^z\frac{\Gamma(z)\Gamma(k+1)}{\Gamma(z+k+1)}\right)$$

which is Gauss's definition (154). In contrast to Euler's integral (147) for $\mathrm{Re}(z) > 0$, the functional equation (152) is valid for all z and so is Gauss's product (156).

[30] Gauss proceeds further in [16, p. 148] and derives the reflection formula from his classical result [1, 15.1.20] for the hypergeometric series at $z = 1$, for $c \neq -k$ (k integer) and $\mathrm{Re}(c-a-b) > 0$,

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} \tag{155}$$

He also deduces his multiplication formula, compares his theory with Stirling and Euler's logarithmic expansion in terms of Bernoulli numbers, studies the digamma function, derives his fractional argument digamma function, deduces an integral for the digamma function and complements Euler's computations. In short, an amazing sequence of beautifully derived deep results that constitute our current basis of the Gamma function. In line with his genius, Gauss even laid the basis of prime factors of an entire function of which Weierstass has given the functional theory [47, Chapter VIII].

[31] Both Klein [27, p. 74] and Whittaker and Watson [55, p. 237] mention that Euler has given (156) in a letter to Goldbach in 1729, but that Gauss has provided the first rigorous analysis in [16].

A.2 Deductions from Gauss's product (156) for $\Gamma(z)$

64. *Weierstrass's product.* Weierstrass's product (149) can be obtained from Gauss's definition $\Gamma(z+1) = \lim_{k\to\infty} \Pi(k,z)$. Following Erdélyi *et al.* [11, p. 2], Gauss's definition of the Gamma function can be written, with (150), as

$$\begin{aligned}
\frac{1}{\Gamma(z+1)} &= \lim_{k\to\infty} (1+z)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\dots\left(1+\frac{z}{k}\right)e^{-z\log k} \\
&= \lim_{k\to\infty} (1+z)e^{-z}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\left(1+\frac{z}{3}\right)e^{-\frac{z}{3}}\dots\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}e^{z\left(\sum_{n=1}^{k}\frac{1}{n}-\log k\right)} \\
&= \lim_{k\to\infty}\prod_{n=1}^{k}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\lim_{k\to\infty}e^{z\left(\sum_{n=1}^{k}\frac{1}{n}-\log k\right)}
\end{aligned}$$

Introducing[32] Euler's constant [1, 6.1.3]

$$\gamma = \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n}-\log k\right) = 0.57721\dots \tag{158}$$

leads to Weierstrass's product (149) which illustrates that Euler's constant γ plays a fundamental role in the theory of the Gamma function.

Whittacker and Watson [55, p. 235] elegantly demonstrate that the limit in (158) exists. They define

$$u_n = \int_0^1 \frac{t}{n(n+t)}dt = \frac{1}{n}-\log\frac{n+1}{n}$$

With $\sum_{n=1}^{k-1}\log\frac{n+1}{n} = \sum_{n=1}^{k-1}\log(n+1)-\sum_{n=1}^{k-1}\log n = \log(k)$ and

$$\begin{aligned}
\sum_{n=1}^{k}\frac{1}{n}-\log k &= \sum_{n=1}^{k}\left(\frac{1}{n}-\log\frac{n+1}{n}\right)+\log\frac{k+1}{k} \\
&= \sum_{n=1}^{k}u_n+\log\left(1+\frac{1}{k}\right)
\end{aligned}\tag{159}$$

an alternative representation of Euler's constant (158) is obtained as $\gamma = \sum_{n=1}^{\infty}u_n$. Since $0 < u_n = \int_0^1\frac{t}{n(n+t)}dt < \int_0^1\frac{1}{n^2}dt = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} = 1.64493$, it holds that $0 < \gamma < \frac{\pi^2}{6}$.

The bounds can be sharpened by the inequality $\frac{1}{n+1} < \int_n^{n+1}\frac{dx}{x} = \log\frac{n+1}{n} < \frac{1}{n}$ for any $n > 0$. Indeed, since $\frac{1}{n}-\log\frac{n+1}{n} > 0$, the identity (159) provides the lower bound

$$\sum_{n=1}^{k}\frac{1}{n}-\log k = 1-\log 2+\sum_{n=2}^{k}\left(\frac{1}{n}-\log\frac{n+1}{n}\right)+\log\frac{k+1}{k} > 1-\log 2 = 0.30683$$

Similarly, rewriting the identity (159) and using $\log\frac{n+1}{n}-\frac{1}{n+1} > 0$ gives us the upper bound

$$\begin{aligned}
\sum_{n=1}^{k}\frac{1}{n}-\log k &= 1+\sum_{n=2}^{k}\frac{1}{n}-\sum_{n=1}^{k-1}\log\frac{n+1}{n} = 1-\sum_{n=1}^{k-1}\left(\log\frac{n+1}{n}-\frac{1}{n+1}\right) \\
&< 1-\left(\log 2-\frac{1}{2}\right) = 0.80683
\end{aligned}$$

[32] Gauss [16, 154, footnote] gives the Euler-Mascheroni constant γ up to 40 decimals accurate. Gauss provided the method (i.e. the power series of the digamma function $\psi(z)$) and Fredericus Bernhardus Gothofredus Nicolai has performed the computation. Whittaker and Watson [55, p. 235] mention that J. C. Adams computed γ up to 260 decimals accurate.

In summary, we find $1 - \log 2 < \gamma < \frac{3}{2} - \log 2$. Sharper bounds follows from Poisson's integral (175) in **art**. 68.

65. *Reflection formula.* Gauss [16, p. 148] derives the reflection formula for the Gamma function via his contiguous relations of the hypergeometric function, that, for specific parameters, reduce to the sinus function. Gauss then finds the infinite product of the sinus function [1, 4.3.89]

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \tag{160}$$

Reversely, if we consider the infinite product (160) as known[33], then it follows from Gauss's infinite product (156) for $\Gamma(z)$ that

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{\sin \pi z}{\pi z}$$

which establishes the reflection formula [1, 6.1.17] of the Gamma function, valid for all z,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{161}$$

For example, if $z = \frac{1}{2}$, then the reflection formula (161) shows that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

66. *Multiplication formula.* Gauss [16, p. 149] derives his elegant[34] multiplication formula [1, 6.1.20]

$$\Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)} n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) \tag{162}$$

as follows. Gauss observes that

$$\frac{n^{nz} \prod_{j=0}^{n-1} \Pi\left(k, z - \frac{j}{n}\right)}{\Pi(nk, nz)} = \frac{(\Gamma(k+1))^n}{\Gamma(nk+1)\, k^{\frac{(n-1)}{2}}} \tag{163}$$

does not depend on z. However, he does not give the derivation of (163), but we do. Multiplying the Gauss factors (150),

$$\begin{aligned}\prod_{j=0}^{n-1} \Pi\left(k, z - \frac{j}{n}\right) &= \prod_{j=0}^{n-1} \frac{\Pi(k)\, \Pi\left(z - \frac{j}{n}\right)}{\Pi\left(k + z - \frac{j}{n}\right)} k^{z - \frac{j}{n}} \\ &= (\Pi(k))^n k^{nz} k^{-\frac{1}{n}\sum_{j=1}^{n-1} j} \prod_{j=0}^{n-1} \frac{\Pi\left(z - \frac{j}{n}\right)}{\Pi\left(z + k - \frac{j}{n}\right)}\end{aligned}$$

[33] After integration of the Taylor series of $\pi \cot(\pi x) = \frac{1}{x} - 2\sum_{n=1}^{\infty} \zeta(2n)\, x^{2n-1}$, valid for $|x| < 1$, leading to $\log \frac{\pi x}{\sin(\pi x)} = 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} x^{2n}$, in which the Zeta functions $\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}}$, one deduces that

$$\log \frac{\pi x}{\sin(\pi x)} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x}{k}\right)^{2n} = -\sum_{k=1}^{\infty} \log\left(1 - \frac{x^2}{k^2}\right)$$

from which (160) follows. Although derived under the restriction $|x| < 1$, (160) can be shown to hold for any complex x.

[34] Gauss writes "Unde habemus theorema elegans".

and, changing the notation $\Pi(z) = \Gamma(z+1)$ and with $\sum_{j=1}^{n-1} j = \frac{n(n-1)}{2}$, we have

$$\prod_{j=0}^{n-1} \Pi\left(k, z - \frac{j}{n}\right) = (\Gamma(k+1))^n k^{nz} k^{-\frac{(n-1)}{2}} \prod_{j=0}^{n-1} \frac{\Gamma\left(z+1-\frac{j}{n}\right)}{\Gamma\left(z+k+1-\frac{j}{n}\right)}$$

Introducing Gauss's infinite product (156) yields

$$\begin{aligned}
\prod_{j=0}^{n-1} \frac{\Gamma\left(z+1-\frac{j}{n}\right)}{\Gamma\left(z+k+1-\frac{j}{n}\right)} &= \prod_{j=0}^{n-1} \prod_{m=1}^{\infty} \frac{m^k}{(m+1)^k} \prod_{m=0}^{\infty} \frac{(nm+nz+nk+n-j)}{(nm+nz+n-j)} \\
&= \prod_{m=1}^{\infty} \frac{m^{nk}}{(m+1)^{nk}} \prod_{m=0}^{\infty} \prod_{l=1}^{n} \frac{(nm+nz+nk+l)}{(nm+nz+l)} \\
&= \prod_{m=1}^{\infty} \frac{m^{nk}}{(m+1)^{nk}} \prod_{m=0}^{\infty} \prod_{l=1}^{n} \left(1 + \frac{nk}{nm+nz+l}\right)
\end{aligned}$$

Because $r = nm + l$ runs over all integers, we observe that

$$\prod_{m=0}^{\infty} \prod_{l=1}^{n} \left(1 + \frac{nk}{nm+l+nz}\right) = \prod_{r=1}^{\infty} \left(1 + \frac{nk}{r+nz}\right)$$

and we arrive at

$$\prod_{j=0}^{n-1} \Pi\left(k, z - \frac{j}{n}\right) = (\Gamma(k+1))^n k^{nz} k^{-\frac{(n-1)}{2}} \prod_{m=1}^{\infty} \frac{m^{nk}}{(m+1)^{nk}} \prod_{r=1}^{\infty} \left(1 + \frac{nk}{r+nz}\right)$$

Gauss divides $\prod_{j=0}^{n-1} \Pi\left(k, z - \frac{j}{n}\right)$ by $\Pi(nk, nz)$, which equals

$$\begin{aligned}
\Pi(nk, nz) &= \frac{\Pi(nk)\,\Pi(nz)}{\Pi(nk+nz)} n^{nz} k^{nz} = \frac{\Gamma(nk+1)\,\Gamma(nz+1)}{\Gamma(nk+nz+1)} n^{nz} k^{nz} \\
&= n^{nz} k^{nz} \Gamma(nk+1) \prod_{m=1}^{\infty} \frac{m^{nk}}{(m+1)^{nk}} \prod_{m=1}^{\infty} \frac{m+nz+nk}{m+nz} \\
&= n^{nz} k^{nz} \Gamma(nk+1) \prod_{m=1}^{\infty} \frac{m^{nk}}{(m+1)^{nk}} \prod_{r=1}^{\infty} \left(1 + \frac{nk}{r+nz}\right)
\end{aligned}$$

resulting in (163) and demonstrating that the left-hand side of (163) is independent of z. As follows from the Gauss factors (150), $\Pi(k, 0) = \Pi(nk, 0) = 1$ and the choice of $z = 0$ in the left-hand side of (163) is

$$\frac{n^{nz} \prod_{j=0}^{n-1} \Pi\left(k, z - \frac{j}{n}\right)}{\Pi(nk, nz)} = \prod_{j=0}^{n-1} \Pi\left(k, -\frac{j}{n}\right)$$

After taking the limit of $k \to \infty$ of both sides, the definition $\Pi(z) = \Gamma(z+1) = \lim_{k\to\infty} \Pi(k, z)$ leads to

$$\frac{n^{nz} \prod_{j=0}^{n-1} \Gamma\left(z+1-\frac{j}{n}\right)}{\Gamma(nz+1)} = \prod_{j=1}^{n-1} \Gamma\left(1 - \frac{j}{n}\right)$$

Let $l = n - j$, then $\prod_{j=1}^{n-1} \Gamma\left(1 - \frac{j}{n}\right) = \prod_{j=1}^{n-1} \Gamma\left(\frac{n-j}{n}\right) = \prod_{l=1}^{n-1} \Gamma\left(\frac{l}{n}\right)$ so that

$$\left(\prod_{j=1}^{n-1} \Gamma\left(1 - \frac{j}{n}\right)\right)^2 = \prod_{j=1}^{n-1} \Gamma\left(1 - \frac{j}{n}\right)\Gamma\left(\frac{j}{n}\right) = \prod_{j=1}^{n-1} \frac{\pi}{\sin \pi \frac{j}{n}} = \frac{\pi^{n-1}}{\prod_{j=1}^{n-1} \sin \frac{\pi j}{n}}$$

where the reflection formula (161) has been invoked. Using Euler's formula[35]

$$\sin x = 2^{n-1} \prod_{k=0}^{n-1} \sin \frac{(\pi k + x)}{n} \tag{164}$$

for $x \to 0$ gives $\prod_{k=1}^{n-1} \sin \frac{\pi k}{n} = \frac{n}{2^{n-1}}$ and finally leads to

$$\Gamma(nz+1) = (2\pi)^{-\frac{n-1}{2}} n^{nz+\frac{1}{2}} \prod_{k=1}^{n} \Gamma\left(z + \frac{k}{n}\right)$$

that equals (162), because $\prod_{k=1}^{n} \Gamma\left(z + \frac{k}{n}\right) = \Gamma(z+1) \prod_{k=1}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = z \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$.

67. *Gauss's integral for the digamma function.* We will demonstrate Gauss's integral [**?**, p. 160, formula [78]Gauss1813

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\right) dt \tag{165}$$

Instead of following Gauss's deduction, a more elegant derivation [55, p. 247] is obtained from Weierstrass's infinite product (149). The logarithm of Weierstrass's infinite product (149) is

$$-\log \Gamma(z+1) = \gamma z + \sum_{n=1}^{\infty} \left(\log\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right)$$

and differentiation yields

$$\psi(z+1) = \frac{d}{dz} \log \Gamma(z+1) = -\gamma - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) = -\gamma + z \sum_{n=1}^{\infty} \frac{1}{(z+n)\, n}$$

from which $\psi(1) = -\gamma$. The functional equation (148) of $\psi(z)$ then shows that

$$\psi(z) = -\gamma - \frac{1}{z} - \lim_{k \to \infty} \sum_{n=1}^{k} \left(\frac{1}{z+n} - \frac{1}{n}\right) \tag{166}$$

The polygamma functions, defined as $\psi^{(n)}(z) = \frac{d^{n+1} \ln \Gamma(z)}{dz^{n+1}}$ with $\psi^{(0)}(z) = \psi(z)$, follow immediately from (166) as [1, 6.4.10]

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \tag{167}$$

Substituting $\int_0^\infty e^{-t(z+n)} dt = \frac{1}{z+n}$, valid for $\operatorname{Re}(t) > 0$, into (166) yields

$$\psi(z) = -\gamma - \int_0^\infty e^{-tz} dt - \lim_{k \to \infty} \int_0^\infty \left(e^{-tz} - 1\right) \sum_{n=1}^{k} e^{-tn} dt$$

[35]The polynomial $z^n - 1$ has as zeros the n-th roots of unity, $z^n - 1 = (z-1) \prod_{k=1}^{n-1} \left(1 - z\, e^{-\frac{2\pi i k}{n}}\right)$. Choosing $z = e^{\frac{2ix}{n}}$ leads, after some manipulations, to (164).

With $\sum_{n=1}^{k} e^{-tn} = e^{-t}\frac{1-e^{-kt}}{1-e^{-t}}$, we have

$$\begin{aligned}
\psi(z) &= -\gamma - \int_0^\infty e^{-tz}dt + \lim_{k\to\infty}\int_0^\infty \frac{\left(e^{-t}-e^{-t(z+1)}\right)\left(1-e^{-kt}\right)}{1-e^{-t}}dt \\
&= -\gamma - \int_0^\infty e^{-tz}dt + \int_0^\infty \frac{e^{-t}-e^{-t(z+1)}}{1-e^{-t}}dt - \lim_{k\to\infty}\int_0^\infty \frac{\left(1-e^{-tz}\right)}{1-e^{-t}}e^{-(k+1)t}dt
\end{aligned}$$

and

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-t}-e^{-tz}}{1-e^{-t}}dt + \lim_{k\to\infty}\int_0^\infty \frac{\left(1-e^{-tz}\right)}{1-e^{-t}}e^{-(k+1)t}dt$$

Since $\left|\int_0^\infty \frac{\left(1-e^{-tz}\right)}{1-e^{-t}}e^{-(k+1)t}dt\right| \leq \int_0^\infty \left|\frac{\left(1-e^{-tz}\right)}{1-e^{-t}}\right| e^{-(k+1)t}dt \leq \max_{t\geq 0}\left|\frac{\left(1-e^{-tz}\right)}{1-e^{-t}}\right|\frac{1}{k+1} \to 0$ for large k, because the function $\frac{\left(1-e^{-tz}\right)}{1-e^{-t}}$ is bounded for $t\geq 0$, we arrive at

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-t}-e^{-tz}}{1-e^{-t}}dt$$

Similarly as above, we rewrite Euler's constant γ in (158) by invoking again $\frac{1}{s} = \int_0^\infty e^{-st}dt$ for $\mathrm{Re}(t) > 0$ and by using $\log x = \int_1^x \frac{1}{s}ds = \int_1^x\int_0^\infty e^{-st}dtds = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}dt$,

$$\begin{aligned}
\sum_{n=1}^{k}\frac{1}{n} - \log k &= \int_0^\infty e^{-t}\frac{1-e^{-kt}}{1-e^{-t}}dt - \int_0^\infty \frac{e^{-t}-e^{-kt}}{t}dt \\
&= \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}dt - \int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}} - \frac{1}{t}\right)e^{-kt}dt
\end{aligned}$$

The generating function of the Bernoulli numbers, written as

$$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \sum_{n=1}^{\infty} B_{2n}\frac{t^{2n-1}}{(2n)!} \qquad \text{for } |t| \leq 2\pi \tag{168}$$

shows that $\frac{e^{-t}}{1-e^{-t}} - \frac{1}{t}$ is continuous at $t=0$ and bounded for $t \geq 0$, leading for $k\to\infty$ to

$$\gamma = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}dt \tag{169}$$

and to Gauss's integral (165).

68. *Stirling's asymptotic formula.* Inspired by the Bernoulli generating function (168), we rewrite Gauss's integral (165) for $z \to z+1$,

$$\begin{aligned}
\frac{d}{dz}\log\Gamma(z+1) &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{e^t-1}\right)dt \\
&= \int_0^\infty \left(\frac{e^{-t}-e^{-zt}}{t} + \frac{e^{-zt}}{2} - e^{-zt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\right)dt \\
&= \log z + \frac{1}{2z} - \int_0^\infty e^{-zt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt
\end{aligned}$$

Since $0 \leq \frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \leq \frac{1}{2}$ is bounded for $t \geq 0$ and continuous at $t = 0$, the last integral is uniformly convergent for $\mathrm{Re}(z) > 0$ and, hence, can be integrated from 1 to z,

$$\log\Gamma(z+1) = z\log z - z + 1 + \frac{1}{2}\log z + \int_0^\infty \frac{e^{-zt}-e^{-t}}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt$$

With the functional equation (146), we find

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + 1 + \int_0^\infty \frac{e^{-zt}}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt - \int_0^\infty \frac{e^{-t}}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt$$

The last integral can be evaluated [55, p. 249] as $1-\frac{1}{2}\log(2\pi)$ and we arrive at Binet's first integral for $\operatorname{Re}(z)>0$,

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \frac{e^{-zt}}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt \tag{170}$$

We observe from (170) that

$$\left|\int_0^\infty \frac{e^{-zt}}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt\right| \le \max_{t\ge 0}\left|\frac{1}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\right|\frac{1}{z} = \frac{B_2}{2z} = \frac{1}{12z}$$

so that, for any $\operatorname{Re}(z)>0$, a sharp upper bound is found,

$$|\log\Gamma(z)| \le \left|\left(z-\frac{1}{2}\right)\log z - z\right| + \frac{1}{2}\log(2\pi) + \frac{1}{12z} \tag{171}$$

In particular, the integral in (170) is positive for positive real $x>0$, leading to the bounds

$$\left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) \le \log\Gamma(x) \le \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} \tag{172}$$

Invoking the generating function of the Bernoulli numbers (168) in (170), ignoring the restriction $|t|<2\pi$, yields Stirling's approximation [1, 6.1.40],

$$\begin{aligned}\log\Gamma(z) &= \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{n=1}^\infty \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \\ &= \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + O\left(\frac{1}{z^5}\right)\end{aligned} \tag{173}$$

By substituting the partial fraction expansion $\frac{1}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right) = \sum_{n=1}^\infty \frac{2}{t^2+4\pi^2n^2}$ in (170),

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + 2\int_0^\infty e^{-zt}\sum_{n=1}^\infty \frac{1}{t^2+4\pi^2n^2}dt$$

and changing the integration variable,

$$\begin{aligned}\int_0^\infty e^{-zt}\sum_{n=1}^\infty \frac{1}{t^2+4\pi^2n^2}dt &= \int_0^\infty \sum_{n=1}^\infty \frac{e^{-zt}}{\left(1+\left(\frac{t}{2\pi n}\right)^2\right)(2\pi n)^2}dt = \frac{1}{2\pi}\int_0^\infty \frac{du}{1+u^2}\sum_{n=1}^\infty \frac{e^{-2\pi nzu}}{n} \\ &= -\frac{1}{2\pi}\int_0^\infty \frac{du}{1+u^2}\log\left(1-e^{-2\pi zu}\right)\end{aligned}$$

we find, after partial integration, Binet's second form [55, p. 251],[44, p. 217],

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + 2z\int_0^\infty \frac{\arctan t}{e^{2\pi tz}-1}dt \tag{174}$$

Substituting in Binet's second form (174), the Taylor series $\arctan t = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1}t^{2n+1}$ around $t=0$, valid for $|t|<1$, reversing the integral and summation while ignoring the restriction $|t|<1$, leads, with $B_{2n} = (-1)^{n-1}4n\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}dt$ for $n\ge 1$, again to Stirling's approximation (173).

Titchmarsh [47, p. 151] gives a third form

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \frac{[t]-t+\frac{1}{2}}{t+z}dt$$

where $[t]$ is the nearest integer smaller than or equal to t. Blagouchine [8] gives seven series for $\log\Gamma(z)$. Edwards [9, p. 106-114] derives the Stirling approximation (173) for $\log\Pi(z) = \log\Gamma(z+1)$ by Euler-Maclaurin summation (60).

Differentiating Binet's second form (174) results in a companion of Gauss's integral (165) for the digamma function,

$$\psi(z) = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{e^{2\pi tz}-1}\frac{dt}{1+t^2} = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{e^{2\pi t}-1}\frac{dt}{z^2+t^2} \tag{175}$$

from which we find that Euler's constant $\gamma = -\psi(1) = \frac{1}{2}+2\int_0^\infty \frac{t}{e^{2\pi t}-1}\frac{dt}{1+t^2} > \frac{1}{2}$. Since $\int_0^\infty \frac{t}{e^{2\pi t}-1}\frac{dt}{1+t^2} < \int_0^\infty \frac{tdt}{e^{2\pi t}-1}$ and $\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}dt}{e^t-1}$, an upper bound follows as $\gamma < \frac{1}{2} + \frac{2}{(2\pi)^2}\zeta(2) = 0.5833$.

69. *Asymptotic behavior of* $\log\Gamma(z)$. For large r and $\theta \neq \pi$, Stirling's formula (173) shows that

$$\log\Gamma\left(b+re^{i\theta}\right) = \left(b+re^{i\theta}-\frac{1}{2}\right)\log\left(b+re^{i\theta}\right) - b - re^{i\theta} + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{r}\right)$$

With

$$\begin{aligned}\log\left(b+re^{i\theta}\right) &= \log\left(re^{i\theta}\left(1+r^{-1}e^{-i\theta}b\right)\right) = \log re^{i\theta} + \log\left(1+r^{-1}e^{-i\theta}b\right)\\ &= \log r + i\theta + r^{-1}e^{-i\theta}b + O\left(\frac{1}{r^2}\right)\end{aligned}$$

we have

$$\begin{aligned}\log\Gamma\left(b+re^{i\theta}\right) &= \left(b+re^{i\theta}-\frac{1}{2}\right)\log r + i\theta\left(b+re^{i\theta}-\frac{1}{2}\right) + \left(br^{-1}+e^{i\theta}-\frac{1}{2}r^{-1}\right)e^{-i\theta}b\\ &\quad - b - re^{i\theta} + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{r}\right)\\ &= \left(b-\frac{1}{2}+r\cos\theta\right)\log r - \theta(r\sin\theta) - r\cos\theta + \frac{1}{2}\log(2\pi)\\ &\quad + i\left\{r\ln r\sin\theta + \left(b-\frac{1}{2}+r\cos\theta\right)\theta - r\sin\theta\right\} + O\left(\frac{1}{r}\right)\end{aligned}$$

Thus,

$$\Gamma\left(b+re^{i\theta}\right) = \sqrt{2\pi}e^{r(\log(r)-1)\cos\theta-\theta r\sin\theta+\left(b-\frac{1}{2}\right)\log(r)}e^{i\left(r(\log(r)-1)\sin\theta+r\theta\cos\theta+\left(b-\frac{1}{2}\right)\theta\right)}\left(1+O\left(\frac{1}{r}\right)\right)$$

from which

$$\left|\Gamma\left(b+re^{i\theta}\right)\right| = r^{b-\frac{1}{2}}e^{r(\log(r)-1)\cos\theta-\theta r\sin\theta}\left(1+O\left(\frac{1}{r}\right)\right)$$

Hence, it holds that

$$\frac{1}{\left|\Gamma\left(b+re^{i\theta}\right)\right|} = \frac{r^{\frac{1}{2}-b}}{\sqrt{2\pi}}e^{-r(\ln(r)-1)\cos\theta+\theta r\sin\theta}\left(1+O\left(\frac{1}{r}\right)\right) \tag{176}$$

illustrating that $\lim_{r\to\infty}\frac{1}{\left|\Gamma\left(b+re^{i\theta}\right)\right|} = 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. However, if $\theta = \frac{\pi}{2}$, then in contrast an exponential increase in r is witnessed,

$$\frac{1}{|\Gamma(b+ir)|} = \frac{r^{\frac{1}{2}-b}}{\sqrt{2\pi}}e^{\frac{\pi}{2}r}\left(1+O\left(\frac{1}{r}\right)\right). \tag{177}$$

A.3 Complex integrals for the Gamma function

70. *Hankel's integral.* We derive Hankel's integral

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C w^{-z}e^w dw \tag{178}$$

where the contour C starts at $-\infty$ below the real axis, encircles the origin at $z=0$ and returns above the negative real axis again to $-\infty$. Such a contour around the branch-cut (here the negative real axis) is "classical" in integrals containing w^α such as Mellin transforms [46]. If ε is the radius of a circle at the origin, then evaluation of the integral along the contour C yields

$$\begin{aligned}\int_C w^{-z}e^w dw &= -\int_\infty^\varepsilon \left(xe^{-i\pi}\right)^{-z} e^{-x}dx + i\int_{-\pi}^{\pi}\left(\varepsilon e^{i\theta}\right)^{-z} e^{\varepsilon e^{i\theta}}\varepsilon e^{i\theta}d\theta - \int_\varepsilon^\infty \left(xe^{i\pi}\right)^{-z} e^{-x}dx \\ &= \left(e^{i\pi z}-e^{-i\pi z}\right)\int_\varepsilon^\infty x^{-z}e^{-x}dx + i\varepsilon^{1-z}\int_{-\pi}^{\pi} e^{i\theta(1-z)}e^{\varepsilon e^{i\theta}}d\theta\end{aligned}$$

If $\mathrm{Re}(1-z)>0$, then we take the limit $\varepsilon\to 0$ and find with Euler's integral (147)

$$\int_C w^{-z}e^w dw = 2i\sin\pi z\int_0^\infty x^{1-z-1}e^{-x}dx = 2i\sin\pi z\,\Gamma(1-z)$$

After replacing z by $1-z$, we obtain a contour integral for the Gamma function,

$$\Gamma(z) = \frac{\pi}{\sin\pi z}\frac{1}{2\pi i}\int_C w^{z-1}e^w dw \tag{179}$$

The reflection formula (161), $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$, of the Gamma function leads to Hankel's contour integral (178). Although the derivation was restricted to $\mathrm{Re}(z)>0$ in (179), by analytic continuation (see e.g. [47, Chapter IV],[13, Chapter III]), (178) as well as (179) hold for all $z\in\mathbb{C}$. The contour integral (179) for the Gamma function demonstrates that $\Gamma(z)$ has simple poles at $z=-n$ for each integer n (due to $\sin\pi z$), in agreement with Gauss's product (156).

We can deform[36] the contour C into C_ϕ by tilting the straight line above the negative real axis over an angle ϕ and the straight line below the negative real axis over an angle $-\phi$. Indeed, consider the contour L consisting of the contour C, the circle segment at infinity from the angle π to the angle ϕ, followed by the line $w=re^{i\phi}$, where r ranges from infinity towards ρ, the circle centered at $w=0$ with radius ρ turning from angle ϕ towards $-\phi$ and complemented by the line $w=re^{-i\phi}$ and infinite circle segment towards the begin of the contour C. The integral $\int_L w^{-z}e^w dw = 0$, because the contour L encloses an analytic region of the function $w^{-z}e^w$. If $\phi>\frac{\pi}{2}$, then the part of L along the circle segment at infinity, $\lim_{r\to\infty} i\int_\pi^\phi r^{1-z}e^{(1-z)i\theta}e^{re^{i\theta}}d\theta = 0$. Hence, combining all contributions results in

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{C_\phi} w^{-z}e^w dw \tag{180}$$

where the contour C_ϕ starts at infinity on the straight line at the angle $-\frac{\pi}{2}>\phi>-\pi$ below the real axis until the circle at the origin with radius ρ that turns over the angle $-\phi$ to ϕ until hitting the line $w=re^{i\phi}$ along which it passes towards infinity again. The contour in (178) is the particular case where $C=C_\pi$ and $\rho=\varepsilon$. Hankel's integral in (180) can be written as

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{c-\infty e^{-i\phi}}^{c+\infty e^{i\phi}} w^{-z}e^w dw \qquad \text{with } c>0 \text{ and } \frac{\pi}{2}<\phi<\pi \tag{181}$$

[36]The same deformation holds for the contour integral (179) as well.

Since multiplying $c' = ac > 0$ provided $a > 0$, the map $w \to aw$ for any real, positive number a yields

$$\frac{a^{z-1}}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c'-\infty e^{-i\phi}}^{c'+\infty e^{i\phi}} w^{-z} e^{aw} dw \qquad \text{with } c' > 0 \text{ and } \frac{\pi}{2} < \phi < \pi \tag{182}$$

B Complex integrals due to Cauchy and Mellin

We evaluate integrals of a general kind.

71. *A Cauchy-type integral.*

Theorem 1 *Let $f(z)$ be an entire function, that is real on the real axis and $\lim_{r\to\infty} \frac{f(re^{i\theta})}{r^2} = 0$ for $\theta = 0$ and $\theta = \frac{\pi}{2}$. If σ is a positive real number, then it holds that*

$$\int_{-\infty}^{\infty} \frac{f(\sigma + it)}{|\sigma + it|^2} dt = \frac{\pi}{\sigma}\, f(2\,\sigma) \tag{183}$$

Proof: Consider the integral $I = \frac{1}{2\pi i} \int_L \frac{f(z)}{z(2\sigma - z)} dz$ where the contour L is taken counter-clockwise round the rectangle formed by the lines $\text{Im}(z) = -T$, $\text{Re}(z) = a > 2\sigma$, $\text{Im}(z) = T$ and $\text{Re}(z) = \sigma > 0$. The contour L encloses the pole on the real axis at $x = 2\sigma$. Since the entire function $f(z)$ is analytic inside and on L, the integral equals $I = -\frac{f(2\sigma)}{2\sigma}$. On the other hand, evaluating the integral I along the contour L yields,

$$\begin{aligned} I = &\frac{1}{2\pi i} \int_{\sigma}^{a} \frac{f(x - iT)}{(x - iT)(2\sigma - x + iT)}\, dx + \frac{1}{2\pi} \int_{-T}^{T} \frac{f(a + it)}{(a + it)(2\sigma - a - it)} dt \\ &- \frac{1}{2\pi i} \int_{\sigma}^{a} \frac{f(x + iT)}{(x + iT)(2\sigma - x - iT)}\, dx - \frac{1}{2\pi} \int_{-T}^{T} \frac{f(\sigma + it)}{|\sigma + it|^2}\, dt \end{aligned}$$

Combined and rewritten leads to

$$\begin{aligned} f(2\sigma) = &\frac{\sigma}{\pi} \int_{-T}^{T} \frac{f(\sigma + it)}{\sigma^2 + t^2} dt - \frac{\sigma}{\pi} \int_{-T}^{T} \frac{f(a + it)}{(a + it)(2\sigma - a - it)} dt \\ &+ \frac{\sigma}{\pi i} \int_{\sigma}^{a} \text{Im} \left[\frac{f(x + iT)}{(x + iT)(2\sigma - x - iT)} \right] dx \end{aligned}$$

Since $\lim_{z\to\infty} \frac{f(z)}{z^2} = 0$ for $z = re^{i\theta}$ with $\theta = 0$ and $\theta = \frac{\pi}{2}$, the second and third integral vanish, demonstrating the Theorem. $\square$

72. *Mellin transform of a product of Gamma functions.* Let $p_1, p_2, \ldots, p_n$ be different real, positive numbers, then the Mellin transform

$$\prod_{j=1}^{n} \Gamma(s + p_j) = \int_0^{\infty} u^{s-1} g\left(u; \{p_j\}_{1\leq j\leq n}\right) du \tag{184}$$

has an inverse

$$g\left(u; \{p_j\}_{1\leq j\leq n}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{n} \Gamma(s + p_j)\, u^{-s} ds \qquad \text{with } c > 0 \tag{185}$$

which can be evaluated. The contour in (185) can be closed over the negative $\mathrm{Re}\,(s)$-plane and encloses the simple poles at $s=-q_j-p_j$ for integers $q_j\geq 0$ for $1\leq j\leq n$ on the negative real axis. Cauchy's residue theorem [47] leads to

$$g\left(u;\{p_j\}_{1\leq j\leq n}\right)=\sum_{j=1}^{n}\sum_{q_j=0}^{\infty}\lim_{s\to -q_j-p_j}(s+q_j+p_j)\,\Gamma(s+p_j)\prod_{m=1;m\neq j}^{n}\Gamma(s+p_m)\,u^{-s}$$

Iterating the functional equation (146) of the Gamma function (q_j+1)-times gives

$$\begin{aligned}\lim_{s\to -q_j-p_j}(s+q_j+p_j)\,\Gamma(s+p_j)&=\lim_{s\to -q_j-p_j}\frac{\Gamma(s+p_j+q_j+1)\,(s+p_j+q_j)}{(s+p_j)\,(s+p_j+1)\ldots(s+p_j+q_j-1)\,(s+p_j+q_j)}\\&=\frac{\Gamma(1)}{(-q_j)\,(-q_j+1)\ldots(-1)}=\frac{(-1)^{q_j}}{(q_j)!}\end{aligned}$$

Hence, we obtain

$$g\left(u;\{p_j\}_{1\leq j\leq n}\right)=\sum_{j=1}^{n}\sum_{q_j=0}^{\infty}\frac{(-1)^{q_j}\,u^{q_j+p_j}}{(q_j)!}\prod_{m=1;m\neq j}^{n}\Gamma(-q_j+p_m-p_j)$$

With the reflection formula (161)

$$\Gamma(-q_j+p_m-p_j)=\frac{\pi\,(-1)^{q_j}}{\sin\pi\,(p_m-p_j)\,\Gamma(1+q_j+p_j-p_m)}$$

we have

$$\begin{aligned}\prod_{m=1;m\neq j}^{n}\Gamma(-q_j+p_m-p_j)&=\prod_{m=1;m\neq j}^{n}\frac{\pi\,(-1)^{q_j}}{\sin\pi\,(p_m-p_j)\,\Gamma(1+q_j+p_j-p_m)}\\&=\frac{\pi^{n-1}\,(-1)^{(n-1)q_j}}{\prod_{m=1}^{j-1}\sin\pi\,(p_m-p_j)\prod_{m=j+1}^{n}\sin\pi\,(p_m-p_j)}\\&\quad\times\frac{1}{\prod_{m=1}^{j-1}\Gamma(1+q_j+p_j-p_m)\prod_{m=j+1}^{n}\Gamma(1+q_j+p_j-p_m)}\end{aligned}$$

we arrive at the series

$$g\left(u;\{p_j\}_{1\leq j\leq n}\right)=\sum_{j=1}^{n}\frac{(-1)^{j-1}\,\pi^{n-1}u^{p_j}}{\prod_{m=1}^{j-1}\sin\pi\,(p_j-p_m)\prod_{m=j+1}^{n}\sin\pi\,(p_m-p_j)}\sum_{q_j=0}^{\infty}\frac{(-1)^{nq_j}\,u^{q_j}}{(q_j)!\prod_{m=1;m\neq j}^{n}\Gamma(1+q_j+p_j-p_m)}\tag{186}$$

with the convention that $\prod_{m=a}^{b}f_m=1$ if $b<a$.

Examples If $n=1$, then we retrieve the classical Mellin transform of the pair e^{-u} and $\Gamma(s)$,

$$g(u;p_1)=u^{p_1}\sum_{q_1=0}^{\infty}\frac{(-1)^{q_1}\,u^{q_1}}{(q_1)!}=u^{p_1}e^{-u}$$

If $p_j = \frac{j}{n}$ for $1 \leq j \leq n-1$ as in Gauss's multiplication formula (162) without $j=0$ factor, then (186), denoted as $g\left(u; \left\{\frac{j}{n}\right\}_{1\leq j\leq n-1}\right) = h_n(u)$ reduces to

$$h_n(u) = \pi^{n-2} \sum_{j=1}^{n-1} \frac{(-1)^{j-1} u^{\frac{j}{n}}}{\prod_{k=1}^{j-1} \sin\left(\frac{\pi k}{n}\right) \prod_{k=1}^{n-j-1} \sin\left(\frac{\pi k}{n}\right)} \sum_{q_j=0}^{\infty} \frac{\left((-1)^{n-1}\right)^{q_j} u^{q_j}}{(q_j)! \prod_{k=1}^{j-1} \Gamma\left(1+q_j+\frac{k}{n}\right) \prod_{m=1}^{n-j-1} \Gamma\left(1+q_j-\frac{m}{n}\right)} \tag{187}$$

In particular for $n=3$, then

$$h_3(u) = \pi \frac{u^{\frac{1}{3}}}{\sin\left(\frac{\pi}{3}\right)} \sum_{q_1=0}^{\infty} \frac{u^{q_1}}{(q_j)!\Gamma\left(1+q_1-\frac{1}{3}\right)} - \pi \frac{u^{\frac{2}{3}}}{\sin\left(\frac{\pi}{3}\right)} \sum_{q_2=0}^{\infty} \frac{u^{q_2}}{(q_j)!\Gamma\left(1+q_2+\frac{1}{3}\right)}$$

We briefly summarize the theory of the modified Bessel function $K_\nu(z)$, but strongly advise to consult the monumental treatise of Watson [54]. The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are defined [54, p. 77-78] as the two independent solutions of the modified Bessel differential equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} - \left(z^2+\nu^2\right) y = 0$$

Both $I_\nu(z)$ and $K_\nu(z)$ are entire functions in v for $z \neq 0$ and analytic in z, except for a cut along the negative real axis. The function $I_\nu(z)$ in z possesses the Taylor series

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \tag{188}$$

The modified Bessel functions $K_\nu(z)$ is defined as

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\left(I_{-\nu}(z) - I_\nu(z)\right) \tag{189}$$

clearly even in the order ν, $K_\nu(z) = K_{-\nu}(z)$ for all $z \neq 0$, with Taylor series

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\left(\left(\frac{1}{2}z\right)^{-\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(1-\nu+k)} - \left(\frac{1}{2}z\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(1+\nu+k)}\right) \tag{190}$$

The Taylor series (190) shows that

$$h_3(u) = 2u^{\frac{1}{2}} K_{\frac{1}{3}}(2\sqrt{u}) \tag{191}$$

On the other hand, the integral [41, eq. (28.73), p. 55], valid for $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(p) > 0$,

$$\Gamma(s)\Gamma(s+p) = 2\int_0^\infty K_p\left(2\sqrt{x}\right) x^{\frac{p}{2}+s-1} dx$$

is a special case of (184) and corresponds to $h_3(u)$ in (191) with $p = \frac{1}{3}$ and $s \to s + \frac{1}{3}$.

C Inverse Laplace transform

The Laplace transform for complex z is defined (see e.g. [46], [13, Chapter VII], [56]) as

$$\varphi(z) = \int_0^\infty e^{-zt} f(t) dt \tag{192}$$

with the inverse transform,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi(z)e^{zt}dz \tag{193}$$

where c is the smallest real value of $\mathrm{Re}(z)$ for which the integral in (192) converges.

We evaluate the integral in (193) along the line $z = c + iw$,

$$f(t) = \frac{e^{tc}}{2\pi}\int_{-\infty}^{\infty} \varphi(c+iw)e^{itw}dw$$

After writing the integrand in a real and an imaginary part,

$$\begin{aligned} f(t) = {} & \frac{e^{tc}}{2\pi}\int_{-\infty}^{\infty} \{\mathrm{Re}\,\varphi(c+iw)\cos tw - \mathrm{Im}\,(\varphi(c+iw))\sin tw\}\,dw \\ & + i\frac{e^{tc}}{2\pi}\int_{-\infty}^{\infty} \{\mathrm{Re}\,(\varphi(c+iw))\sin tw + \mathrm{Im}\,(\varphi(c+iw))\cos tw\}\,dw \end{aligned}$$

we find, after separating the real and imaginary part, that

$$\begin{cases} f(t) = \frac{e^{tc}}{2\pi}\int_{-\infty}^{\infty} \{\mathrm{Re}\,\varphi(c+iw)\cos tw - \mathrm{Im}\,\varphi(c+iw)\sin tw\}\,dw \\ 0 = \int_{-\infty}^{\infty} \{\mathrm{Re}\,\varphi(c+iw)\sin tw + \mathrm{Im}\,\varphi(c+iw)\cos tw\}\,dw \end{cases}$$

On the other hand, it follows from (192) that

$$\begin{cases} \mathrm{Re}\,\varphi(c+iw) = \int_0^{\infty} e^{-ct}f(t)\cos wt dt \\ \mathrm{Im}\,\varphi(c+iw) = -\int_0^{\infty} e^{-ct}f(t)\sin wt dt \end{cases}$$

and that $\mathrm{Re}\,\varphi(c+iw)\cos tw$ and $\mathrm{Im}\,\varphi(c+iw)\sin tw$ are even in w. Likewise, $\mathrm{Re}\,\varphi(c+iw)\sin tw$ and $\mathrm{Im}\,\varphi(c+iw)\cos tw$ are odd in w. Hence, we arrive at

$$\begin{cases} f(t) = \frac{e^{tc}}{\pi}\int_0^{\infty} \{\mathrm{Re}\,\varphi(c+iw)\cos tw - \mathrm{Im}\,\varphi(c+iw)\sin tw\}\,dw \\ \int_{-\infty}^{\infty} \mathrm{Re}\,\varphi(c+iw)\sin tw dw = \int_{-\infty}^{\infty} \mathrm{Im}\,\varphi(c+iw)\cos tw dw = 0 \end{cases} \tag{194}$$

The derivation is a corrected version of [3].

Berberan-Santos[37] suggests to proceed a step further by defining $f(t) = 0$ for $t < 0$. In that case, it follows from (194) that

$$0 = \int_0^{\infty} \{\mathrm{Re}\,\varphi(c+iw)\cos tw - \mathrm{Im}\,\varphi(c+iw)\sin tw\}\,dw \qquad \text{for } t < 0$$

which leads, after replacing $t \to -t$, to

$$\int_0^{\infty} \mathrm{Re}\,\varphi(c+iw)\cos tw dw = -\int_0^{\infty} \mathrm{Im}\,\varphi(c+iw)\sin tw dw \qquad \text{for } t > 0$$

The final resulting set of integral equations, subject to "$f(t) = 0$ for $t < 0$", simplifies to

$$\begin{cases} f(t) = \frac{2e^{tc}}{\pi}\int_0^{\infty} \mathrm{Re}\,\varphi(c+iw)\cos tw dw & \text{for } t > 0 \\ f(t) = -\frac{2e^{tc}}{\pi}\int_0^{\infty} \mathrm{Im}\,\varphi(c+iw)\sin tw dw & \text{for } t > 0 \\ \int_{-\infty}^{\infty} \mathrm{Re}\,\varphi(c+iw)\sin tw dw = \int_{-\infty}^{\infty} \mathrm{Im}\,\varphi(c+iw)\cos tw dw = 0 & \end{cases} \tag{195}$$

[37] Private communication.

The general form (193) does not impose the restriction "$f(t) = 0$ for $t < 0$" and is continuous[38] at $t = 0$. The restriction "$f(t) = 0$ for $t < 0$" is only continuous at $t = 0$ if $f(0) = 0$. In spite of this concern, Berberan-Santos [3] evaluates various Laplace inverses via $f(t) = \frac{2e^{tc}}{\pi} \int_0^\infty \operatorname{Re} \varphi(c + iw) \cos twdw$.

In Fourier transforms, the inverse has a similar form as the transform itself and tables of Fourier transforms can thus be used in two directions. Gross [21] has written[39] a note on the question when the inverse Laplace transform (193) is of the same form as the Laplace transform (192) itself. In particular, if $\varphi(z) = \int_0^\infty e^{-zt} f(t)dt = \mathcal{L}[f(t)]$, then Gross [21] asks when the inverse is of the form

$$f(z) = \int_0^\infty e^{-zt} g(t)dt = \mathcal{L}[g(t)]$$

for real f and g. Formal substitution of the latter into the former and reversing the integrals yields a Stieltjes transform [56, Chapter VIII],[46, 9.15, p. 269]

$$\varphi(z) = \int_0^\infty \left(\int_0^\infty e^{-(z+s)t} dt \right) g(s)ds = \int_0^\infty \frac{g(s)}{z+s} ds$$

which is inverted as $g(s) = \frac{1}{\pi} \operatorname{Im}\left(\varphi(se^{-i\pi})\right) = -\frac{1}{\pi} \operatorname{Im}\left(\varphi(se^{i\pi})\right)$ for real s. Hence, if the Laplace transform and its inverse are of the same form, then it holds that

$$\varphi(z) = \mathcal{L}[f(t)] \iff f(z) = \mathcal{L}\left[\tfrac{1}{\pi} \operatorname{Im}\left(\varphi(te^{-i\pi})\right)\right] \tag{196}$$

and Gross [21] briefly states conditions on the validity of (196), which essentially relate to Theorem 1. Berberan-Santos' set (195) of equations is thus a further development of Gross's Laplace pair (196).

D Mittag-Leffler function and fractional calculus

The k-th order derivative of a complex function can be deduced from Cauchy's integral as

$$\left.\frac{d^k f(z)}{dz^k}\right|_{z=z_0} = \frac{\Gamma(1+k)}{2\pi i} \int_{C(z_0)} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{k+1}} \tag{197}$$

where $C(z_0)$ is a contour around the point z_0 in a region of the complex plane where the function $f(z)$ is analytic. The integer number k at the right-hand side in (197) can be *formally* extended to a complex number α, which then defines the left-hand side as a *complex fractional derivative*,

$$D^\alpha f(z_0) \equiv \left.\frac{d^\alpha f(z)}{dz^\alpha}\right|_{z=z_0} = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{C(z_0)} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{\alpha+1}} \tag{198}$$

[38] Indeed,

$$\begin{aligned} |f(\varepsilon) - f(-\varepsilon)| &\leq \frac{1}{2\pi} \int_{-\infty}^\infty |\varphi(c+iw)| \left| e^{\varepsilon(c+iw)} - e^{-\varepsilon(c+iw)} \right| dw \\ &= \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^T |\varphi(c+iw)| \left|\sinh\left(\varepsilon\left(c+iw\right)\right)\right| dw \end{aligned}$$

Since $|\sinh(\varepsilon(c+iw))| = |\sinh \varepsilon c \cos \varepsilon w + \sin \varepsilon w \cosh \varepsilon c| < |\sinh \varepsilon c| + |\sin \varepsilon w| + O\left(\varepsilon^2\right)$ and choosing $\varepsilon = T^{-2-\delta}$,

$$\begin{aligned} \int_{-T}^T |\varphi(c+iw)| \left|\sinh\left(\varepsilon\left(c+iw\right)\right)\right| dw &\leq cT^{-2-\delta} \int_{-T}^T |\varphi(c+iw)|\, dw + T^{-2-\delta} \int_{-T}^T |w\varphi(c+iw)|\, dw \\ &\leq 2cT^{-1-\delta} \max|\varphi(c+iw)| + T^{-\delta} \max|\varphi(c+iw)| \leq AT^{-\delta} \end{aligned}$$

which can be made arbitrarily small for large T so that $|f(\varepsilon) - f(-\varepsilon)| \to 0$ when $\varepsilon \to 0$.

[39] Professor Apelblat has informed me about this note.

By choosing or deforming the contour $C(z_0)$ in an appropriate manner (which is one of the core ingenuities in complex integration), we can transform (198) to

$$D_m^\alpha f(t) = \left.\frac{d^\alpha f(z)}{dz^\alpha}\right|_{z=t} = \frac{1}{\Gamma(m-\alpha)} \int_p^t \frac{f^{(m)}(x)}{(t-x)^{\alpha+1-m}} dx \quad \text{for } \operatorname{Re}(\alpha) < m \tag{199}$$

which is known as the Caputo fractional derivative. As explained in [52], the integral in (199) is a convolution of the function $g(x) = f^{(m)}(x)$, the m-th derivative of $f(x)$ and the function $x^{-\alpha-1+m}$, which is a power law.

After the extension of "classical" derivative (197) to a "fractional" derivative (199), we replace or extend the derivative in the linear matrix differential equation

$$\frac{ds(t)}{dt} = Qs(t) \tag{200}$$

where Q is an $N \times N$ matrix and $s(t)$ is an $N \times 1$ vector, to a Caputo fractional derivative (199)

$$D_m^\alpha s_\alpha(t) = -Qs_\alpha(t) \tag{201}$$

The classical solution of (200) for *any* $N \times N$ matrix Q is

$$s(t) = e^{-Qt} s(0)$$

while, for $0 < \alpha < 1$ and[40] $m = 0$, the "fractional α process" is described by

$$s_\alpha(t) = E_\alpha(-Qt^\alpha) s_\alpha(0) \tag{202}$$

where the matrix $E_\alpha(-Qt^\alpha) = \sum_{k=0}^\infty \frac{Q^k(-t^a)^k}{\Gamma(1+ak)}$ is also defined for any matrix Q. This exact analytic result (202) generalizes a large number of physical processes that are described by a linear differential equation of order N.

[40]The more general form for $m > 0$ is also analytically known (see [52, Appendix C])