The Arithmetic-Geometric Mean: A Pearl of Gauss

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5 June 2023

Abstract

The arithmetic-geometric mean (AGM) algorithm has an amazingly fast convergence. We try to follow Gauss on his remarkable path of mathematical discoveries, posthumously published in Gauss's Nachlass in 1866.

1 Introduction

Carl Friedrich Gauss (1777-1855) was a titan of science [9]. Words fall short to describe his phenomenal mathematical creations: just as paintings and musical pieces by the greatest artists[1], his mathematics fills an impressive gallery of the finest art. The only difference between music and paintings compared to mathematical art is that the latter requires more effort to understand, before its penetrating light embraces human emotions. After all, just as in music and paintings, it requires much technical skills, before creations and art occur. Art is all about emotion. The most beautiful mathematical art shines by its simplicity, which often shields its depth. It may sounds odd that I speak about mathematical art, while most people associate mathematics with a cool and logical system, void of any human emotion. And, yet, there is an ocean of beauty in which the logical pieces are built towards a magnificent castle.

The sequel here is devoted to one type of painting, one style of Gaussian symphony. I have tried to unravel his unpublished work, posthumously collected in his Nachlass (Gauss Werke, band 3), about the arithmetic-geometric mean (AGM). While the AGM algorithm, explained in (2) in Section 2, is rather basic and before Gauss discovered by another genius Langrange, it was Gauss, who created an astonishing piece of art. The first part in the Nachlass [14] in Latin is the easiest, because it is sufficiently well explained. That first part also shows Gauss's trajectory towards his first fundamental result (28) via elegant Taylor series expansions. I have expanded that part in Section 3 and rederived Gauss's series based on our current theory of Taylor series. The second part [15] in German is challenging and difficult, because Gauss has merely left sketches or just a list of formulae without any clue nor derivation. Of course, we cannot blame Gauss: he never found the time to publish his work

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[1]Ludwig Van Beethoven (1770-1827), with Flemish roots from Mechelen, was a German contemporary of Carl Friedrich Gauss.

on AGM in his near to perfect style, based on his adagium[2] "pauca, sed matura". After Gauss has seen the work of Abel and Jacobi, who found independently parts of his own discoveries about thirty years later, Gauss seemed to have been content that Abel and Jacobi had relieved him from publishing his work on elliptic functions. King [18] has provided a little more steps, but still insufficient. I found King often insulting to a reader by the arrogant "it is easy to see", which has cost me, unfortunately, much time. King proposed many exercises with only a little hint. I found him mimicking the great master, perhaps, frustrated by his time spent to unearth Gauss's second part? King could have written an attractive book on AGM, but his notes of merely 42 pages require from a reader so much that I think that most people after him have just neglected his work. Cox [8, Section 2], on the other hand, was a relief and pleasure to read. Thus, I have not copied Cox deductions in Section 10, especially his Theorem 1, where Cox has rederived Gauss's incredible achievement with the current knowledge of the theory of elliptic functions. Let me stress this point: only with a current machinery that vastly overreached the knowledge at Gauss's time, Cox was able to fill in the many details, which Gauss in [15] did not supply. Initially, I have also benefited from Borwein & Borwein [5] and Almkvist and Berndt [2].

I end with the quote of Edwards in [10], "read the masters", an advice that is still most valuable. Sometimes, even after understanding the logic of Gauss's derivations after laborious hours of trying, it took me more time to really understand what he was telling, after rereading his work. In order to safe a future reader both time and frustration – nothing is worse than struggling with a puzzle and not finding the solution, because it confronts you with own stupidity –, I have provided here ample, perhaps too many, intermediate steps. I hope that the story may read fluently, while opposing the master's "pauca" in his adagium.

2 The arithmetic-geometric mean (AGM)

1. *Definition of the arithmetic-geometric mean* $M(a,b)$. The arithmetic mean of two numbers a and b is defined by $m_A = \frac{a+b}{2}$ and their geometric mean by $m_G = \sqrt{ab}$. We first assume that a and b are non-negative real numbers, to avoid complications with the squareroot in $m_G = \sqrt{ab}$. An immediate bound $m_A \geq m_G$, with equality only if $a = b$, follows from

$$0 \leq \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 = \frac{a+b}{2} - \sqrt{ab} \tag{1}$$

After taking the logarithm of the geometric mean, we observe that $\log m_G = \frac{\log a + \log b}{2}$ is an arithmetic mean.

Gauss [14] studies the sequence $\{(a_n, b_n)\}_{n\geq 0}$, where[3]

$$a_n = \tfrac{a_{n-1}+b_{n-1}}{2} \quad \text{and} \quad b_n = \sqrt{a_{n-1}b_{n-1}} \tag{2}$$

[2] "Pauca, sed matura" is Latin and means "Few, but ripe". Gauss wrote clearly and briefly in a Ciceroan Latin style. He avoided unnecessary proza, in contrast to the then ruling French scientists of the Academy Francaise in Paris. Each sentence in his work plays a role; it is hard to further condense or skip parts without missing the idea. When contempories complained to him that he shielded the way in which he has found his discoveries, Gauss briefly replied: "Have you ever seen a beautiful building, to which the scaffold is still attached?"

"Pauca" does not imply that Gauss wrote only few articles. In fact, he was very productive, but he could have published more if his high writing standard was reduced. His work on AGM is an example; he did not publish this pearl.

[3] Gauss writes accents instead of subscripts in n; thus $a' = a_1, a'' = a_2, a''' = a_3$, etc.

starting from $(a_0, b_0) = (a, b)$. Explicitly, the arithmetic-geometric mean (AGM) algorithm (2), written in two columns, is

$$\begin{array}{ll} a_0 = a & b_0 = b \\ a_1 = \frac{a+b}{2} & b_1 = \sqrt{ab} \\ a_2 = \frac{1}{2}\left(\frac{a+b}{2} + \sqrt{ab}\right) & b_2 = \sqrt{\frac{a+b}{2}\sqrt{ab}} \\ a_3 = \cdots & b_3 = \cdots \end{array}$$

Invoking the inequality $m_A \geq m_G$ to (2) illustrates that $a_n \geq b_n$ for any integer $n \geq 1$. In other words, for $n \geq 1$, the left column with $\{a_n\}_{n\geq 1}$ will contain numbers that are always larger than the right column with $\{b_n\}_{n\geq 1}$, if we exclude the uninteresting case that $a = b$, for which $a_n = b_n = a$ and nothing changes with n. If $n = 0$, we obviously have that $a_0 < b_0$ if $a < b$, but for $n \geq 1$, it holds that $a_n \geq b_n$. In the sequel, therefore, we assume that $a > b$, so that the inequality $a_n > b_n$ holds for any integer $n \geq 0$. Using (1), Tannery and Molk [30, p. 269] mention $a_n - b_n = \frac{1}{2}\left(\sqrt{a_{n-1}} - \sqrt{b_{n-1}}\right)^2$, implying that $a_n > b_n$ for any integer $n \geq 1$, as found above. Combining $a_n > b_n$ with the arithmetic mean $a_n = \frac{a_{n-1}+b_{n-1}}{2} < a_{n-1}$ then shows, for any integer $n \geq 1$, that $a_n < a_{n-1}$, while the geometric mean $b_n = \sqrt{a_{n-1}b_{n-1}} > b_{n-1}$ shows that $b_n > b_{n-1}$.

2. *Convergence of the AGM algorithm in (2).* Gauss observes, for $a \geq b \geq 0$, that

$$\begin{aligned} \frac{a_n - b_n}{a_{n-1} - b_{n-1}} &= \frac{(a_n - b_n)(a_n + b_n)}{(a_{n-1} - b_{n-1})(a_n + b_n)} = \frac{a_n^2 - b_n^2}{(a_{n-1} - b_{n-1})(a_n + b_n)} \\ &= \frac{\left(\frac{a_{n-1}+b_{n-1}}{2}\right)^2 - a_{n-1}b_{n-1}}{(a_{n-1} - b_{n-1})(a_n + b_n)} = \frac{(a_{n-1} - b_{n-1})^2}{4(a_{n-1} - b_{n-1})(a_n + b_n)} \\ &= \frac{a_{n-1} - b_{n-1}}{2(a_{n-1} + b_{n-1}) + 4b_n} \leq \frac{1}{2}\frac{a_{n-1} - b_{n-1}}{a_{n-1} + b_{n-1}} \leq \frac{1}{2} \end{aligned}$$

where equality only holds if $b = 0$. Only if $b = 0$, the AGM algorithm (2) reduces to $b_n = 0$ and $a_n = \frac{1}{2}a_{n-1}$ with solution $a_n = \frac{a}{2^n}$ for $n \geq 0$. Hence, for $a > b > 0$, Gauss obtains the inequality $a_n - b_n < \frac{1}{2}(a_{n-1} - b_{n-1})$, which after iteration on $n \geq 0$ shows[4] that

$$a_n - b_n < \frac{1}{2^n}(a - b) \tag{3}$$

With $a_n \pm b_n = \frac{1}{2}\left(\sqrt{a_{n-1}} \pm \sqrt{b_{n-1}}\right)^2$, we have

$$\frac{a_n - b_n}{a_n + b_n} = \left(\frac{\sqrt{a_{n-1}} - \sqrt{b_{n-1}}}{\sqrt{a_{n-1}} + \sqrt{b_{n-1}}}\right)^2 = \left(\frac{a_{n-1} - b_{n-1}}{\left(\sqrt{a_{n-1}} + \sqrt{b_{n-1}}\right)^2}\right)^2 < \left(\frac{a_{n-1} - b_{n-1}}{a_{n-1} + b_{n-1}}\right)^2$$

because $\left(\sqrt{a_{n-1}} + \sqrt{b_{n-1}}\right)^4 = \left(a_{n-1} + b_{n-1} + 2\sqrt{a_{n-1}b_{n-1}}\right)^2 > (a_{n-1} + b_{n-1})^2$ for $a > b > 0$. Iterated p times,

$$\frac{a_n - b_n}{a_n + b_n} < \left(\frac{a_{n-1} - b_{n-1}}{a_{n-1} + b_{n-1}}\right)^2 < \cdots < \left(\frac{a_{n-p} - b_{n-p}}{a_{n-p} + b_{n-p}}\right)^{2^p}$$

[4]Indeed, denote the difference $v_n = a_n - b_n$, then the recursion is $v_n < \frac{1}{2}v_{n-1}$. Applying the recursion p-times yields $v_n < \frac{1}{2}v_{n-1} < \frac{1}{4}v_{n-2} < \frac{1}{8}v_{n-3} < \cdots$ or $v_n < \frac{1}{2^p}v_{n-p}$. If $n - p = 0$, then $v_0 = a - b$ and we arrive, after choosing $p = n$ to (3).

leads, after choosing $p = n$,

$$\frac{a_n - b_n}{a_n + b_n} < \left(\frac{a-b}{a+b}\right)^{2^n} \tag{4}$$

Since $a_{n+1} = 2(a_n + b_n)$ and $a_{n+1} < a_n$, we find that $a_n - b_n < 2a_1 \left(\frac{a-b}{a+b}\right)^{2^n}$, which tends to zero considerably faster than $\frac{a-b}{2^n}$ in (3) for $n > n_0$, where n_0 is a threshold value.

In summary, the difference $a_n - b_n$ in the sequence $\{(a_n, b_n)\}_{n\geq 0}$ tends to zero with $n \to \infty$. In other words, the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ converge to the same limit $M(a,b)$, which Gauss calls the *arithmetic-geometric mean (AGM)*,

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{5}$$

For $a > b > 0$, the above analysis shows that

$$a > a_1 > \cdots > a_n > a_{n+1} \geq \cdots \geq M(a,b) \geq \cdots \geq b_{n+1} > b_n > \cdots > b_1 > b$$

while

$$M(a,a) = a \qquad M(a,0) = 0$$

Since any pair (a_n, b_n) in the sequence $\{(a_n, b_n)\}_{n\geq 0}$ converges to the same limit, we also conclude that[5]

$$M(a,b) = M(a_1, b_1) = \cdots = M(a_n, b_n) = \cdots \tag{6}$$

3. *Specifying the convergence rate of the AGM algorithm.* Gauss [14, p. 375], followed by Borwein & Borwein [5] and Almkvist and Berndt [2], defines, for $n \geq 0$, a more quantitative measure of convergence

$$c_n = \sqrt{a_n^2 - b_n^2} \tag{7}$$

that obeys, with the AGM recursion (2),

$$\begin{aligned} c_n &= \sqrt{\left(\frac{a_{n-1} + b_{n-1}}{2}\right)^2 - (a_{n-1} b_{n-1})} = \sqrt{\left(\frac{a_{n-1} - b_{n-1}}{2}\right)^2} \\ &= \frac{a_{n-1} - b_{n-1}}{2} = \frac{a_{n-1}^2 - b_{n-1}^2}{2(a_{n-1} + b_{n-1})} \end{aligned}$$

and

$$c_n = \frac{c_{n-1}^2}{4a_n} \tag{8}$$

After p iterations of (8), we find that $c_n = \frac{(c_{n-p})^{2^p}}{\prod_{j=0}^{p-1}(4a_{n-j})^{2^j}}$ and choosing $p = n$ yields

$$\begin{aligned} c_n &= \frac{(c_0)^{2^n}}{\prod_{j=0}^{n-1} 2^{2^{j+1}} (a_{n-j})^{2^j}} = \frac{\left(a^2 - b^2\right)^{2^{n-1}}}{\prod_{j=0}^{n-1} 2^{2^{j+1}} \prod_{j=0}^{n-1} (a_{n-j})^{2^j}} = \frac{(a-b)^{2^{n-1}} (a+b)^{2^{n-1}}}{2^{2\sum_{j=0}^{n-1} 2^j} \prod_{j=0}^{n-1} (a_{n-j})^{2^j}} \\ &= 4\frac{(a-b)^{2^{n-1}} (a+b)^{2^{n-1}}}{2^{2^{n+1}} \prod_{j=0}^{n-1} (a_{n-j})^{2^j}} = 4\frac{\left(\frac{a-b}{2}\right)^{2^{n-1}} \left(\frac{a+b}{2}\right)^{2^{n-1}}}{2^{2^n} \prod_{j=0}^{n-1} (a_{n-j})^{2^j}} = 4\frac{\left(\frac{a-b}{2}\right)^{2^{n-1}} (a_1)^{2^{n-1}}}{2^{2^n} (a_1)^{2^{n-1}} \prod_{l=2}^{n} (a_l)^{2^{n-l}}} \end{aligned}$$

[5] The *harmonic-geometric mean* of a and b is $\frac{1}{M\left(\frac{1}{a}, \frac{1}{b}\right)} = \frac{ab}{M(a,b)}$, where the equality follows from the scaling property (11) below.

and

$$c_n = 4\frac{\left(\frac{a-b}{8}\right)^{2^{n-1}}}{\prod_{l=2}^{n}\left(a_l\right)^{2^{n-l}}}$$

Since $a > a_1 > \cdots \geq M(a,b)$, we obtain, with $a_2^{2^{n-1}-1} > \prod_{l=2}^{n}\left(a_l\right)^{2^{n-l}} > \left(M(a,b)\right)^{2^{n-1}-1}$, that

$$4a_2\left(\frac{a-b}{8a_2}\right)^{2^{n-1}} < c_n < 4M(a,b)\left(\frac{a-b}{8M(a,b)}\right)^{2^{n-1}}$$

where the upper bound is tighter than the lower bound. Both upper and lower bound converge to zero extremely rapidly. A recursion inequality follows from (8) with $a_n \geq M(a,b)$ as

$$c_n \leq \frac{c_{n-1}^2}{4M(a,b)} \tag{9}$$

4. *Convergence of order m.* If the sequence $\{\alpha_n\}_{n\geq 0} = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ converges to α, i.e. $\lim_{n\to\infty}\alpha_n = \alpha$, and assume that there exist constants $d > 0$ and $m \geq 1$ such that

$$|\alpha_n - \alpha| \leq d\,|\alpha_{n-1} - \alpha|^m \text{ for } n \geq 2$$

then the convergence of the sequence $\{\alpha_n\}_{n\geq 0}$ is of m-th order. Thus, c_n in (8) tends to zero quadratically and the convergence of $\{c_n\}_{n\geq 0}$ is of second order. The difference $a_n - b_n = 2c_{n+1} < 8M(a,b)\left(\frac{a-b}{8M(a,b)}\right)^{2^n}$ is a measure of the speed of convergence and of the number of common digits as a function of n. A convergence of second order means that each iteration in the AGM algorithm (2) for positive a and b approximately doubles the number of correct decimal digits. Indeed, since $b_n < M(a,b) < a_m$ for any pair of finite integers n and m and $a > b > 0$, the inequality (4) with $\frac{a-b}{a+b} = 10^{-r} < 1$ and $r > 0$ (thus excluding that $b = 0$), shows that

$$\frac{a_n}{a_{n+1}} < \frac{b_n}{a_{n+1}} + \frac{1}{2}\left(10^{-r}\right)^{2^n} < 1 + 10^{-r.2^n} \qquad \text{where } r = -\log_{10}\left(\frac{a-b}{a+b}\right) \tag{10}$$

which implies that the ratio $\frac{a_n}{a_{n+1}}$ starts tending to one quadratically for $n > n_0$, where $r2^{n_0} \geq 1$; thus for $n > -\frac{\log r}{\log 2} = -\log_2 r$. Hence, quadratic convergence starts immediately when $r \geq 2$. Only if $r = 0$, thus only if $b = 0$, then the AGM algorithm converges linearly to zero.

If $t_n(z) = \sum_{k=0}^{n} f_k(z_0)(z-z_0)^k$ is the n-order Taylor polynomial, then

$$|t_n(z) - \alpha| = |t_{n-1}(z) + f_n(z_0)(z-z_0)^n - \alpha| \leq |t_{n-1}(z) - \alpha| + |f_n(z_0)|\,|(z-z_0)^n|$$

shows that the convergence of Taylor series is, at best if $|f_n(z_0)|\,|(z-z_0)^n| < \varepsilon$, of first order or the convergence is linear.

5. *Scaling of $M(a,b)$.* If we multiply both a_n and b_n by a positive real number β, then the AGM recursion (2) shows that also a_{n+1} and b_{n+1} are multiplied by β. Hence, $\lim_{n\to\infty}\beta a_n = \beta M(a,b)$ and

$$M(a\beta, b\beta) = \beta M(a,b) \tag{11}$$

Taking $\beta = \frac{1}{a}$ and subsequently $\beta = \frac{1}{b}$ in (11) yields $M(a,b) = aM\left(1, \frac{b}{a}\right) = bM\left(\frac{a}{b}, 1\right)$. The scaling (11) means that the study of $M(a,b)$ can be reduced to $M(1,x)$, where $0 \leq x \leq 1$, because we assume

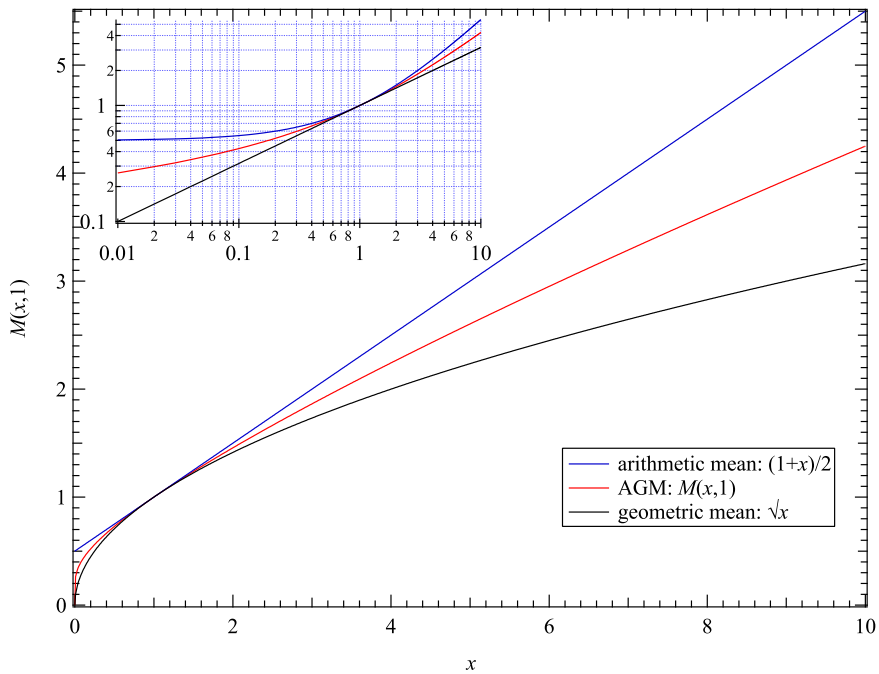

Figure 1: The three means of x and 1: the arithmetic mean $\frac{1+x}{2}$, Gauss's AGM $M(x,1)$ and the geometric mean $\sqrt{x}$, both on lin-lin and log-log scale (inset).

as Gauss that $a > b$. However, interchanging a and b in the iterative algorithm (2) does not impact the limit, i.e. $M(a,b) = M(b,a)$. Thus, alternatively $M(a,b)$ can be reduced to $M(x,1)$, where $x \geq 1$. Fig. 1 draws $M(x,1)$ together with its upper bound $\frac{1+x}{2}$ and lower bound $\sqrt{x}$ on a lin-lin and log-log scale.

6. *Backward AGM algorithm.* Gauss [14, art. 2; p. 362-363] also inverts the AGM recursion (2) as

$$\begin{cases} 2a_n = a_{n-1} + b_{n-1} \\ b_n^2 = a_{n-1} b_{n-1} \end{cases}$$

Substituting $b_{n-1} = \frac{b_n^2}{a_{n-1}}$ into $2a_n = a_{n-1} + b_{n-1}$ yields $2a_n = a_{n-1} + \frac{b_n^2}{a_{n-1}}$. After multiplying both sides by a_{n-1}, we obtain a quadratic equation in $x = a_{n-1}$,

$$x^2 - 2a_n x + b_n^2 = 0$$

with solution

$$x = a_n \pm \sqrt{a_n^2 - b_n^2}$$

The quadratic equation illustrates that the sum of the roots is $2a_n$ and their product is b_n^2 so that, since $a_n > b_n$, the roots are

$$a_{n-1} = a_n + \sqrt{a_n^2 - b_n^2} \quad \text{and} \quad b_{n-1} = a_n - \sqrt{a_n^2 - b_n^2}$$

and, written in terms of $c_n = \sqrt{a_n^2 - b_n^2}$,

$$a_{n-1} = a_n + c_n \quad \text{and} \quad b_{n-1} = a_n - c_n \tag{12}$$

This recursion (12) can be formulated in the opposite direction with $m = -n$, where $m \geq 0$,

$$a_{-m-1} = a_{-m} + \sqrt{a_{-m}^2 - b_{-m}^2} \quad \text{and} \quad b_{-m-1} = a_{-m} - \sqrt{a_{-m}^2 - b_{-m}^2} \tag{13}$$

and provides the opposite direction or backward iteration of the AGM algorithm (2).

Since $\sqrt{a_{-m}^2 - b_{-m}^2} > 0$, it holds that $a_{-m-1} > a_{-m} > 0$ for all $m \geq 0$, which implies that $\lim_{m\to\infty} a_{-m} = \infty$, while $b_{n-1} = \frac{b_n^2}{a_{n-1}}$ indicates that $\frac{b_{-m-1}}{b_{-m}} = \frac{b_{-m}}{a_{-m-1}} < \frac{b_{-m}}{a_{-m}}$ and $\frac{b_{-m-1}}{b_{-m}}$ is decreasing for all $m \geq 0$, so that $\lim_{m\to\infty} b_{-m} = 0$.

With $c_n = \frac{a_{n-1}-b_{n-1}}{2}$ and also $c_n = \frac{a_{n-1}^2-b_{n-1}^2}{4a_n}$, it follows from $M(a_n, c_n) = M\left(\frac{a_n+c_n}{2}, \sqrt{a_n c_n}\right)$ that

$$\begin{aligned} M(a_n, c_n) &= M\left(\frac{1}{2}\left(a_n + \frac{a_{n-1}-b_{n-1}}{2}\right), \sqrt{\frac{a_{n-1}^2 - b_{n-1}^2}{4a_n} a_n}\right) \\ &= M\left(\frac{1}{2}a_{n-1}, \frac{1}{2}\sqrt{a_{n-1}^2 - b_{n-1}^2}\right) = \frac{1}{2}M(a_{n-1}, c_{n-1}) \end{aligned}$$

Iteration in n shows that

$$M(a, c) = 2^n M(a_n, c_n) \tag{14}$$

and also $M(a, c) = 2^{-m} M(a_{-m}, c_{-m})$. The definition (5) of $M(a, b)$ then indicates that

$$M(a, c) = \lim_{m\to\infty} \frac{a_{-m}}{2^m} = \lim_{m\to\infty} \frac{c_{-m}}{2^m}$$

illustrating, because $M(a, c)$ is finite, that the sequence $a_0, a_{-1}, a_{-2}, \ldots$ and $c_0, c_{-1}, c_{-2}, \ldots$ grow as a geometric series in 2^m, which leads, with the definition $c_{-m} = \sqrt{a_{-m}^2 - b_{-m}^2}$, again to $\lim_{m\to\infty} b_{-m} = 0$. If $a = b\sqrt{2}$, then $c = \sqrt{a^2 - b^2} = b$, so that $M(a, b) = M(a, c)$.

The backward iteration can be avoided by altering the starting values in the AGM algorithm (2). Indeed, let a'_n and b'_n satisfy the AGM recursion (2) with initial values $a'_0 = a_0$ and $b'_0 = c_0 = \frac{1}{2}(a_{-1} - b_{-1})$, then $c'_0 = \sqrt{(a'_0)^2 - (b'_0)^2} = b_0$. For $n = 1$, we find

$$\begin{aligned} a'_1 &= \frac{1}{2}\left(a_0 + \frac{1}{2}(a_{-1} - b_{-1})\right) = \frac{1}{2}a_{-1} \\ b'_1 &= \sqrt{\frac{a_{-1}^2 - b_{-1}^2}{4a_n} a_n} = \frac{1}{2}c_{-1} \\ c'_1 &= \sqrt{(a'_1)^2 - (b'_1)^2} = \frac{1}{2}\sqrt{a_{-1}^2 - c_{-1}^2} = \frac{1}{2}b_{-1} \end{aligned}$$

and an induction[6] shows, for any integer n, that

$$a'_n = 2^{-n} a_{-n} \qquad b'_n = 2^{-n} c_{-n} \qquad c'_n = 2^{-n} b_{-n} \tag{15}$$

[6]Let us assume that (15) holds for n. We have already shown that (15) is satisfied for $n = 1$. Using (12) and $c_{-m-1} = 2\sqrt{a_{-m}c_{-m}}$ in (8), the case for $n + 1$,

$$\begin{aligned} a'_{n+1} &= \frac{1}{2}\left(a'_n + b'_n\right) = 2^{-n-1}(a_{-n} + c_{-n}) = 2^{-n-1}a_{-n-1} \\ b'_{n+1} &= \sqrt{a'_n b'_n} = 2^{-n}\sqrt{a_{-n}c_{-n}} = 2^{-n-1}c_{-n-1} \\ c'_{n+1} &= \sqrt{\left(a'_{n+1}\right)^2 - \left(b'_{n+1}\right)^2} = 2^{-n-1}\sqrt{a_{-n-1}^2 - c_{-n-1}^2} = 2^{-n-1}b_{-n-1} \end{aligned}$$

is also demonstrated. By the induction principle, (15) holds for any $n \geq 1$.

We conclude that the forward AGM algorithm with the set $\{a_n, b_n, c_n\}$ and with the complementary set $\{a'_n, b'_n, c'_n\}$, connected via (15), combine both forward and backward algorithm and no other information is gained by searching for another set.

Gauss [14, art. 3] computes 4 examples with different (a, b) up to 20 decimal digits (!) to illustrate divergence in (13) on the one hand and how fast the recursion (2) converges on the other hand. Numerical computations confirm a convergence of second order and that the iteration n in (2) has about twice the number of correct digits than the iteration $n-1$, which is numerically an amazingly fast convergence. This very fast convergence of the sequence in (2) likely attracted Gauss to explore the properties of the arithmetic-geometric mean $M(a, b)$.

3 Power series for $M(a, b)$

7. *Power series expansions.* Gauss [14, art. 5] computes the Taylor expansion of

$$M(1+x, 1) = \sum_{k=0}^{\infty} h_k x^k \tag{16}$$

rather than $M(1, x)$, because a Taylor only converges in a circle around a point in the complex plane where the function is analytic. The geometric mean $m_G = \sqrt{x}$, corresponding to $M(1, x) = M\left(\frac{1+x}{2}, \sqrt{x}\right)$ by $M(a, b) = M(a_1, b_1)$, has a branch cut along the negative real axis. Using $M(a, a) = a$ for $a = 1$, shows that $h_0 = M(1, 1) = 1$. Since $M(1+x, 1) = M\left(1 + \frac{x}{2}, \sqrt{1+x}\right)$, Gauss proposes to take $x = 2t + t^2$, resulting in

$$M(1+x, 1) = M\left(1 + t + \frac{t^2}{2}, 1 + t\right) = (1+t)\, M\left(1 + \frac{t^2}{2(1+t)}, 1\right)$$

where the last step follows from scaling in (11). Applying the Taylor series (16) to the left- and right-hand side yields

$$\sum_{k=0}^{\infty} h_k \left(2t + t^2\right)^k = (1+t) \sum_{k=0}^{\infty} h_k \left(\frac{t^2}{2(1+t)}\right)^k \tag{17}$$

As usual, Gauss almost directly gives the sequence of h_k in [14, art. 5], but we proceed to simplify the resulting series in (17) and write them as powers in t. First, invoking Newton's binomium

$$\begin{aligned}\sum_{k=0}^{\infty} h_k \left(2t + t^2\right)^k &= \sum_{k=0}^{\infty} h_k t^k (2+t)^k = \sum_{k=0}^{\infty} \sum_{m=0}^{k} h_k \binom{k}{m} 2^{k-m} t^{m+k} \\ &= \sum_{k=0}^{\infty} \sum_{m=k}^{2k} h_k \binom{k}{m-k} 2^{2k-m} t^m\end{aligned}$$

and reversing the k- and m-sum, yields

$$\sum_{k=0}^{\infty} h_k \left(2t + t^2\right)^k = \sum_{m=0}^{\infty} \left\{ \sum_{k=\left[\frac{m}{2}\right]}^{m} h_k \binom{k}{m-k} 2^{2k-m} \right\} t^m$$

Next, we compute the right-hand side in (17)

$$(1+t) \sum_{k=0}^{\infty} h_k \left(\frac{t^2}{2(1+t)}\right)^k = \sum_{k=0}^{\infty} \frac{h_k}{2^k} t^{2k} \frac{1}{(1+t)^{k-1}}$$

and apply the Taylor series $\frac{1}{(1-z)^{k+1}} = \sum_{m=k}^{\infty} \binom{m}{k} z^{m-k}$ for $|z| < 1$ (see e.g. [1, 24.1.1.B], [23, 26.3.4])

$$\frac{1}{(1+t)^{k-2+1}} = \sum_{m=k-2}^{\infty} \binom{m}{k-2} (-1)^{m-k} t^{m-k+2} = \sum_{m=k}^{\infty} \binom{m-2}{k-2} (-1)^{m-k} t^{m-k}$$

to obtain

$$\begin{aligned}(1+t) \sum_{k=0}^{\infty} h_k \left(\frac{t^2}{2(1+t)} \right)^k &= \sum_{k=0}^{\infty} \frac{h_k}{2^k} \sum_{m=k}^{\infty} \binom{m-2}{k-2} (-1)^{m-k} t^{m+k} \\ &= \sum_{k=0}^{\infty} \frac{h_k}{2^k} \sum_{m=2k}^{\infty} \binom{m-k-2}{k-2} (-1)^m t^m\end{aligned}$$

Reversing the k- and m-sum results in

$$(1+t) \sum_{k=0}^{\infty} h_k \left(\frac{t^2}{2(1+t)} \right)^k = \sum_{m=0}^{\infty} \left\{ (-1)^m \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{h_k}{2^k} \binom{m-k-2}{k-2} \right\} t^m$$

Equating corresponding powers in t^m at both sides of (17) gives, for $m \geq 0$,

$$\sum_{k=\left[\frac{m}{2}\right]}^{m} h_k \binom{k}{m-k} 2^{2k-m} = (-1)^m \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{h_k}{2^k} \binom{m-k-2}{k-2}$$

If $m = 1$, then $2h_1 = h_0 = 1$. It is better to rewrite the above as a recursion in h_m

$$h_m = \frac{1}{2^m} \left\{ (-1)^m \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{h_k}{2^k} \binom{m-k-2}{k-2} - \sum_{k=\left[\frac{m}{2}\right]}^{m-1} h_k \binom{k}{m-k} 2^{2k-m} \right\} \tag{18}$$

After iterating $m > 0$, we find $h_1 = \frac{1}{2}$, $h_2 = -\frac{1}{16}$, $h_3 = \frac{1}{32}$, $h_4 = \frac{21}{1024}$, $h_5 = \frac{31}{2048}$, $h_6 = -\frac{195}{16384}$, $h_7 = \frac{319}{32768}$, $h_8 = \frac{-34325}{4194304}$, $h_9 = \frac{58899}{8388608}$ and $h_{10} = \frac{-410771}{67108864}$. Gauss [14, art. 5] has computed the coefficients up to h_6, of course correctly, and he concludes that there is no obvious simple law that produces these coefficients.

Subsequently, Gauss [14, art. 6] considers $M(1+x, 1-x) = (1-x) M\left(1 + \frac{2x}{1-x}, 1\right)$. Invoking the Taylor (16) of $M(1+x, 1)$ shows that

$$M\left(1 + \frac{2x}{1-x}, 1\right) = \sum_{k=0}^{\infty} h_k \left(\frac{2x}{1-x} \right)^k$$

With the Taylor series $\frac{1}{(1-z)^k} = \sum_{m=k-1}^{\infty} \binom{m}{k-1} z^{m-k+1}$ for $|z| < 1$, we have

$$M\left(1 + \frac{2x}{1-x}, 1\right) = \sum_{k=0}^{\infty} 2^k h_k \sum_{m=k-1}^{\infty} \binom{m}{k-1} x^{m+1} = \sum_{k=0}^{\infty} 2^k h_k \left(x^k + \sum_{m=k}^{\infty} \binom{m}{k-1} x^{m+1} \right)$$

After reversing the k- and m-sum, we find

$$\begin{aligned}M\left(1 + \frac{2x}{1-x}, 1\right) &= \sum_{m=0}^{\infty} 2^m h_m x^m + \sum_{m=0}^{\infty} \left\{ \sum_{k=0}^{m} 2^k h_k \binom{m}{k-1} \right\} x^{m+1} \\ &= 1 + \sum_{m=1}^{\infty} \left\{ 2^m h_m + \sum_{k=0}^{m-1} 2^k h_k \binom{m-1}{k-1} \right\} x^m\end{aligned}$$

We arrive at

$$M\left(1+\frac{2x}{1-x},1\right)=1+\sum_{m=1}^{\infty}c_m x^m$$

where the Taylor coefficients are

$$c_m=\sum_{k=0}^{m}\binom{m-1}{k-1}2^k h_k$$

Using the recursion (18) of the coefficients h_k, we explicitly obtain the Taylor coefficients $c_m=\sum_{k=0}^{m}\binom{m-1}{k-1}2^k h_k$ of $M\left(1+\frac{2x}{1-x},1\right)$ as $c_0=c_1=1$, $c_2=c_3=\frac{3}{4}$, $c_4=c_5=\frac{43}{64}$, $c_6=c_7=\frac{161}{256}$, $c_8=c_9=\frac{9835}{16384}$, $c_{10}=c_{11}=\frac{37961}{65536}$, etc. We observe that $c_m=c_{2\left[\frac{m}{2}\right]}$.

Computing $M(1+x,1-x)=(1-x)M\left(1+\frac{2x}{1-x},1\right)=(1-x)\sum_{m=0}^{\infty}c_m x^m$ and observing that $M(1+x,1-x)$ is even in x, results in the Taylor series

$$M(1+x,1-x)=\sum_{m=0}^{\infty}g_m x^m \tag{19}$$

where the Taylor coefficients $g_m=c_m-c_{m-1}$ of $M(1+x,1-x)$ are $g_{2m+1}=0$ and $g_2=-\frac{1}{4}$, $g_4=-\frac{5}{64}$, $g_6=-\frac{11}{256}$, $g_8=-\frac{469}{16384}$, $g_{10}=-\frac{1379}{65536}$, etc. Gauss [14, art. 6] also remarks that the Taylor series of the even function $M(1+x,1-x)=\sum_{m=0}^{\infty}g_{2m}x^{2m}$ can be computed similarly as the Taylor series $M(1+x,1)=\sum_{k=0}^{\infty}h_k x^k$ in (16) by the substitution $x=\frac{2t}{1+t^2}$, that transforms $M(1+x,1-x)$ into $M\left(1+\frac{2t}{1+t^2},1-\frac{2t}{1+t^2}\right)$. Invoking the property $M(a,b)=M(a_1,b_2)$ leads to

$$M\left(1+\frac{2t}{1+t^2},1-\frac{2t}{1+t^2}\right)=M\left(\frac{1-t^2}{1+t^2},1\right)=\frac{1}{1+t^2}M\left(1+t^2,1-t^2\right) \tag{20}$$

where the last equality follows from scaling in (11). Consequently, the Taylor coefficients g_{2m} of $M(1+x,1-x)$ satisfy

$$\sum_{m=0}^{\infty}g_{2m}\left(\frac{2t}{1+t^2}\right)^{2m}=\frac{1}{1+t^2}\sum_{m=0}^{\infty}g_{2m}t^{4m}$$

We omit the explicit computation, because the method is similar to that of (17). Gauss again says that the Taylor coefficients $\{g_{2m}\}_{m\geq 0}$ do not exhibit a simple law.

8. However, in the next sentence on [14, p. 367] and just at the end of art. 6, Gauss gives the Taylor series of

$$\frac{1}{M(1+x,1-x)}=\sum_{m=0}^{\infty}f_{2m}x^{2m} \tag{21}$$

with $f_0=1$ and tells that the Taylor coefficients $\{f_{2m}\}_{m\geq 0}$ obey an interesting law!

Gauss [14, art. 7] inverts the previous relation (20)

$$\left.\frac{1}{M(1+x,1-x)}\right|_{x=\frac{2t}{1+t^2}}=\frac{1+t^2}{M\left(1+t^2,1-t^2\right)}$$

substitutes at both sides the Taylor series $\frac{1}{M(1+x,1-x)}=\sum_{m=0}^{\infty}f_{2m}x^{2m}$ and obtains

$$\begin{aligned}\sum_{m=0}^{\infty}f_{2m}\left(\frac{2t}{1+t^2}\right)^{2m}&=\left(1+t^2\right)\sum_{m=0}^{\infty}f_{2m}t^{4m}=\sum_{m=0}^{\infty}f_{2m}\left(t^2\right)^{2m}+\sum_{m=0}^{\infty}f_{2m}\left(t^2\right)^{2m+1}\\&=\sum_{k=0}^{\infty}f_{2\left[\frac{k}{2}\right]}t^{2k}\end{aligned}$$

The Taylor series $\frac{1}{(1-z)^k} = \sum_{m=k-1}^{\infty} \binom{m}{k-1} z^{m-k+1}$ for $|z| < 1$ indicates that

$$\begin{aligned}\sum_{m=0}^{\infty} f_{2m} \left(\frac{2t}{1+t^2}\right)^{2m} &= 1 + \sum_{m=1}^{\infty} f_{2m} 2^{2m} t^{2m} \sum_{j=2m-1}^{\infty} \binom{j}{2m-1} (-1)^{j+1} \left(t^2\right)^{j-2m+1} \\ &= 1 + \sum_{m=1}^{\infty} f_{2m} 2^{2m} \sum_{j=m}^{\infty} \binom{j+m-1}{2m-1} (-1)^{j+m} t^{2j}\end{aligned}$$

Reversing the summations yields

$$\sum_{m=0}^{\infty} f_{2m} \left(\frac{2t}{1+t^2}\right)^{2m} = 1 + \sum_{j=1}^{\infty} \left\{ \sum_{m=1}^{j} f_{2m} 2^{2m} \binom{j+m-1}{2m-1} (-1)^{j+m} \right\} t^{2j}$$

Equating corresponding powers in t^{2j} yields, for $j \geq 1$,

$$\begin{aligned} f_{2\left[\frac{j}{2}\right]} &= (-1)^j \sum_{m=1}^{j} f_{2m} 2^{2m} \binom{j+m-1}{2m-1} (-1)^m \\ &= f_{2j} 2^{2j} + (-1)^j \sum_{m=1}^{j-1} f_{2m} 2^{2m} \binom{j+m-1}{2m-1} (-1)^m \end{aligned}$$

from which the recursion in f_{2j}, starting at $f_0 = 1$, follows as

$$f_{2j} = \frac{1}{2^{2j}} \left(f_{2\left[\frac{j}{2}\right]} - (-1)^j \sum_{m=1}^{j-1} f_{2m} 2^{2m} \binom{j+m-1}{2m-1} (-1)^m \right)$$

Evaluating the first few coefficients yields $f_2 = \frac{1}{4}$, $f_4 = \frac{9}{64}$, $f_6 = \frac{25}{256}$, $f_8 = \frac{1225}{16384}$, $f_{10} = \frac{3969}{65536}$, etc. Gauss recognizes that the Taylor coefficients $\{f_{2m}\}_{m\geq 0}$ of $\frac{1}{M(1+x,1-x)}$ are the squares of $\sqrt{f_2} = \frac{1}{2}$, $\sqrt{f_4} = \frac{3}{8} = \frac{3}{2.4}$, $\sqrt{f_6} = \frac{5}{16} = \frac{1}{2}\frac{3}{4}\frac{5}{6}$, $\sqrt{f_8} = \frac{35}{128} = \frac{1.3.5.7}{2.4.6.8}$, $\sqrt{f_{10}} = \frac{63}{256} = \frac{1.3.5.7.9}{2.4.6.8.10}$, etc. In general, Gauss finds that

$$f_{2m} = \left(\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots(2m)}\right)^2 = \left(\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots(2m)} . \frac{2.4.6\ldots(2m)}{2.4.6\ldots(2m)}\right)^2 = \left(\frac{(2m)!}{\left(2^m m!\right)^2}\right)^2 \tag{22}$$

resulting in the beautiful Taylor series around $x = 0$ of

$$\frac{1}{M(1+x,1-x)} = \sum_{m=0}^{\infty} \left(\frac{(2m)!}{\left(2^m m!\right)^2}\right)^2 x^{2m} = 1 + \sum_{m=1}^{\infty} \left(\prod_{j=1}^{m} \frac{2j-1}{2j}\right)^2 x^{2m} \tag{23}$$

I think that the discovery of (23) with the Taylor coefficients (22) for f_{2m} must have been one of the many "Eureka[7]" moments in Gauss's life! After many trials or is it genial insights in the Taylor coefficients?, he eventually succeeded in his first great step. Gauss in [14, art. 7] spends efforts in rewriting the recursion above as a set of equations in $\{f_{2m}\}_{m\geq 0}$ and deduces, in his characteristic genial style, the set $0 = f_0 - 4f_2$; $0 = 9f_2 - 16f_4$; $0 = 25f_4 - 36f_6$; $0 = 49f_6 - 64f_8$; etc. of which the general form is $0 = (2m-1)^2 f_{2m-2} - (2m)^2 f_{2m}$, for $m \geq 2$. That general form again leads to the explicit Taylor series (22).

[7] "Eureka" ($\eta\upsilon\rho\eta\kappa\alpha$) means "I have found it" and is the Greek perfectum of $\varepsilon\upsilon\rho\iota\sigma\kappa\varepsilon\iota\nu$ (to find).

9. *Taylor coefficients f_m and g_m.* We found by numerical computations that both $-2^{4m-2}g_{2m}$ and $2^{4m-2}f_{2m}$ are integers for $m>0$. Since a binomial coefficient $\binom{m}{k}$ is an integer for integer m and k, it follows from $\sqrt{f_{2m}}=\frac{(2m)!}{2^{2m}m!m!}=\frac{\binom{2m}{m}}{2^{2m}}$ in (22) that

$$2^{4m}f_{2m}=\binom{2m}{m}^2$$

Introducing the binomial series $(1+z)^\alpha=\sum_{m=0}^\infty\binom{\alpha}{m}z^m$, convergent for $|z|<1$ and for any complex α, in the integral of $\arcsin z=\int_0^z\frac{du}{\sqrt{1-u^2}}$ leads to

$$\arcsin z=\sum_{k=0}^\infty\binom{-\frac{1}{2}}{k}\frac{(-1)^k}{2k+1}z^{2k+1}\qquad\text{for }|z|<1 \tag{24}$$

Using $\binom{-\frac{1}{2}}{k}=(-1)^k\binom{k-\frac{1}{2}}{k}=(-1)^k\frac{\Gamma(\frac{1}{2}+k)}{k!\,\Gamma(\frac{1}{2})}$ and the duplication formula of the Gamma function, $\Gamma(2z)=\frac{1}{2\sqrt{\pi}}2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})$, shows with $\Gamma(\frac{1}{2})=\sqrt{\pi}$ that

$$\binom{-\frac{1}{2}}{k}=\frac{(-1)^k\,\Gamma(2k)}{2^{2k-1}k!\Gamma(k)}=\frac{(-1)^k\,(2k-1)!}{2^{2k-1}k!\,(k-1)!}=\frac{(-1)^k\,(2k)!}{2^{2k}(k!)^2}=\frac{(-1)^k}{2^{2k}}\binom{2k}{k} \tag{25}$$

illustrating that $\binom{-\frac{1}{2}}{k}$ are rational numbers[8] with denominator at most 2^{2k-1}, because the integer $\binom{2k-1}{k}$ can be even and, hence, divisible by 2. In summary, this argument proves that $2^{4m-2}f_{2m}$ are integers. We have not spent time to prove the case for $-2^{4m-2}g_{2m}$.

With (25), we also have

$$\frac{1}{M\,(1+x,1-x)}=\sum_{m=0}^\infty\binom{-\frac{1}{2}}{m}^2x^{2m} \tag{26}$$

while $\binom{-\frac{1}{2}}{k}^2=\left(\frac{\Gamma(\frac{1}{2}+k)}{k!\,\Gamma(\frac{1}{2})}\right)^2$ leads to

$$\frac{1}{M\,(1+x,1-x)}=F\left(\frac{1}{2},\frac{1}{2},1;x^2\right) \tag{27}$$

where the hypergeometric function [23, 15.2.1] is $F\,(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}z^k$.

We found that the largest prime number in the decomposition of $2^{4m-2}f_{2m}$ is the integer just smaller than $2m$, i.e. $p_{\pi(2m)}$, where p_n is the n-th prime number and $\pi\,(x)$ denotes the number of primes smaller than or equal to x. Since f_{2m} is a square, in contrast to g_{2m}, all prime numbers have multiplicity at least two and $\sqrt{f_{2m}}$ contains many small primes up $p_{\pi(2m)}$. The integers $\sqrt{f_{2m}}$ and $\prod_{j=1}^{\pi(2m)}p_j$ are even comparable: we found that $\sqrt{f_{2m}}$ always contains the factor $\prod_{j=\pi(m)+1}^{\pi(2m)}p_j$ with the largest primes in the prime decomposition of $\sqrt{f_{2m}}$.

In contrast, $-g_{2m}<f_{2m}$ contains a few large primes larger than $p_{\pi(2m)}$ with multiplicity 1: one such prime for $m=2$ up to $m=8$, precisely two larger primes than $p_{\pi(2m)}$ for $m=9$ up to $m=15$, at most three larger primes (sometimes two or 1) for $m=16$ up to $m=25$, at most four larger primes for $m=26$ up to $m=36$, at most 5 larger primes (actually only 1 at $m=37$) for $m=37$ up to

[8] Interestingly, $\binom{\frac{1}{2}}{k}=\frac{\Gamma(\frac{1}{2}+1)}{k!\Gamma(\frac{1}{2}-k+1)}=\frac{1}{(1-2k)}\binom{-\frac{1}{2}}{k}$ are also rational numbers with the same denominator as $\binom{-\frac{1}{2}}{k}=\frac{\Gamma(\frac{1}{2})}{k!\Gamma(\frac{1}{2}-k)}$, because $\binom{-\frac{1}{2}}{k}$ is divisible by $(2k-1)$.

$m = 48$, at most 6 larger primes (actually only 1 at $m = 49$) from $m = 49$ up to more than $m = 66$ and 7 larger primes at $m = 67$ (we stopped at $m = 70$).

By fitting up to $m = 500$, we found the accurate fit for $m > 0$:

$$-\frac{f_{2m}}{g_{2m}} \simeq 0.615425 + 0.787049 \log(2m) + 0.0798748 \log^2(2m)$$

Also, $y = \log\left(\frac{\sqrt{f_{2m}}}{\prod_{j=\pi(m)+1}^{\pi(2m)} p_j}\right)$ linearly correlates with $x = 2m$ and the fitted line (up to $m = 30$) through the scattered data is $y \approx -0.793422 + 0.226858x$ and $y \approx -1.61084 + 0.223784x$ (up to $m = 120$) revealing that the data y irregularly "oscillates" around the line. In other words, apart from the clear linear correlation, the finer details of y as a function of $x = 2m$ are complicated.

4 The arithmetic-geometric mean and elliptic integrals

10. *An integral for $M(a,b)$.* After the discovery of the Taylor series for $\frac{1}{M(1+x,1-x)} = 1+\sum_{m=1}^{\infty} f_{2m}x^{2m}$ in (23) and (26), Gauss dived deeper, incredibly much deeper as we will see soon and as elaborated in Cox [8, Section 2]. A next fundamental result, called a "tour de force" by McKean and Moll, is

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = \frac{\pi}{2}\frac{1}{M(a,b)} \tag{28}$$

which is proved in the literature in several ways. Gauss himself provides two proofs, the first via a power series and the second via an integral transformation that he has just stated, without any clue how he has found it. We will give a couple of demonstrations, but they are essentially variations on Gauss's proofs.

In his second proof, Gauss[9] proves (28), but we follow Cox [8, p. 278] and define

$$I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \tag{29}$$

Substituting $\theta = \frac{\pi}{2} - u$ and employing $\sin\left(\frac{\pi}{2} - u\right) = \cos u$ and $\cos\left(\frac{\pi}{2} - u\right) = \sin u$ indicates that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{a^2\cos^2 u + b^2\sin^2 u}}$$

and that $I(a,b) = I(b,a)$. The key is to demonstrate that $I(a,b) = I(a_1,b_1)$, because iteration of the AGM algorithm (2) then gives

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = \cdots = I(a_n,b_n)$$

for all n. Taking the limit then yields

$$I(a,b) = \lim_{n\to\infty} I(a_n,b_n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\lim_{n\to\infty}\sqrt{a_n^2\sin^2\theta + b_n^2\cos^2\theta}} = \frac{1}{M(a,b)}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

and

$$I(a,b) = \frac{\pi}{2}\frac{1}{M(a,b)} \tag{30}$$

[9] Gauss Werke, Band 3, p. 352-353. In fact, Gauss shows that $\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \frac{1}{M(a,b)}$.

We will devote many articles below and in appendix C to deduce $I(a,b) = I(a_1,b_1)$ from its integral definition (29) by substitutions.

11. *A verification of Gauss's fundamental integral (29).* Before giving Gauss's first proof, we provide a *verification* proof by series expansion of the integral $I(a,b)$ in (29). We rewrite the integral as

$$I(a,b) = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{a^2\cos^2 u + b^2 \sin^2 u}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 - (a^2-b^2)\sin^2\theta}} = \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{(a^2-b^2)}{a^2}\sin^2\theta}}$$

Introducing the binomial expansion $(1+z)^\alpha = \sum_{m=0}^{\infty}\binom{\alpha}{m} z^m$, valid for $|z|<1$, and denoting $k = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} < 1$ yields

$$I(a,b) = \frac{1}{a}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m} k^{2m}\int_0^{\frac{\pi}{2}} \sin^{2m}\theta d\theta$$

Invoking the Beta integral $\frac{1}{2}B(a,b) = \int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2b-1}\theta d\theta$ in [23, 5.12.2] indicates that

$$\int_0^{\frac{\pi}{2}} \sin^{2m}\theta d\theta = \frac{1}{2}B\left(m+\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+1)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(m+\frac{1}{2}\right)}{m!}$$

while also $\int_0^{\frac{\pi}{2}} \cos^{2m}\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^{2m}\theta d\theta$. Further, with $\binom{-\frac{1}{2}}{k} = (-1)^k \frac{\Gamma(\frac{1}{2}+k)}{k!\,\Gamma(\frac{1}{2})}$, we find

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{2m}\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^{2m}\theta d\theta &= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(m+\frac{1}{2}\right)}{m!} = \frac{\pi}{2}(-1)^m\binom{-\frac{1}{2}}{m} \\ &= \frac{\pi}{2}\frac{\binom{2m}{m}}{2^{2m}} = \frac{\pi}{2}\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots(2m)} = \frac{\pi}{2}\prod_{j=1}^{m}\frac{2j-1}{2j}\end{aligned} \tag{31}$$

where the last equalities follow from (22) and (25). Thus, we obtain

$$I(a,b) = \frac{\pi}{2}\frac{1}{a}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2 k^{2m} = \frac{\pi}{2}\frac{1}{a}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2\left(1-\frac{b^2}{a^2}\right)^m$$

which converges for $|k|<1$, but diverges for $k\to 1$ logarithmically (**art**. 47). The Taylor series $\frac{1}{M(1+x,1-x)} = \sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2 x^{2m}$ in (26) then shows that

$$I(a,b) = \frac{\pi}{2}\frac{1}{a}\frac{1}{M\left(1+\sqrt{1-\frac{b^2}{a^2}}, 1-\sqrt{1-\frac{b^2}{a^2}}\right)}$$

Finally, $M(a,b) = M(a_1,b_1)$ leads to

$$I(a,b) = \frac{\pi}{2}\frac{1}{bM\left(1,\sqrt{\left(1-\sqrt{1-\frac{b^2}{a^2}}\right)\left(1+\sqrt{1-\frac{b^2}{a^2}}\right)}\right)} = \frac{\pi}{2}\frac{1}{aM\left(1,\frac{b}{a}\right)} = \frac{\pi}{2}\frac{1}{M(a,b)}$$

12. *Gauss's first proof or fundamental integral (29).* Now, we return to Gauss's great insight. From his Taylor series $\frac{1}{M(1+x,1-x)} = \sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2 x^{2m}$ in (26), Gauss [14, art. 8] observes that $\int_0^{\pi}\cos^2\varphi d\varphi = \frac{1}{2}\pi$,

$\int_0^\pi \cos^4\varphi d\varphi = \frac{1}{2}\frac{3}{4}\pi$, $\int_0^\pi \cos^6\varphi d\varphi = \frac{1}{2}\frac{3}{4}\frac{5}{8}\pi$ etc., which are instances of (31). Introduced in the Taylor series (26) gives

$$\begin{aligned}\frac{1}{M(1+x,1-x)} &= \sum_{m=0}^{\infty}\left(\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots(2m)}\right)^2 x^{2m} = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots(2m)}\int_0^\pi \cos^{2m}\varphi d\varphi\, x^{2m} \\ &= \frac{1}{\pi}\int_0^\pi d\varphi\left\{\sum_{m=0}^{\infty}\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots(2m)}(x\cos\varphi)^{2m}\right\} = \frac{1}{\pi}\int_0^\pi d\varphi\left\{\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}(x\cos\varphi)^{2m}\right\} \\ &= \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{\sqrt{1-(x\cos\varphi)^2}}\end{aligned}$$

which is an instance of his fundamental result (28). Gauss further observes, for coefficients $v_k(x)$ only dependent on x, that

$$\frac{1}{\sqrt{1-(x\cos\varphi)^2}} = P + 2\sum_{k=1}^{\infty} v_k(x)\cos(2k\varphi)$$

and that, after integrations with respect to φ, it holds that

$$\frac{1}{\pi}\int_0^\varphi \frac{d\varphi}{\sqrt{1-(x\cos\varphi)^2}} = P\frac{\varphi}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{v_k(x)}{k}\sin(2k\varphi)$$

which we have specified in (143) on p. 72. After letting $\varphi = \pi$, Gauss deduces

$$\frac{1}{M(1+x,1-x)} = P = \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{\sqrt{1-(x\cos\varphi)^2}} \tag{32}$$

Gauss [14, art. 8] proceeds with the more general form, with positive real α, β and $\gamma \le \beta$,

$$W = \frac{\alpha}{\sqrt{\beta-\gamma\cos^2\varphi}} = \frac{\alpha}{\sqrt{\beta}\sqrt{1-\frac{\gamma}{\beta}\cos^2\varphi}} = \frac{\alpha}{\sqrt{\beta}}\left(P + 2\sum_{k=1}^{\infty} b_k\left(\sqrt{\frac{\gamma}{\beta}}\right)\cos(2k\varphi)\right)$$

which has a maximum $W_{\max} = \frac{\alpha}{\sqrt{\beta-\gamma}}$ and a minimum $W_{\min} = \frac{\alpha}{\sqrt{\beta}}$, while $\frac{1}{\pi}\int_0^\pi \frac{\alpha d\varphi}{\sqrt{\beta-\gamma\cos^2\varphi}} = \frac{\alpha}{\sqrt{\beta}}P$ corresponds to an average W_{av} of φ over the interval $[0,\pi]$ and, due to periodicity of the cosine $\cos\varphi$, thus over the entire real φ axis. With $W_{\min} \le W_{\text{av}} \le W_{\max}$ and (32), we find that $\frac{\alpha}{\sqrt{\beta}} \le \frac{\alpha}{\sqrt{\beta}}P \le \frac{\alpha}{\sqrt{\beta-\gamma}}$ and

$$\frac{\sqrt{\beta-\gamma}}{\alpha} \le \frac{\sqrt{\beta}}{\alpha}M\left(1+\sqrt{\frac{\gamma}{\beta}}, 1-\sqrt{\frac{\gamma}{\beta}}\right) \le \frac{\sqrt{\beta}}{\alpha}$$

Using scaling $cM(a,b) = M(ca,cb)$ in (11) and the property $M(a,b) = M(a_1,b_1)$, it holds that

$$\frac{\sqrt{\beta}}{\alpha}M\left(1+\sqrt{\frac{\gamma}{\beta}}, 1-\sqrt{\frac{\gamma}{\beta}}\right) = M\left(\frac{\sqrt{\beta}}{\alpha}+\frac{\sqrt{\gamma}}{\alpha}, \frac{\sqrt{\beta}}{\alpha}-\frac{\sqrt{\gamma}}{\alpha}\right) = M\left(\frac{\sqrt{\beta}}{\alpha}, \frac{\sqrt{\beta-\gamma}}{\alpha}\right) = M\left(\frac{1}{W_{\min}}, \frac{1}{W_{\max}}\right)$$

and

$$\frac{1}{W_{\max}} \le M\left(\frac{1}{W_{\min}}, \frac{1}{W_{\max}}\right) \le \frac{1}{W_{\min}}$$

Consequently, we arrive at the known bounds $b \le M(a,b) \le a$ for $a > b$. Gauss ends part 1 in [14, art. 8] and appreciates the divine beauty[10] of (28) over its applications to astronomy.

[10] "...veritatum aeternarum sublimitatem atque divinam venustatem...", in English "... sublimity and divine beauty of eternal truths..."

13. *Newman's approach to (28).* Before turning to elliptic integrals in Section 8, we give a first example of an integral substitution that demonstrates $I(a,b) = I(a_1,b_1)$. After substitution $t = \tan\theta$, Gauss's fundamental formula (28) becomes

$$\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\tan^2\theta}d\theta}{\sqrt{a^2\tan^2\theta+b^2}} = \frac{2}{\pi}\int_0^{\infty} \frac{1}{\sqrt{1+t^2}\sqrt{a^2t^2+b^2}}dt$$

Letting $u = at$ results in

$$\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\infty} \frac{du}{\sqrt{u^2+a^2}\sqrt{u^2+b^2}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a^2)(u^2+b^2)}}$$

Newman [22] proves $M(a,b) = M(a_1,b_1)$, which equals

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{\left(t^2+\left(\frac{a+b}{2}\right)^2\right)(t^2+ab)}} = \int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a^2)(u^2+b^2)}}$$

after the substitution $t = \frac{1}{2}\left(u - \frac{ab}{u}\right)$ or $u = t \pm \sqrt{t^2+ab}$. If we take the plus sign, then the interval $t \in (-\infty,\infty)$ is mapped on the interval $u \in [0,\infty)$. The negative sign results in the interval $u \in (-\infty,0]$. Proceeding with the plus sign, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dt}{\sqrt{\left(t^2+\left(\frac{a+b}{2}\right)^2\right)(t^2+ab)}} &= \frac{1}{2}\int_0^{\infty} \frac{\left(1+\frac{ab}{u^2}\right)du}{\sqrt{\left(\frac{1}{4}\left(u^2-2ab+\frac{a^2b^2}{u^2}\right)+\left(\frac{a+b}{2}\right)^2\right)\left(\frac{1}{4}\left(u^2-2ab+\frac{a^2b^2}{u^2}\right)+ab\right)}} \\
&= \int_0^{\infty} \frac{u\left(1+\frac{ab}{u^2}\right)du}{\sqrt{\left(u^4+(a^2+b^2)u^2+a^2b^2\right)\left(\frac{1}{4}\left(u^2+2ab+\frac{a^2b^2}{u^2}\right)\right)}} \\
&= 2\int_0^{\infty} \frac{\left(u+\frac{ab}{u}\right)du}{\sqrt{\left((u^2+a^2)(u^2+b^2)\right)\left(u+\frac{ab}{u}\right)^2}} = 2\int_0^{\infty} \frac{du}{\sqrt{\left((u^2+a^2)(u^2+b^2)\right)}} \\
&= \int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a^2)(u^2+b^2)}}
\end{aligned}$$

After n iterations, we arrive at

$$\int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a^2)(u^2+b^2)}} = \int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a_n^2)(u^2+b_n^2)}}$$

and taking the limit $n \to \infty$, recalling that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a,b) = \mu$, leads to

$$\int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a^2)(u^2+b^2)}} = \lim_{n\to\infty}\int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2+a_n^2)(u^2+b_n^2)}} = \int_{-\infty}^{\infty} \frac{du}{u^2+\mu^2} = \frac{\pi}{\mu}$$

which is again Gauss's fundamental result (28).

5 Series and identities for $M(a,b)$

We prove a number of formulae for the arithmetic-geometric mean $M(a,b)$, which Gauss has just listed without any explanation in [15, p. 376]. The first articles in this section are elementary, except for **art**. 16-17. These formulae illustrate Gauss's versatility with his AGM algorithm (2).

14. Iterating the recursion $a_n = a_{n+1} + c_{n+1}$ in (12) p times,

$$a_n = a_{n+p} + \sum_{k=1}^{p} c_{n+k}$$

leads with $M(a,b) = \lim_{p\to\infty} a_{n+p}$ to [15, p. 376]

$$M(a,b) = a_n - \lim_{p\to\infty} \sum_{k=1}^{p} c_{n+k} = a_n - c_{n+1} - c_{n+2} - \cdots - c_{n+p} - \cdots$$

while $b_n = a_{n+1} - c_{n+1}$ similarly results in

$$M(a,b) = b_n + \lim_{p\to\infty} \sum_{k=1}^{p} c_{n+k} = b_n + c_{n+1} - c_{n+2} - \cdots - c_{n+p} - \cdots$$

15. After one iteration, the recursion $c_{n+1} = \frac{c_n^2}{4a_{n+1}}$ in (8) equals

$$c_{n+2} = \frac{c_{n+1}^2}{4a_{n+2}} = \frac{1}{4a_{n+2}} \frac{c_n^4}{16a_{n+1}^2}$$

Since $a_{n+2} = \frac{1}{2}(a_{n+1} + b_{n+1}) = \frac{1}{2}\left(\frac{1}{2}(a_n + b_n) + \sqrt{a_n b_n}\right) = \left(\frac{\sqrt{a_n}+\sqrt{b_n}}{2}\right)^2$, we obtain

$$\sqrt{c_{n+2}} = \frac{1}{8\sqrt{a_{n+2}}} \frac{c_n^2}{a_{n+1}} = \frac{1}{2\left(\sqrt{a_n} + \sqrt{b_n}\right)} \frac{(a_n - b_n)(a_n + b_n)}{(a_n + b_n)}$$

Hence, we find that $\sqrt{a_{n+2}} = \frac{1}{2}\left(\sqrt{a_n} + \sqrt{b_n}\right)$ and that $\sqrt{c_{n+2}} = \frac{1}{2}\left(\sqrt{a_n} - \sqrt{b_n}\right)$. Their combination leads to

$$\sqrt{c_{n+2}} = \sqrt{a_n} - \sqrt{a_{n+2}} \tag{33}$$

After p iterations of $\sqrt{a_n} = \sqrt{a_{n+2}} + \sqrt{c_{n+2}}$ in (33),

$$\sqrt{a_n} = \sqrt{a_{n+2p}} + \sum_{k=1}^{p} \sqrt{c_{n+2k}}$$

and passing to the limit $p \to \infty$, the following series originates [15, p. 376]

$$\sqrt{M(a,b)} = \sqrt{a_n} - \lim_{p\to\infty} \sum_{k=1}^{p} \sqrt{c_{n+2k}} = \sqrt{a_n} - \sqrt{c_{n+2}} - \sqrt{c_{n+4}} - \cdots - \sqrt{c_{n+2p}} - \cdots$$

Since $\sqrt{b_n} = \sqrt{a_n} - 2\sqrt{c_{n+2}}$, we also have

$$\sqrt{M(a,b)} = \sqrt{b_n} + \sqrt{c_{n+2}} - \sqrt{c_{n+4}} - \cdots - \sqrt{c_{n+2p}} - \cdots$$

16. From $c_n = \frac{a_{n-1}-b_{n-1}}{2}$, it follows that

$$2b_{n-1}c_n = b_{n-1}a_{n-1} - b_{n-1}^2 = b_n^2 - b_{n-1}^2 \tag{34}$$

The right-hand side can be written in terms of c_n with $b_n^2 = a_n^2 - c_n^2$ as

$$b_n^2 - b_{n-1}^2 = a_n^2 - a_{n-1}^2 - c_n^2 + c_{n-1}^2 \tag{35}$$

With the AGM formulae (2),

$$a_n^2 - a_{n-1}^2 = \frac{1}{4}a_{n-1}^2 + \frac{1}{2}a_{n-1}b_{n-1} + \frac{1}{4}b_{n-1}^2 - a_{n-1}^2 = -\frac{3}{4}a_{n-1}^2 + \frac{1}{2}b_n^2 + \frac{1}{4}b_{n-1}^2 = -\frac{3}{4}c_{n-1}^2 + \frac{1}{2}\left(b_n^2 - b_{n-1}^2\right)$$

substitution into (35) yields

$$b_n^2 - b_{n-1}^2 = -c_n^2 + c_{n-1}^2 - \frac{3}{4}c_{n-1}^2 + \frac{1}{2}\left(b_n^2 - b_{n-1}^2\right)$$

and, simplified,

$$b_n^2 - b_{n-1}^2 = -2c_n^2 + \frac{1}{2}c_{n-1}^2 \tag{36}$$

After introduction of (36) into (34), we arrive at

$$2b_{n-1}c_n = \frac{1}{2}c_{n-1}^2 - 2c_n^2 \tag{37}$$

Iterating $c_n^2 = 4b_nc_{n+1} + 4c_{n+1}^2$ in (37) p times,

$$c_n^2 = 4b_nc_{n+1} + 4^2b_{n+1}c_{n+2} + 4^3b_{n+2}c_{n+3} + \cdots + 4^pb_{n+p-1}c_{n+p} + 4^pc_{n+p}^2 = \sum_{k=1}^{p} 4^kb_{n+(k-1)}c_{n+k} + 4^pc_{n+p}^2$$

from which we find, for $n = 0$, that $c^2 = \sum_{k=1}^{\infty} 4^kb_{(k-1)}c_k$. Summing (37) from $n = m$ and $n = p$ yields

$$2\sum_{n=m}^{p} b_{n-1}c_n = \frac{1}{2}\sum_{n=m}^{p} c_{n-1}^2 - 2\sum_{n=m}^{p} c_n^2 = \frac{1}{2}\sum_{n=m-1}^{p-1} c_n^2 - 2\sum_{n=m}^{p} c_n^2$$

and

$$2\sum_{n=m}^{p} b_{n-1}c_n = \frac{1}{2}c_{m-1}^2 - \frac{3}{2}\sum_{n=m}^{p-1} c_n^2 - 2c_p^2 \tag{38}$$

while summing (36) gives

$$\sum_{n=m}^{p} b_n^2 - \sum_{n=m}^{p} b_{n-1}^2 = -2\sum_{n=m}^{p} c_n^2 + \frac{1}{2}\sum_{n=m}^{p} c_{n-1}^2$$

and simplifies to

$$b_p^2 - b_{m-1}^2 = \frac{1}{2}c_{m-1}^2 - \frac{3}{2}\sum_{n=m}^{p-1} c_n^2 - 2c_p^2$$

With (38), we obtain

$$b_p^2 = b_{m-1}^2 + 2\sum_{n=m}^{p} b_{n-1}c_n$$

and also

$$a_p^2 = b_{m-1}^2 + 2\sum_{n=m}^{p} b_{n-1}c_n + c_p^2$$

Again with (38), we arrive at

$$a_p^2 = b_{m-1}^2 + \frac{1}{2}c_{m-1}^2 - \frac{3}{2}\sum_{n=m}^{p-1} c_n^2 - c_p^2$$

and, if $p \to \infty$, [15, p. 376]

$$M^2(a,b) = b_{m-1}^2 + \frac{1}{2}c_{m-1}^2 - \frac{3}{2}\sum_{n=m}^{\infty} c_n^2$$

17. From $c_n = \frac{c_{n-1}^2}{4a_n}$ in (8), it follows that $\log\left(\frac{a_n}{c_n}\right) = \log\left(\frac{4a_n^2}{c_{n-1}^2}\right) = 2\log\left(\frac{2a_n}{c_{n-1}}\frac{a_{n-1}}{a_{n-1}}\right)$, leading to the recursion

$$\log\left(\frac{a_n}{c_n}\right) = 2\log\left(\frac{2a_n}{a_{n-1}}\right) + 2\log\left(\frac{a_{n-1}}{c_{n-1}}\right) \tag{39}$$

After p iterations, we obtain

$$\log\left(\frac{a_n}{c_n}\right) = \sum_{j=1}^{p} 2^j \log\left(\frac{2a_{n+1-j}}{a_{n-j}}\right) + 2^p \log\left(\frac{a_{n-p}}{c_{n-p}}\right)$$

Since $\frac{\pi}{2}\frac{K(k')}{K(k)} = \lim_{n\to\infty}\frac{1}{2^n}\log\left(\frac{a_n}{c_n}\right)$ as shown by King[11] [18, eq. (60) on p. 13], we find, after dividing both sides by 2^n,

$$\begin{aligned}\frac{1}{2^n}\log\left(\frac{a_n}{c_n}\right) &= \sum_{j=1}^{p}\frac{1}{2^{n-j}}\log\left(\frac{2a_{n+1-j}}{a_{n-j}}\right) + \frac{1}{2^{n-p}}\log\left(\frac{a_{n-p}}{c_{n-p}}\right)\\ &= \frac{1}{2^{n-p}}\log\left(\frac{a_{n-p}}{c_{n-p}}\right) + \sum_{j=1}^{p}\frac{1}{2^{n-j}}\log\left(\frac{a_{n+1-j}}{a_{n-j}}\right) + \log(2)\sum_{j=1}^{p}\frac{1}{2^{n-j}}\end{aligned}$$

After executing the geometric series $\sum_{j=1}^{p}\frac{1}{2^{n-j}} = 2^{1-n}\sum_{j=0}^{p-1}2^j = 2^{1-n}(2^p-1)$ and letting $l = n-j$, we obtain

$$\begin{aligned}\frac{1}{2^n}\log\left(\frac{a_n}{c_n}\right) &= \frac{1}{2^{n-p}}\log\left(\frac{a_{n-p}}{c_{n-p}}\right) + \sum_{l=n-p}^{n-1}\frac{1}{2^l}\log\left(\frac{a_{l+1}}{a_l}\right) + 2\left(2^{p-n}-2^{-n}\right)\log(2)\\ &= \frac{1}{2^{n-p}}\log\left(\frac{4a_{n-p}}{c_{n-p}}\right) - \sum_{l=n-p}^{n-1}\frac{1}{2^l}\log\left(\frac{a_l}{a_{l+1}}\right) - 2^{1-n}\log(2)\end{aligned}$$

Finally[12], choosing first $p = n-m$ and then $n \to \infty$ leads to Gauss's expansion [15, p. 377],

$$\frac{\pi}{2}\frac{M(a,b)}{M(a,c)} = \frac{1}{2^m}\log\left(\frac{4a_m}{c_m}\right) - \sum_{l=m}^{\infty}\frac{1}{2^l}\log\left(\frac{a_l}{a_{l+1}}\right) \tag{40}$$

where the sum in (40) contains only non-negative terms because $a_l \geq a_{l+1}$.

[11] King's derivation, similar in framework as his Section V that we have entirely derived in **art**. 57 based on a few earlier articles, is equivalent to Gauss's in [15, p. 388], illustrating the difficulty to follow the sketches of Gauss in the Nachlass in sequential order. The formulae (40) and (42) are listed on p. 377.

[12] Gauss's fundamental integral $\frac{1}{M(a,b)} = \frac{2}{a\pi}K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)$ in (84) and $k^2 = \frac{c^2}{a^2} = 1 - \frac{b^2}{a^2} = 1-(k')^2$ indicates that

$$\frac{M(a,b)}{M(a,c)} = \frac{K\left(\sqrt{1-\left(\frac{c}{a}\right)^2}\right)}{K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)} = \frac{K\left(\sqrt{1-k^2}\right)}{K(k)} = \frac{K(k')}{K(k)}$$

Gauss's alternative expansion is deduced from $\log\left(\frac{a_{n-1}}{a_n}\right)+\log\left(\frac{b_{n-1}}{b_n}\right)=\log\left(\frac{a_{n-1}b_{n-1}}{a_nb_n}\right)=2\log\left(\frac{b_n}{b_{n+1}}\right)$, which leads the recurrence formula

$$\log\left(\frac{a_{n-1}}{a_n}\right)=\log\left(\frac{b_n}{b_{n-1}}\right)-2\log\left(\frac{b_{n+1}}{b_n}\right) \tag{41}$$

Substituting (41) into (40),

$$\begin{aligned}\frac{\pi}{2}\frac{M(a,b)}{M(a,c)}&=\frac{1}{2^m}\log\left(\frac{4a_m}{c_m}\right)-\sum_{l=m}^{\infty}\frac{1}{2^l}\left(\log\left(\frac{b_{l+1}}{b_l}\right)-2\log\left(\frac{b_{l+2}}{b_{l+1}}\right)\right)\\&=\frac{1}{2^m}\log\left(\frac{4a_m}{c_m}\right)-\sum_{l=m}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right)+4\sum_{l=m+1}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right)\\&=\frac{1}{2^m}\log\left(\frac{4a_m}{c_m}\right)-\log\left(\frac{b_{m+1}}{b_m}\right)-\sum_{l=m+1}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right)+4\sum_{l=m+1}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right)\end{aligned}$$

and simplifying $\frac{a_mb_m}{c_mb_{m+1}}=\frac{b_{m+1}}{c_m}$ gives Gauss's alternative expansion [15, p. 377]

$$\frac{\pi}{2}\frac{M(a,b)}{M(a,c)}=\frac{1}{2^m}\log\left(\frac{4b_{m+1}}{c_m}\right)+3\sum_{l=m+1}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right) \tag{42}$$

where the sum in (42) also contains only non-negative terms because $b_{l+1}\geq b_l$. Since a_n and b_n converge very rapidly towards each other by the AGM algorithm (2) and $\frac{a_n}{a_{n-1}}=1+O\left(10^{-r.2^n}\right)$ in (10), it holds, using $\log(1+x)=x+O\left(x^2\right)$, that $\log\left(\frac{a_n}{a_{n-1}}\right)=O\left(10^{-r.2^n}\right)$. Thus, the series $\sum_{l=m}^{\infty}\frac{1}{2^l}\log\left(\frac{a_l}{a_{l+1}}\right)$ in (40) as well as $\sum_{l=m+1}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right)$ in (42) also converge quadratically as the AGM algorithm (2). In **art**. 42, we derive, following Gauss, from (40) and (42) an upper and lower bound for π, that illustrates the AGM algorithm's characteristic approximate doubling in decimal digits per iteration m.

From (40) and (42) together with $M(a,c)=2^mM(a_m,c_m)$ in (14) and $M(a,b)=M(a_m,b_m)$ in (6), we deduce that

$$\frac{\pi}{2}=\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4a_m}{c_m}\right)-\frac{M(a,c)}{M(a,b)}\sum_{l=m}^{\infty}\frac{1}{2^l}\log\left(\frac{a_l}{a_{l+1}}\right)=\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4a_m}{c_m}\right)+O\left(10^{-r.2^m}\right)$$

$$\frac{\pi}{2}=\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4b_{m+1}}{c_m}\right)+3\frac{M(a,c)}{M(a,b)}\sum_{l=m+1}^{\infty}\frac{1}{2^l}\log\left(\frac{b_{l+1}}{b_l}\right)=\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4b_{m+1}}{c_m}\right)+O\left(10^{-r.2^m}\right)$$

and

$$\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4b_{m+1}}{c_m}\right)\leq\frac{\pi}{2}\leq\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4a_m}{c_m}\right) \tag{43}$$

Thus, Gauss notes that $\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(\frac{4a_m}{c_m}\right)$ and $\frac{M(a_m,c_m)}{M(a_m,b_m)}\log\left(4\frac{b_{m+1}}{c_m}\right)$, as well as the two backward variants that we omit, tend to the same limit $\frac{\pi}{2}$ if m grows large, from which he concludes[13] that

[13] With the scaling (11) property of $M(a,b)$ and observing that $\log\frac{b_{m+1}}{a_m}<0$ for finite m, the inequality (43) becomes

$$\frac{M\left(1,\frac{c_m}{a_m}\right)}{M\left(1,\frac{b_m}{a_m}\right)}\left(\log\left(\frac{4a_m}{c_m}\right)+\log\frac{b_{m+1}}{a_m}\right)\leq\frac{\pi}{2}\leq\frac{M\left(1,\frac{c_m}{a_m}\right)}{M\left(1,\frac{b_m}{a_m}\right)}\left(\log\left(\frac{4a_m}{c_m}\right)\right)$$

If $m\to\infty$, then $a_m=b_m\to M(a,b)$ and $c_m\to 0$ so that replacing $\frac{c_m}{a_m}=\varepsilon$ and using $M(1,1)=1$ demonstrates that $\lim_{\varepsilon\to 0}M(1,\varepsilon)\log\frac{4}{\varepsilon}=\frac{\pi}{2}$.

$\lim_{\varepsilon\to 0} M(1,\varepsilon)\log\frac{4}{\varepsilon}=\frac{\pi}{2}$. This observation is verified in **art**. 57 in Appendix B, where we show that $K(k')=O\left(\log\frac{4}{k}\right)$ if $k\to 0$.

Gauss also states that his equations hold for complex values of a, b and c. The sequel of his Nachlass is difficult, contains gaps and leads in the following page [15, p. 378] already to Theorem 2, which is deciphered and treated in extenso by Cox (see **art**. 37). About his AGM system $\{a_n, b_n, c_n\}$ where $n\in\mathbb{Z}$, thus comprising both the forward and backward AGM algorithm and the complementary modulus $k'=\sqrt{1-k^2}$ with $k=\frac{c}{a}$ and thus $k'=\frac{b}{a}$, Gauss had a very deep knowledge. Without King's $\frac{\pi}{2}\frac{K(k')}{K(k)}=\lim_{n\to\infty}\frac{1}{2^n}\log\left(\frac{a_n}{c_n}\right)$ and the relation $\frac{K(k')}{K(k)}=\frac{M(a,b)}{M(a,c)}$ via (84), it is rather difficult to demonstrate (40). Section 7, in particular **art**. 25, will illustrate the importance of the series (40) and (42), as well as Gauss's limit argument for large m.

6 Differential calculus on the arithmetic-geometric mean

18. *Differential equation for* $y(x)=\frac{1}{M(1+x,1-x)}$. Gauss in [14, art. 8] derives a differential equation for $y(x)=\frac{1}{M(1+x,1-x)}$, starting from

$$\begin{aligned} y(x)&=\sum_{m=0}^{\infty} f_{2m}x^{2m}=\sum_{m=0}^{\infty}\left(\frac{(2m)!}{(2^m m!)^2}\right)^2 x^{2m}=1+\sum_{m=1}^{\infty}\left(\prod_{j=1}^{m}\frac{2j-1}{2j}\right)^2 x^{2m}\\ &=1+\frac{1}{4}x^2+\frac{1}{4}\frac{9}{16}x^4+\frac{1}{4}\frac{9}{16}\frac{25}{36}x^6+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{64}x^8+\cdots \end{aligned}$$

The first and second derivative

$$\begin{aligned} \frac{dy(x)}{dx}&=\sum_{m=0}^{\infty}2mf_{2m}x^{2m-1}=\frac{1}{2}x+\frac{1}{4}\frac{9}{4}x^3+\frac{1}{4}\frac{9}{16}\frac{25}{6}x^5+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{8}x^7+\cdots\\ \frac{d^2y(x)}{dx^2}&=\sum_{m=0}^{\infty}2m(2m-1)f_{2m}x^{2m-2}=\frac{1}{2}+\frac{1}{4}\frac{9}{4}3x^2+\frac{1}{4}\frac{9}{16}\frac{25}{6}5x^4+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{8}7x^6+\cdots \end{aligned}$$

lead to

$$x\frac{dy(x)}{dx}=\sum_{m=0}^{\infty}2mf_{2m}x^{2m}=\frac{1}{2}x^2+\frac{1}{4}\frac{9}{4}x^4+\frac{1}{4}\frac{9}{16}\frac{25}{6}x^6+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{8}x^8+\cdots$$

and

$$\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right)=x\frac{d^2y(x)}{dx^2}+\frac{dy(x)}{dx}=x+\frac{1}{4}\frac{9}{1}x^3+\frac{1}{4}\frac{9}{16}\frac{25}{1}x^5+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{1}x^7+\cdots$$

Hence,

$$\frac{1}{x^2}\left(x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right)=1+\frac{1}{4}9x^2+\frac{1}{4}\frac{9}{16}25x^4+\frac{1}{4}\frac{9}{16}\frac{25}{36}49x^6+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{64}81x^8+\cdots$$

which also equals the sum of the series

$$\begin{aligned} x^2\frac{d^2y(x)}{dx^2}&=\frac{1}{4}2x^2+\frac{1}{4}\frac{9}{16}12x^4+\frac{1}{4}\frac{9}{16}\frac{25}{36}30x^6+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{64}56x^8+\cdots\\ 3x\frac{dy(x)}{dx}&=\frac{1}{4}6x^2+\frac{1}{4}\frac{9}{16}12x^4+\frac{1}{4}\frac{9}{16}\frac{25}{36}18x^6+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{64}24x^8+\cdots\\ y(x)-1&=\frac{1}{4}1x^2+\frac{1}{4}\frac{9}{16}x^4+\frac{1}{4}\frac{9}{16}\frac{25}{36}x^6+\frac{1}{4}\frac{9}{16}\frac{25}{36}\frac{49}{64}x^8+\cdots \end{aligned}$$

Gauss then finds that $x^2\frac{d^2y(x)}{dx^2}+3x\frac{dy(x)}{dx}+y(x)=\frac{1}{x^2}\left(x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right)$ from which the differential equation follows

$$\left(x^3-x\right)\frac{d^2y(x)}{dx^2}+\left(3x^2-1\right)\frac{dy(x)}{dx}+xy(x)=0 \tag{44}$$

Without proof, Gauss [14, art. 8] states that the general solution of the second order differential equation (44) is

$$y(x)=\frac{A}{M(1+x,1-x)}+\frac{B}{M(1,x)} \tag{45}$$

Gauss[14] has explored the differential equation a little further.

The differential equation (44) can be reduced into the hypergeometric differential equation [23, 15.10.1]

$$z(1-z)\frac{d^2w}{dz^2}+(c-(a+b+1)z)\frac{dw}{dz}-abw(z)=0 \tag{46}$$

by the transformation $z=x^2$ as mentioned in [9, p. 460]. Indeed, let $w(z)=y(\sqrt{z})$, then $\frac{dy(x)}{dx}=\frac{dy(\sqrt{z})}{dz}\frac{dz}{dx}=2x\frac{dw}{dz}=2\sqrt{z}\frac{dw}{dz}$ and

$$\begin{aligned}\frac{d^2y(x)}{dx^2}&=\frac{d}{dz}\left(\frac{dy(x)}{dx}\right)\frac{dz}{dx}=2\sqrt{z}\frac{d}{dz}\left(2\sqrt{z}\frac{dw}{dz}\right)=2\sqrt{z}\left\{\frac{1}{\sqrt{z}}\frac{dw}{dz}+2\sqrt{z}\frac{d^2w}{dz^2}\right\}\\&=2\frac{dw}{dz}+4z\frac{d^2w}{dz^2}\end{aligned}$$

The differential equation (44) becomes

$$\sqrt{z}(z-1)\left(2\frac{dw}{dz}+4z\frac{d^2w}{dz^2}\right)+(3z-1)2\sqrt{z}\frac{dw}{dz}+\sqrt{z}w(z)=0$$

and simplifies to

$$z(1-z)\frac{d^2w}{dz^2}+(1-2z)\frac{dw}{dz}-\frac{1}{4}w(z)=0 \tag{47}$$

which is an instance with $a=b=\frac{1}{2}$ and $c=1$ of the hypergeometric differential equation (46). We observe that, after replacing z by $1-z$, the differential equation (47) remains the same, which indicates that both $w(z)$ and $w(1-z)$ are a solution. The Wronskian[15] [21, p. 524-530] is non-zero, which indicates that both solutions are independent. Hence, the general solution of (46) is a linear combination $w(z)=AF(a,b,c;z)+BF(a,b,c;1-z)$ of two hypergeometric functions $F(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}z^k$ and $F(a,b,c;1-z)$. After invoking (27) and $z=x^2$, we arrive at Gauss's solution (45). Precisely the hypergeometric function has been studied in detail by Gauss in [13]. Anticipating $\frac{1}{M(1,x)}=\frac{2}{\pi}K\left(\sqrt{1-x^2}\right)$ in (85) and $\frac{1}{M(1+x,1-x)}=\frac{2}{\pi}K(x)$, Gauss's general solution (45)

$$y(x)=\frac{2A}{\pi}K(x)+\frac{2B}{\pi}K\left(\sqrt{1-x^2}\right)$$

exhibits that the complete elliptic integral $K(x)$, evaluated at x and at its complementary modulus $\sqrt{1-x^2}$, constitute a set of independent functions, which underlines the importance of c_n in (7), beside a_n and b_n in the AGM algorithm (2).

[14] Gauss Werke, Band 10.1, p. 181-183. Also part of the Nachlass, which is far less clear and collected after Gauss's death.

[15] The Wronskian $\Delta(z)=\Delta(z_0)\exp\left(-\int_{z_0}^{z}p(u)\,du\right)$ of the differential equation $\frac{d^2v(z)}{dz^2}+p(z)\frac{dv(z)}{dz}+q(z)v(z)=0$, applied to (47), equals $\Delta(z)=\frac{c}{z(1-z)}$, where the constant c depends on the initial condition.

19. *The differential* $dM(a,b)$. Part 2 in [14, art. 9] is the last, relatively complete text in Latin[16] of Gauss, in which he studies the differential $dM(a,b) = \frac{\partial M(a,b)}{\partial b}dx + \frac{\partial M(a,b)}{\partial b}db$. Gauss replace a_n and b_n in his AGM algorithm (2) by the variable x_n and y_n, respectively, to emphasize that a and b are given constant numbers, while x and y are variables. Here, we maintain the usual notation, but a and b are now variables. The corresponding differentials of a_n and b_n in the AGM algorithm (2) are

$$da_n = \tfrac{1}{2}\left(da_{n-1} + db_{n-1}\right) \quad \text{and} \quad db_n = \tfrac{1}{2}\left(\sqrt{\tfrac{b_{n-1}}{a_{n-1}}}da_{n-1} + \sqrt{\tfrac{a_{n-1}}{b_{n-1}}}db_{n-1}\right)$$

where the last differential is rewritten with $b_n = \sqrt{a_{n-1}b_{n-1}}$ as $db_n = \frac{1}{2}b_n\left(\frac{da_{n-1}}{a_{n-1}} + \frac{db_{n-1}}{b_{n-1}}\right)$. Gauss remarks that proceeding with these differentials will obscure the underlying law. Therefore, he defines the new variables

$$f_n = \tfrac{da_n}{a_n} + \tfrac{db_n}{b_n} \quad \text{and} \quad g_n = \tfrac{da_n}{a_n} - \tfrac{db_n}{b_n} \tag{48}$$

which shows that $f_n = d\log a_n b_n$ and $g_n = d\log\frac{a_n}{b_n}$. Then, Gauss introduces $da_n = \frac{1}{2}\left(da_{n-1} + db_{n-1}\right)$ and $db_n = \frac{1}{2}b_n\left(\frac{da_{n-1}}{a_{n-1}} + \frac{db_{n-1}}{b_{n-1}}\right)$ in f_n,

$$\begin{aligned} f_n &= \frac{da_n}{a_n} + \frac{db_n}{b_n} = \frac{1}{2}\left(\frac{da_{n-1}}{a_n} + \frac{db_{n-1}}{a_n}\right) + \frac{1}{2}\left(\frac{da_{n-1}}{a_{n-1}} + \frac{db_{n-1}}{b_{n-1}}\right) \\ &= \frac{1}{2}\left(\frac{da_{n-1}}{a_{n-1}} + \frac{db_{n-1}}{b_{n-1}} + \frac{da_{n-1}}{a_n} + \frac{db_{n-1}}{a_n}\right) \end{aligned}$$

and he proceeds to deduce a recursion for f_n as follows,

$$\begin{aligned} f_n &= f_{n-1} + \frac{1}{2}\left(\frac{da_{n-1}}{a_n} + \frac{db_{n-1}}{a_n} - \frac{da_{n-1}}{a_{n-1}} - \frac{db_{n-1}}{b_{n-1}}\right) \\ &= f_{n-1} + \frac{1}{2}\left(\frac{a_{n-1} - a_n}{a_n a_{n-1}}\right)da_{n-1} + \frac{1}{2}\left(\frac{b_{n-1} - a_n}{a_n b_{n-1}}\right)db_{n-1} \end{aligned}$$

Replacing $b_{n-1} = 2a_n - a_{n-1}$ in the last bracket results in

$$\begin{aligned} f_n &= f_{n-1} + \frac{1}{2}\left(\frac{a_{n-1} - a_n}{a_n a_{n-1}}\right)da_{n-1} - \frac{1}{2}\left(\frac{a_{n-1} - a_n}{a_n b_{n-1}}\right)db_{n-1} \\ &= f_{n-1} + \frac{1}{2}\left(\frac{a_{n-1} - a_n}{a_n}\right)\left\{\frac{da_{n-1}}{a_{n-1}} - \frac{db_{n-1}}{b_{n-1}}\right\} \end{aligned}$$

Finally, with $g_n = \frac{da_n}{a_n} - \frac{db_n}{b_n}$ in (48), Gauss arrives at the recursion for f_n,

$$f_n = f_{n-1} + \frac{1}{2}\left(\frac{a_{n-1} - a_n}{a_n}\right)g_{n-1} \tag{49}$$

An analogous computation for g_n,

$$\begin{aligned} g_n &= \frac{da_n}{a_n} - \frac{db_n}{b_n} = \frac{1}{2}\left(\frac{da_{n-1}}{a_n} - \frac{da_{n-1}}{a_{n-1}} + \frac{db_{n-1}}{a_n} - \frac{db_{n-1}}{b_{n-1}}\right) \\ &= \frac{1}{2}\left(\frac{a_{n-1} - a_n}{a_n a_{n-1}}da_{n-1} + \frac{b_{n-1} - a_n}{a_n b_{n-1}}db_{n-1}\right) \end{aligned}$$

[16] Pars II. De Functionibus Transscendentibus quae ex Differentiatione Mediorum Arithmetico-Geometricorium oriuntur.

leads to

$$g_n = \frac{1}{2}\left(\frac{a_{n-1}-a_n}{a_n}\right) g_{n-1} \tag{50}$$

which allows us to simplify the recursion (49) for f_n with (50) as

$$f_n = f_{n-1} + g_n \tag{51}$$

After p iterations of (51), we obtain $f_n = f_{n-p} + \sum_{j=0}^{p-1} g_{n-j}$ and changing $k = n-j$ then yields, for integer $p \geq 0$,

$$f_n = f_{n-p} + \sum_{k=n-p+1}^{n} g_k \tag{52}$$

Similarly, p iterations of (50) resulting in $g_n = \frac{g_{n-p}}{2^p}\prod_{j=0}^{p-1}\left(\frac{a_{n-1-j}-a_{n-j}}{a_{n-j}}\right)$ and choosing $p = n$, we obtain

$$g_n = \frac{g_0}{2^n}\prod_{k=1}^{n}\left(\frac{a_{k-1}-a_k}{a_k}\right) \tag{53}$$

which demonstrates that g_n decreases very rapidly with n, because $a_{k-1}-a_k$ converges to zero quadratically by the AGM algorithm (2). Hence, the limit $n \to \infty$ of (53) is $g_\infty = 0$ and, thus, the series for f_n in (52) converges rapidly. Gauss rewrites $\frac{a_{k-1}-a_k}{a_k} = \frac{a_{k-1}-b_{k-1}}{a_{k-1}+b_{k-1}} = \frac{(a_{k-1}-b_{k-1})^2}{a_{k-1}^2-b_{k-1}^2} = \frac{(a_{k-1}+b_{k-1})^2-4a_{k-1}b_{k-1}}{a_{k-1}^2-b_{k-1}^2} = 4\frac{a_k^2-b_k^2}{a_{k-1}^2-b_{k-1}^2} = 4\frac{c_k^2}{c_{k-1}^2}$. Since

$$\prod_{k=1}^{n}\left(\frac{a_{k-1}-a_k}{a_k}\right) = 2^{2n}\frac{\prod_{k=1}^{n} c_k^2}{\prod_{k=1}^{n} c_{k-1}^2} = 2^{2n}\frac{\prod_{k=1}^{n} c_k^2}{\prod_{k=0}^{n-1} c_k^2} = 2^{2n}\frac{c_n^2}{c_0^2}$$

relations (53) and (52) simplify to

$$g_n = g_0 2^n \frac{c_n^2}{c^2} \tag{54}$$

and

$$f_{n+p} = f_n + \frac{g_0}{c^2}\sum_{k=n+1}^{n+p} 2^k c_k^2 \tag{55}$$

Gauss now returns to $M(a,b)$ by noting that $\lim_{n\to\infty}\frac{da_n}{a_n} = \lim_{n\to\infty}\frac{db_n}{b_n} = \frac{dM(a,b)}{M(a,b)}$, while inverting the definitions of f_n and g_n in (48) indicates that

$$2\frac{da_n}{a_n} = f_n + g_n \quad \text{and} \quad 2\frac{db_n}{b_n} = f_n - g_n$$

Hence, $\frac{dM(a,b)}{M(a,b)} = \lim_{n\to\infty}\frac{da_n}{a_n} = \frac{1}{2}\lim_{n\to\infty}(f_n+g_n) = \frac{1}{2}f_\infty$ and invoking (55) for $p = 0$ with the definitions of f_n and g_n in (48)

$$\begin{aligned}\frac{dM(a,b)}{M(a,b)} &= \frac{1}{2}\left(\frac{da}{a}+\frac{db}{b}\right) + \frac{1}{2c^2}\left(\frac{da}{a}-\frac{db}{b}\right)\sum_{n=1}^{\infty} 2^n c_n^2 \\ &= \frac{1}{2}\frac{da}{a}\left(1+\frac{1}{c^2}\sum_{n=1}^{\infty} 2^n c_n^2\right) + \frac{1}{2}\frac{db}{b}\left(1-\frac{1}{c^2}\sum_{n=1}^{\infty} 2^n c_n^2\right)\end{aligned}$$

Gauss arrives at the beautiful expansion for the differential of the logarithm of $M(a,b)$

$$\frac{dM(a,b)}{M(a,b)} = d\left(\log M(a,b)\right) = \frac{1}{2c^2}\left\{\frac{da}{a}\left(\sum_{n=0}^{\infty} 2^n c_n^2\right) + \frac{db}{b}\left(c^2 - \sum_{n=1}^{\infty} 2^n c_n^2\right)\right\} \tag{56}$$

In [14, art. 10], Gauss gives a numerical example that illustrates how fast the series $\sum_{n=0}^{\infty} 2^n c_n^2$ in (56) converges. After art. 10, the Nachlass becomes very incomplete: art. 11 only contains "Facile iam etiam coefficientes sequentes series". Art. 12 in [15] continues in German and is a report of others, who have collected Gauss's unfinished and unpublished work after his death.

20. *Partial differential equation.* Since $d\log M(a,b) = \frac{\partial \log M(a,b)}{\partial a} da + \frac{\partial \log M(a,b)}{\partial b} db$, we find from (56) that

$$\begin{cases} \frac{\partial \log M(a,b)}{\partial a} = \frac{1}{2c^2 a}\left(c^2 + \sum_{n=1}^{\infty} 2^n c_n^2\right) \\ \frac{\partial \log M(a,b)}{\partial b} = \frac{1}{2c^2 b}\left(c^2 - \sum_{n=1}^{\infty} 2^n c_n^2\right) \end{cases}$$

which can be compared with $\frac{K(k)-E(k)}{K(k)} = \frac{1}{2a^2}\sum_{n=0}^{\infty} 2^n c_n^2$ below in (96). Hence, it holds that

$$a\frac{\partial \log M(a,b)}{\partial a} + b\frac{\partial \log M(a,b)}{\partial b} = 1 \tag{57}$$

which is also written as $\frac{\partial \log M(a,b)}{\partial \log a} + \frac{\partial \log M(a,b)}{\partial \log b} = 1$. If we replace $a = e^u$ and $b = e^v$ and denote $T(u,v) = \log(M(e^u, e^v))$, then the first order partial differential equation (57) becomes

$$\frac{\partial T(u,v)}{\partial u} + \frac{\partial T(u,v)}{\partial v} = 1$$

Taking the partial derivative of (57) with respect to a yields

$$\frac{\partial \log M(a,b)}{\partial a} + a\frac{\partial^2 \log M(a,b)}{\partial a^2} + b\frac{\partial^2 \log M(a,b)}{\partial b \partial a} = 0$$

and similarly for b,

$$\frac{\partial \log M(a,b)}{\partial b} + b\frac{\partial^2 \log M(a,b)}{\partial b^2} + a\frac{\partial^2 \log M(a,b)}{\partial a \partial b} = 0$$

Elimination of $\frac{\partial^2 \log M(a,b)}{\partial a \partial b}$ results in

$$a\frac{\partial \log M(a,b)}{\partial a} + a^2\frac{\partial^2 \log M(a,b)}{\partial a^2} = b\frac{\partial \log M(a,b)}{\partial b} + b^2\frac{\partial^2 \log M(a,b)}{\partial b^2}$$

which equals

$$a\frac{\partial}{\partial a}\left(a\frac{\partial \log M(a,b)}{\partial a}\right) = b\frac{\partial}{\partial b}\left(b\frac{\partial \log M(a,b)}{\partial b}\right)$$

or

$$\frac{\partial^2 \log M(a,b)}{\partial (\log a)^2} = \frac{\partial^2 \log M(a,b)}{\partial (\log b)^2}$$

Again, a change in variable $a = e^u$ and $b = e^v$ is, functionally, more convenient. If we denote $T(u,v) = \log(M(e^u, e^v))$, then the above partial differential equation becomes

$$\frac{\partial^2 T(u,v)}{\partial u^2} - \frac{\partial^2 T(u,v)}{\partial v^2} = 0$$

A change in variable $w = iv$, then results in a Laplacian equation

$$\frac{\partial^2 T(u,w)}{\partial u^2} + \frac{\partial^2 T(u,w)}{\partial w^2} = 0$$

which implies that $T(u,w) = \log\left(M\left(e^u, e^{-iw}\right)\right)$ represents an analytic function in the complex plane $z = u + iw$.

21. *The constant* Δ. Rewriting (54) as $\frac{g_0}{c^2} = \frac{g_n}{2^n c_n^2}$ indicates that the ratio $\frac{g_n}{2^n c_n^2}$ is independent of n and, thus, a constant. Gauss [15, art. 13, p.379-380] defines $\Delta = \frac{g_0}{c^2} = \frac{1}{c^2} d\log\frac{a}{b}$ and with the definition of $g_n = d\log\frac{a_n}{b_n}$ in (48), he obtains

$$\Delta = \frac{1}{2^n c_n^2} d\log\frac{a_n}{b_n} = \frac{g_n}{2^n c_n^2}$$

The constant Δ will play a prominent role as shown below. From $a_n^2 = b_n^2 + c_n^2$, we find the differential $a_n da_n = b_n db_n + c_n dc_n$, which allows us to eliminate da_n in

$$\begin{aligned} d\log\frac{a_n}{b_n} &= \frac{da_n}{a_n} - \frac{db_n}{b_n} = \frac{b_n db_n + c_n dc_n}{a_n^2} - \frac{db_n}{b_n} \\ &= \left(\frac{b_n^2 - a_n^2}{a_n^2 b_n}\right) db_n + \frac{c_n}{a_n^2} dc_n = \frac{c_n^2}{a_n^2}\left(\frac{dc_n}{c_n} - \frac{db_n}{b_n}\right) \end{aligned}$$

and

$$d\log\frac{a_n}{b_n} = \frac{c_n^2}{a_n^2} d\log\frac{c_n}{b_n}$$

Similarly, we eliminate db_n in $d\log\frac{a_n}{b_n} = \frac{da_n}{a_n} - \frac{db_n}{b_n}$ and find

$$d\log\frac{a_n}{b_n} = \frac{c_n^2}{b_n^2} d\log\frac{c_n}{a_n}$$

In conclusion, the different representations lead, with $\Delta = \frac{1}{c^2} d\log\frac{a}{b}$, to

$$\Delta = \frac{1}{2^n c_n^2} d\log\frac{a_n}{b_n} = \frac{1}{2^n a_n^2} d\log\frac{c_n}{b_n} = \frac{1}{2^n b_n^2} d\log\frac{c_n}{a_n} \tag{58}$$

which holds for all integer values of n and $n = 0$ returns the definition $\Delta = \frac{1}{c^2} d\log\frac{a}{b}$. The second equality in (58) illustrates that a_n and c_n can be reversed, equivalent to the transform $(a_n, b_n, c_n) \to (c_n, b_n, a_n)$; the third equality justifies the reversal of a_n and b_n or the transform $(a_n, b_n, c_n) \to (b_n, a_n, c_n)$, while $\frac{1}{c_n^2} d\log\frac{a_n}{b_n} = \frac{1}{b_n^2} d\log\frac{c_n}{a_n}$ exhibits the circular transformation $(a_n, b_n, c_n) \to (c_n, a_n, b_n)$. The next circular transformation $(c_n, a_n, b_n) \to (b_n, c_n, a_n)$ additionally reverses the sign: $\frac{1}{b_n^2} d\log\frac{c_n}{a_n} = -\frac{1}{a_n^2} d\log\frac{b_n}{c_n}$ and similarly, also the following circular transformation $(b_n, c_n, a_n) \to (a_n, b_n, c_n)$ changes the sign.

22. *Deduction from the constant* Δ *in (58)*. We investigate the constant Δ in (58) for the forward AGM algorithm in (2) and the backward AGM algorithm in **art**. 6, which we denote by a_n', b_n' and c_n' with initial values $a_0' = a_0$, $b_0' = c_0$ and $c_0' = b_0$. Then (58) becomes $\Delta' = \frac{1}{(c_0')^2} d\log\frac{a_0'}{b_0'} = \frac{1}{2^n (a_n')^2} d\log\frac{c_n'}{b_n'}$ and

$$\frac{1}{(c_0')^2} d\log\frac{a_0'}{b_0'} = \frac{1}{b^2} d\log\frac{a}{c} = -\Delta$$

where the last equality follows from the last equality in (58). Hence, for constants α and β, we conclude that

$$\frac{1}{a_n^2} d\left(\frac{1}{2^n}\log\frac{\alpha c_n}{b_n}\right) + \frac{1}{(a_n')^2} d\left(\frac{1}{2^n}\log\frac{\beta c_n'}{b_n'}\right) = 0 \tag{59}$$

Multiplying the identity [18, p. 38, ex. 16]

$$\frac{1}{a_n^2} d\left(\frac{a_n}{a_n'}\right) + \frac{1}{(a_n')^2} d\left(\frac{a_n'}{a_n}\right) = 0$$

by a constant C_n that is independent of $\{a_n, b_n, c_n\}$ and $\{a'_n, b'_n, c'_n\}$ yields, after subtraction from (59),

$$\frac{1}{a_n^2} d\left(\frac{1}{2^n}\log\frac{\alpha c_n}{b_n} - C_n\frac{a_n}{a'_n}\right) + \frac{1}{(a'_n)^2} d\left(\frac{1}{2^n}\log\frac{\beta c'_n}{b'_n} - C_n\frac{a'_n}{a_n}\right) = 0$$

Since $a_n^2 > 0$ and $(a'_n)^2 > 0$, both differentials must be zero and we find that $\frac{1}{2^n}\log\frac{\alpha c_n}{b_n} - C_n\frac{a_n}{a'_n} = r$ and $\frac{1}{2^n}\log\frac{\beta c'_n}{b'_n} - C_n\frac{a'_n}{a_n} = r'$ hold for all n, which is only possible[17] for $r = r' = 0$; thus

$$\frac{1}{2^n}\log\frac{\alpha c_n}{b_n} = C_n\frac{a_n}{a'_n} \qquad \text{for all } n$$

By the AGM property (**art**. 6), we have that $\lim_{n\to\infty} a'_n = M(a_0, c_0) = 2^n M(a_n, c_n) = 2^n a_n M\left(1, \frac{c_n}{a_n}\right)$ as well as $a_n = b_n$ for $n \to \infty$ and

$$\begin{aligned}\lim_{n\to\infty} C_n &= \lim_{n\to\infty}\frac{a'_n}{2^n a_n}\log\frac{\alpha c_n}{b_n} = \lim_{n\to\infty} M\left(1, \frac{c_n}{a_n}\right)\log\frac{\alpha c_n}{b_n}\\ &= \lim_{n\to\infty} M\left(1, \frac{c_n}{a_n}\right)\log\frac{\alpha c_n}{a_n}\end{aligned}$$

With $M(1, w) = \frac{\pi}{2K\left(\sqrt{1-w^2}\right)}$ in (85) and let $w = \frac{c_n}{a_n}$, then $M\left(1, \frac{c_n}{a_n}\right) = \frac{\pi}{2}\frac{1}{K\left(\sqrt{1-\left(\frac{c_n}{a_n}\right)^2}\right)} = \frac{\pi}{2}\frac{1}{\log\left(4\frac{a_n}{c_n}\right)}$ for $c_n \to 0$, where the latter follows from (134). Thus,

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} M\left(1, \frac{c_n}{a_n}\right)\log\frac{\alpha c_n}{a_n} = \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{\log\left(4\frac{a_n}{c_n}\right)}\log\frac{\alpha c_n}{a_n} = -\frac{\pi}{2}$$

because we can choose the constant $\alpha = \frac{1}{4}$. Hence, we conclude that[18]

$$\lim_{n\to\infty}\frac{1}{2^n}\log\frac{4a_n}{c_n} = \frac{\pi}{2}\frac{a_n}{a'_n} \tag{60}$$

We use (60) in $\Delta = \frac{1}{b_n^2} d\left(\frac{1}{2^n}\log\frac{c_n}{a_n}\right)$ for large n,

$$\Delta = \lim_{n\to\infty}\frac{1}{b_n^2} d\left(\frac{1}{2^n}\log\frac{\alpha c_n}{a_n}\right) = -\frac{\pi}{2}\lim_{n\to\infty}\frac{1}{a_n^2} d\left(\frac{a_n}{a'_n}\right) = -\frac{\pi}{2}\lim_{n\to\infty}\frac{1}{a_n^2}\frac{a_n}{a'_n} d\left(\log\frac{a_n}{a'_n}\right)$$

and find [18, p. 39, ex. 17]

$$\Delta = \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{a_n a'_n} d\left(\log\frac{a'_n}{a_n}\right) \tag{61}$$

[17] If $n = 0$, we find $r = \log\frac{c}{b} - C_0$ and $r' = -\log\frac{c}{b} - C_0$ which holds for all b and c while C_0 does not change with b and c, which is impossible, unless $r = r' = 0$.

[18] We remark that the proof of (60) by the Borwein brothers [5, p. 357] is erroneous. They combine Gauss's fundamental result $\frac{\pi}{2} = M(a, b) I(a, b)$ in (30) with $I(a_n, b_n) = I(a, b)$ in **art**. 10 for any integer n, also for negative integers and the complementary AGM algorithm with (15),

$$\frac{\pi}{2} = \lim_{n\to\infty} a'_n I\left(a'_0, b'_0\right) = \lim_{n\to\infty} a'_n I\left(a'_{-n}, b'_{-n}\right) = \lim_{n\to\infty} a'_n I\left(2^n a_n, 2^n c_n\right)$$

The scaling property $M(a\beta, b\beta) = \beta M(a, b)$ in (11) leads to $\frac{\pi}{2} = \lim_{n\to\infty} 2^n a_n a'_n I\left(1, \frac{c_n}{a_n}\right)$, but they [5, p. 357] mention $\frac{\pi}{2} = \lim_{n\to\infty}\frac{2^{-n}}{a_n} a'_n I\left(1, \frac{c_n}{a_n}\right)$, from which, together with $K(k') = O\left(\log\frac{4}{k}\right)$ if $k \to 0$, their claim (60) should follow. However, we find Gauss's bounds in (122) more valuable.

23. *A second order differential.* Taking the differential $2^n c_n^2 = \frac{1}{\Delta} d\log\frac{a_n}{b_n}$ of in (58) is

$$\frac{1}{2}d\left(\frac{1}{\Delta}d\log\frac{a_n}{b_n}\right) = 2^n c_n dc_n = 2^n c_n^2 d\log c_n = \frac{1}{\Delta}d\log\frac{a_n}{b_n}d\log c_n$$

Similarly for the other two equalities in (58), we find

$$\begin{aligned}\frac{1}{2}d\left(\frac{1}{\Delta}d\log\frac{c_n}{b_n}\right) &= \frac{1}{\Delta}d\log\frac{c_n}{b_n}d\log a_n\\ \frac{1}{2}d\left(\frac{1}{\Delta}d\log\frac{c_n}{a_n}\right) &= \frac{1}{\Delta}d\log\frac{c_n}{a_n}d\log b_n\end{aligned}$$

We rewrite the first equation as

$$\frac{1}{2}d\left(\frac{1}{\Delta}d\log\frac{a_n}{b_n}\right) = \frac{1}{2}d\left(\frac{1}{\Delta}d\log\frac{a_nc_n}{b_nc_n}\right) = \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nc_n - \frac{1}{\Delta}d\log b_nc_n\right)$$

which leads to the set

$$\begin{cases}\frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nc_n\right) - \frac{1}{2}d\left(\frac{1}{\Delta}d\log b_nc_n\right) = \frac{1}{\Delta}d\log a_nd\log c_n - \frac{1}{\Delta}d\log b_nd\log c_n\\ \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nc_n\right) - \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nb_n\right) = \frac{1}{\Delta}d\log a_nd\log c_n - \frac{1}{\Delta}d\log a_nd\log b_n\\ \frac{1}{2}d\left(\frac{1}{\Delta}d\log b_nc_n\right) - \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nb_n\right) = \frac{1}{\Delta}d\log b_nd\log c_n - \frac{1}{\Delta}d\log a_nd\log b_n\end{cases}$$

Let $X = \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nb_n\right)$, $Y = \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nc_n\right)$ and $Z = \frac{1}{2}d\left(\frac{1}{\Delta}d\log b_nc_n\right)$, while $x = \frac{1}{\Delta}d\log a_nd\log b_n$, $y = \frac{1}{\Delta}d\log a_nd\log c_n$ and $z = \frac{1}{\Delta}d\log b_nd\log c_n$, then the above set becomes

$$\begin{cases}Y - Z = y - z\\ Y - X = y - x\\ Z - X = z - x\end{cases} \tag{62}$$

whose rank is 2, because subtracting any equation from another gives the third equation. The first two equations are solved in terms of Y as

$$\begin{aligned}Z &= z + (Y - y)\\ X &= x + (Y - y)\end{aligned}$$

After choosing $y - Y = D$ (to comply with Gauss in [15, art. 14, p.380-381]), we arrive at the symmetric solution $x - X = y - Y = z - Z = D$ for any real constant D. In summary, we find for any integer that

$$\begin{aligned}D &= \frac{1}{\Delta}d\log a_nd\log b_n - \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nb_n\right)\\ &= \frac{1}{\Delta}d\log a_nd\log c_n - \frac{1}{2}d\left(\frac{1}{\Delta}d\log a_nc_n\right)\\ &= \frac{1}{\Delta}d\log b_nd\log c_n - \frac{1}{2}d\left(\frac{1}{\Delta}d\log b_nc_n\right)\end{aligned} \tag{63}$$

Gauss has defined two constants Δ (the Greek D) and D. If $n \to \infty$, the first equation becomes

$$\begin{aligned}D &= \frac{1}{\Delta}\left(d\log M(a,b)\right)^2 - d\left(\frac{1}{\Delta}d\log M(a,b)\right)\\ &= \frac{dM(a,b)}{M(a,b)}\left(\frac{1}{\Delta}\frac{dM(a,b)}{M(a,b)}\right) - d\left(\frac{1}{\Delta}\frac{dM(a,b)}{M(a,b)}\right)\end{aligned}$$

After substitution of $\frac{dM(a,b)}{M(a,b)} = -M(a,b)\, d\left(\frac{1}{M(a,b)}\right)$ in the brackets, we have

$$\begin{aligned} D &= -dM(a,b)\left(\frac{1}{\Delta}d\left(\frac{1}{M(a,b)}\right)\right) + d\left(M(a,b)\frac{1}{\Delta}d\left(\frac{1}{M(a,b)}\right)\right) \\ &= M(a,b)\, d\left(\frac{1}{\Delta}d\left(\frac{1}{M(a,b)}\right)\right) \end{aligned} \tag{64}$$

For the backward algorithm, the first equation becomes $D = \frac{1}{\Delta'} d\log a'_n d\log b'_n - \frac{1}{2}d\left(\frac{1}{\Delta'}d\log a'_n b'_n\right)$, which reduces similarly with $\lim_{n\to\infty} a'_n = \lim_{n\to\infty} b'_n = M(a,c)$ to

$$D = M(a,c)\, d\left(\frac{1}{\Delta}d\left(\frac{1}{M(a,c)}\right)\right) \tag{65}$$

As an application of (64), consider the case where $a_0 = a$ is a constant and not variable, then, for $n = 0$,

$$D = \frac{1}{\Delta}d\log a d\log b - \frac{1}{2}d\left(\frac{1}{\Delta}d\log ab\right) = -\frac{1}{2}d\left(\frac{1}{\Delta}d\log b\right)$$

If a is a constant, then Δ in (58) reduces to $\Delta = -\frac{1}{c^2}d\log b = \frac{1}{a^2}d\log\frac{c}{b} = \frac{1}{b^2}d\log c$. With the first equality $\Delta = -\frac{1}{c^2}d\log b$, the constant $D = -\frac{1}{2}d\left(\frac{1}{\Delta}d\log b\right)$ becomes

$$D = -\frac{1}{2}d\left(\frac{c^2}{-d\log b}d\log b\right) = \frac{1}{2}d\left(c^2\right) = cdc = c^2 d\log c$$

Invoking the last equality $\Delta = \frac{1}{b^2}d\log c$ leads to $D = c^2b^2\Delta$, while the first equality $\Delta = -\frac{1}{c^2}d\log b$ gives $D = -b^2\log b$. If a is a constant, then (64) equals

$$D = \frac{M(a,b)}{a}d\left(\frac{1}{\Delta}d\left(\frac{a}{M(a,b)}\right)\right) = M\left(1,\frac{b}{a}\right)d\left(\frac{1}{\Delta}d\left(\frac{1}{M\left(1,\frac{b}{a}\right)}\right)\right)$$

Let $x = \frac{b}{a}$ and denote $y(x) = \frac{1}{M(1,x)}$, then

$$\begin{aligned} D &= yd\left(\frac{1}{\Delta}d\left(\frac{1}{y}\right)\right) = -yd\left(\frac{1}{\Delta}\frac{1}{y^2}dy\right) = -y\frac{1}{\Delta}d\left(\frac{1}{y^2}dy\right) - y\frac{1}{y^2}dyd\left(\frac{1}{\Delta}\right) \\ &= -\frac{1}{\Delta}\frac{1}{y}d^2y + 2y\frac{1}{\Delta}\frac{1}{y^3}(dy)^2 + \frac{1}{y\Delta^2}dyd\Delta \end{aligned}$$

Ignoring the second order differential $(dy)^2$ results in $D = \frac{1}{y\Delta}\left(-d^2y + dy\frac{d\Delta}{\Delta}\right)$. Combined with $D = c^2b^2\Delta$ leads to the differential equation

$$d^2y - \frac{d\Delta}{\Delta}dy + c^2b^2\Delta^2 y = 0 \tag{66}$$

Gauss does not relate this differential equation to the hypergeometric differential equation (47), but just tells that its solutions are $\frac{a}{M(a,b)}$ and $\frac{a}{M(a,c)}$. No interpretation of the coefficients $\frac{d\Delta}{\Delta}$ and[19] $c^2b^2\Delta^2$ are given.

[19] Gauss [15, art. 14, p.381] gives $-c^2b^2\nabla^2$ instead of our $c^2b^2\nabla^2$.

7 Towards Jacobi theta functions

24. *A variation of the second order differential.* Another observation of Gauss in [15, art. 15, p. 382] is more interesting. Consider, besides the variable a, b, c another variable h and

$$D_h = \frac{1}{\Delta}\left(d\log h\right)^2 + d\left(\frac{1}{\Delta}d\log h\right)$$

which is of similar form as the three relations for D in (63), when both appearing variables are the same, except for the plus sign. Gauss computes

$$\begin{aligned} D + D_h &= \frac{1}{\Delta}\left\{d\log a_n d\log b_n + \left(d\log h\right)^2\right\} - d\left(\frac{1}{\Delta}\frac{1}{2}d\log a_n b_n - \frac{1}{\Delta}d\log h\right) \\ &= \frac{1}{\Delta}\left\{d\log a_n d\log b_n + \left(d\log h\right)^2\right\} - d\left(\frac{1}{\Delta}d\log\sqrt{\frac{a_n b_n}{h^2}}\right) \end{aligned}$$

With $d\log\sqrt{\frac{a_n b_n}{h^2}} = d\log\sqrt{\frac{a_n b_n}{h^2}} = \frac{h}{\sqrt{a_n b_n}}d\sqrt{\frac{a_n b_n}{h^2}}$, we obtain

$$\begin{aligned} d\left(\frac{1}{\Delta}d\log\sqrt{\frac{a_n b_n}{h^2}}\right) &= d\left(\frac{1}{\Delta}\frac{h}{\sqrt{a_n b_n}}d\sqrt{\frac{a_n b_n}{h^2}}\right) = d\left(\sqrt{h^2 a_n b_n}\frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}\right) \\ &= \sqrt{h^2 a_n b_n}d\left(\frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}\right) + \frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}d\sqrt{h^2 a_n b_n} \end{aligned}$$

Further, simplifying the last term with

$$\begin{aligned} d\sqrt{\frac{a_n b_n}{h^2}} &= \frac{1}{2}\sqrt{\frac{b_n}{h^2 a_n}}da_n + \frac{1}{2}\sqrt{\frac{a_n}{h^2 b_n}}db_n - \sqrt{a_n b_n}\frac{dh}{h^2} \\ &= \frac{\sqrt{a_n b_n}}{h}\left(\frac{1}{2}d\log a_n b_n - d\log h\right) \end{aligned}$$

and with

$$\begin{aligned} d\sqrt{h^2 a_n b_n} &= \sqrt{a_n b_n}dh + \frac{1}{2}h\left(\sqrt{b_n}\frac{da_n}{\sqrt{a_n}} + \sqrt{a_n}\frac{db_n}{\sqrt{b_n}}\right) = \sqrt{a_n b_n}\left\{dh + \frac{1}{2}hd\log a_n b_n\right\} \\ &= h\sqrt{a_n b_n}\left\{\frac{1}{2}d\log a_n b_n + d\log h\right\} \end{aligned}$$

results in

$$\begin{aligned} \frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}d\sqrt{h^2 a_n b_n} &= \frac{1}{\Delta}\left\{\frac{1}{a_n b_n}\frac{\sqrt{a_n b_n}}{h}\left(\frac{1}{2}d\log a_n b_n - d\log h\right)h\sqrt{a_n b_n}\left(\frac{1}{2}d\log a_n b_n + d\log h\right)\right\} \\ &= \frac{1}{\Delta}\left(\frac{1}{4}\left(d\log a_n b_n\right)^2 - \left(d\log h\right)^2\right) \end{aligned}$$

Substituting all parts into $D + D_h$ yields

$$\begin{aligned} D + D_h &= -\sqrt{h^2 a_n b_n}d\left(\frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}\right) + \frac{1}{\Delta}\left\{d\log a_n d\log b_n + \left(d\log h\right)^2 - \frac{1}{a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}d\sqrt{h^2 a_n b_n}\right\} \\ &= -\sqrt{h^2 a_n b_n}d\left(\frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}\right) + \frac{1}{\Delta}\left\{d\log a_n d\log b_n + \left(d\log h\right)^2 - \frac{1}{4}\left(d\log a_n b_n\right)^2 - \left(d\log h\right)^2\right\} \\ &= -\sqrt{h^2 a_n b_n}d\left(\frac{1}{\Delta a_n b_n}d\sqrt{\frac{a_n b_n}{h^2}}\right) + \frac{1}{\Delta}\left\{d\log a_n d\log b_n - \frac{1}{4}\left(d\log a_n + d\log b_n\right)^2\right\} \end{aligned}$$

and

$$D+D_h=-\sqrt{h^2a_nb_n}d\left(\frac{1}{\Delta a_nb_n}d\sqrt{\frac{a_nb_n}{h^2}}\right)-\frac{1}{4\Delta}\left(d\log\frac{a_n}{b_n}\right)^2 \tag{67}$$

Gauss concludes that the second order differential only contains quotients of variables in the differentials. Moreover, $D+D_h$ does not depend upon n. Also, $D+D_h$ remains the same if the pair (a_n,b_n) is replaced by (a_n,c_n) or by (b_n,c_n) as follows from (63). If $h=a$ in (67) for $n=0$, then Gauss mentions that[20]

$$D+D_a=\Delta b^2c^2$$

while for $h=b$ and $h=c$, it holds that $D+D_b=-\Delta c^2a^2$ and $D+D_c=-\Delta a^2b^2$, respectively. If $n\to\infty$, then (67) becomes

$$D+D_h=-hM(a,b)\,d\left(\frac{1}{\Delta M^2(a,b)}d\frac{M(a,b)}{h}\right) \tag{68}$$

and, after replacing the pair (a,b) by (a,c), also $D+D_h=-hM(a,c)\,d\left(\frac{1}{\Delta M^2(a,c)}d\frac{M(a,c)}{h}\right)$.

From (58) and sufficiently large n, we find that

$$\Delta M^2(a,b)=a_n^2\Delta=\frac{1}{2^n}d\log\frac{c_n}{b_n}$$
$$\Delta M^2(a,b)=b_n^2\Delta=\frac{1}{2^n}d\log\frac{c_n}{a_n}$$

but $\Delta M^2(a,b)=\frac{1}{2^n}\frac{a_n^2}{c_n^2}d\log\frac{a_n}{b_n}$ is of a different form. Therefore, from (58) and $n=0$, it holds that

$$\begin{aligned}\Delta M^2(a,b)&=\left(\frac{M(a,b)}{c}\right)^2d\log\frac{a}{b}=\left(\frac{M(a,b)}{a}\right)^2d\log\frac{c}{b}=\left(\frac{M(a,b)}{b}\right)^2d\log\frac{c}{a}\\&=\left(\frac{M(a,b)}{c}\right)^2d\log\frac{\frac{a}{M(a,b)}}{\frac{b}{M(a,b)}}=\left(\frac{M(a,b)}{a}\right)^2d\log\frac{\frac{c}{M(a,b)}}{\frac{b}{M(a,b)}}=\left(\frac{M(a,b)}{b}\right)^2d\log\frac{\frac{c}{M(a,b)}}{\frac{a}{M(a,b)}}\end{aligned}$$

[20] After putting $h=a$ in (67) for $n=0$, we obtain

$$D+D_a=-a\sqrt{ab}d\left(\frac{1}{\Delta ab}d\sqrt{\frac{b}{a}}\right)-\frac{1}{4\Delta}\left(d\log\frac{a}{b}\right)^2$$

Since Δ in (58) reduces for $n=0$ to $\Delta=\frac{1}{c^2}d\log\frac{a}{b}$, we find that $D+D_a=-a\sqrt{ab}d\left(\frac{1}{\Delta ab}d\sqrt{\frac{b}{a}}\right)-\frac{c^4}{4}\Delta$. With $df^\beta=\beta f^\beta d\log f$ for any $\beta\neq1$, we have for $\beta=\frac{1}{2}$ that $d\sqrt{\frac{b}{a}}=\frac{1}{2}\sqrt{\frac{b}{a}}d\log\frac{b}{a}$ and

$$d\left(\frac{1}{\Delta ab}d\sqrt{\frac{b}{a}}\right)=\frac{1}{2}d\left(\frac{1}{\Delta ab}\sqrt{\frac{b}{a}}d\log\frac{b}{a}\right)=-\frac{1}{2}d\left(\frac{c^2}{a\sqrt{ab}}\right)=-\frac{1}{2}d\left(\frac{a^2-b^2}{a\sqrt{ab}}\right)=-\frac{1}{2}d\sqrt{\frac{a}{b}}+\frac{1}{2}d\left(\frac{b}{a}\right)^{\frac{3}{2}}$$

Again invoking $df^\beta=\beta f^\beta d\log f$,

$$\begin{aligned}d\left(\frac{1}{\Delta ab}d\sqrt{\frac{b}{a}}\right)&=\frac{1}{2}\left\{-\frac{1}{2}\sqrt{\frac{a}{b}}d\log\frac{a}{b}+\frac{3}{2}\left(\frac{b}{a}\right)^{\frac{3}{2}}d\log\frac{b}{a}\right\}=-\frac{1}{4}d\log\frac{a}{b}\left\{\sqrt{\frac{a}{b}}+3\left(\frac{b}{a}\right)^{\frac{3}{2}}\right\}\\&=-\frac{c^2\Delta}{4}\left\{\sqrt{\frac{a}{b}}+3\left(\frac{b}{a}\right)^{\frac{3}{2}}\right\}\end{aligned}$$

Substituted in $D+D_a$ yields, again using $c^2\Delta=d\log\frac{a}{b}$, $D+D_a=\frac{c^2\Delta}{4}\left(a\sqrt{ab}\sqrt{\frac{a}{b}}+3a\sqrt{ab}\sqrt{\frac{b}{a}}\frac{b}{a}-c^2\right)=\Delta b^2c^2$.

Without motivation where the squareroot comes from, Gauss defines in [15, art. 15, p. 382],

$$p=\sqrt{\tfrac{a}{M(a,b)}}\quad q=\sqrt{\tfrac{b}{M(a,b)}}\quad r=\sqrt{\tfrac{c}{M(a,b)}} \tag{69}$$

so that

$$\Delta M^2(a,b)=\frac{2}{r^4}d\log\frac{p}{q}=\frac{2}{p^4}d\log\frac{r}{q}=\frac{2}{q^4}d\log\frac{r}{p} \tag{70}$$

Since $a^2=b^2+c^2$ from (7), the definition (69) indicates that

$$p^4=q^4+r^4 \tag{71}$$

25. *A fourth definition besides (69).* Gauss defines in [15, art. 15, p. 382]

$$\log y=-\pi\frac{M(a,b)}{M(a,c)} \tag{72}$$

which equals $\log y=-\pi\frac{M(a_n,b_n)}{2^nM(a_n,c_n)}$ by (14). Unfortunately, Gauss does not motivate definition (72). Relation (43) is equivalent to $\frac{M(a_n,c_n)}{M(a_n,b_n)}\log\left(\frac{4a_n}{c_n}\right)\to\frac{\pi}{2}$ for large n and we write, for large n,

$$-2^n\log y=\pi\frac{M(a_n,b_n)}{M(a_n,c_n)}=\pi\frac{1}{\frac{M(a_n,b_n)}{2^nM(a_n,c_n)}}\frac{\log\left(\frac{4a_n}{c_n}\right)}{\log\left(\frac{4a_m}{c_n}\right)}$$

to obtain

$$-2^{n-1}\log y=\log\left(\frac{4a_n}{c_n}\right) \tag{73}$$

which is equivalent for large n and any real number k to

$$y^{-2^{n-k}}=\left(\frac{4a_n}{c_n}\right)^{2^{1-k}}$$

If $a>b>0$, then $\frac{a_n}{c_n}>0$ implies that $y>0$. Hence, for large n and choosing $k=2$, Gauss [15, art. 16, p. 383] concludes that

$$\frac{1}{2}y^{-2^{n-2}}\frac{\sqrt{c_n}}{\sqrt{M(a,b)}}\to 1$$

After taking the differential of both sides in (73),

$$d\log y=\frac{1}{2^{n-1}}d\log\left(\frac{c_n}{4a_n}\right)=\frac{1}{2^{n-1}}d\log\left(\frac{c_n}{a_n}\right)$$

and using $\Delta=\frac{1}{2^nb_n^2}d\log\frac{c_n}{a_n}$ in (58), we arrive, for large n, at

$$\Delta M^2(a,b)=\frac{1}{2}d\log y \tag{74}$$

which complements the equalities in (70) for p,q and r.

Returning to $D+D_h=-hM(a,b)\,d\left(\frac{1}{\Delta M^2(a,b)}d\frac{M(a,b)}{h}\right)$ in (68), choosing $h=a$ and invoking the definitions of p,q and r in (69), then $\frac{M(a,b)}{a}=\frac{1}{p^2}$ and $D+D_a=\Delta b^2c^2$ so that

$$\Delta=-\frac{aM(a,b)}{b^2c^2}d\left(\frac{2}{d\log y}d\frac{1}{p^2}\right)=-\frac{2p^2}{r^4q^4M^2(a,b)}d\left(\frac{1}{d\log y}d\frac{1}{p^2}\right)$$

Hence, we arrive at

$$\frac{1}{2}\Delta M^2(a,b) = -\frac{p^2}{r^4q^4}d\left(\frac{1}{d\log y}d\frac{1}{p^2}\right)$$

Similarly, after choosing $h=b$ and $h=c$ with the corresponding $D+D_b=-\Delta c^2a^2$ and $D+D_c=-\Delta c^2a^2$, we find that

$$\frac{1}{2}\Delta M^2(a,b) = \frac{q^2}{r^4p^4}d\left(\frac{1}{d\log y}d\frac{1}{q^2}\right) = \frac{r^2}{p^4q^4}d\left(\frac{1}{d\log y}d\frac{1}{r^2}\right)$$

Denote $U=d\left(\frac{1}{d\log y}d\frac{1}{p^2}\right)$, $V=d\left(\frac{1}{d\log y}d\frac{1}{q^2}\right)$ and $W=d\left(\frac{1}{d\log y}d\frac{1}{r^2}\right)$, then

$$\frac{1}{2}\Delta M^2(a,b) = -\frac{p^2}{r^4q^4}U = \frac{q^2}{r^4p^4}V = \frac{r^2}{p^4q^4}W \tag{75}$$

Gauss finds[21], after elimination of q and r, the differential equation for p (and similarly for q and r as well as backwards variants where $M(a,b)$ in the definition (69) is replaced by $M(a,c)$)

$$\left\{\frac{1}{p^{2?}}\frac{1}{d\log y}d\log\left(\frac{16}{p^{6?}}\frac{1}{d\log y}U\right)\right\}^2 - \frac{16}{p^{6?}}\frac{1}{d\log y}U - 1 = 0 \tag{76}$$

In other words, Gauss [15, art. 15, p. 382] has demonstrated that p, q and r are functions of y, actually of $\log y$.

26. *Series expansions of the functions $p(y)$, $q(y)$ and $r(y)$.* We deduced in **art**. 25 that the quantity $\frac{1}{2}\left(y^{-2^n}\right)^{\frac{1}{4}}\frac{\sqrt{c_n}}{\sqrt{M(a,b)}}\to 1$ with increasing integer n and that $r(y)=\frac{\sqrt{c}}{\sqrt{M(a,b)}}$ is a function of y, which suggests that $\frac{\sqrt{c_n}}{\sqrt{M(a,b)}}=r\left(y^{2^n}\right)$ for $|y|<1$ and $\frac{1}{2}\left(y^{-2^n}\right)^{\frac{1}{4}}r\left(y^{2^n}\right)\to 1$ for large n, which is equivalent to $\frac{1}{2}u^{-\frac{1}{4}}r(u)\to 1$ for $u\to 0$. Alternatively, the function $r(u)=2u^{\frac{1}{4}}g(u)$ if a real, positive u is sufficiently small and $\lim_{u\to 0}g(u)=1$. If we assume that $g(u)$ is an analytic function in some region around $u=0$, which implies that $g(u)=\sum_{k=0}^{\infty}g_ku^k$ possesses a Taylor series, convergent for $|u|<R$ with $g_0=1$, then the corresponding series expansion of $r(u)$ is

$$r(u) = 2u^{\frac{1}{4}}\left(1+\sum_{k=1}^{\infty}r_ku^k\right) \tag{77}$$

In this article **art**. 26, we will specify the Taylor coefficients r_k in (77), following the sketches of Gauss in [15, art. 16, p. 383].

The series $\sqrt{M(a,b)}=\sqrt{a_n}-\sum_{k=1}^{\infty}\sqrt{c_{n+2k}}$ in **art**. 15, rewritten for $n=0$ as

$$1 = \sqrt{\frac{a}{M(a,b)}} - \sum_{k=1}^{\infty}\sqrt{\frac{c_{2k}}{M(a,b)}}$$

becomes, with the definition $p=\sqrt{\frac{a}{M(a,b)}}$ in (69) and $\frac{\sqrt{c_n}}{\sqrt{M(a,b)}}=r\left(y^{2^n}\right)$,

$$p(y) = 1+\sum_{k=1}^{\infty}r\left(y^{2^{2k}}\right) = 1+r\left(y^4\right)+r\left(y^{16}\right)+r\left(y^{64}\right)+\cdots$$

[21] We have skipped the verification of this tedious calculation. Moreover, our copy of the Nachlass was insufficiently sharp to distinguish the powers of p in (76), whence, the ?.

The companion series $\sqrt{M(a,b)} = \sqrt{b_n} + \sqrt{c_{n+2}} - \sum_{k=2}^{\infty} \sqrt{c_{n+2k}}$ in **art**. 15 is similarly transformed into

$$q(y) = 1 - r\left(y^4\right) + \sum_{k=2}^{\infty} r\left(y^{2^{2k}}\right)$$

From $r^4(y) = p^4(y) - q^4(y)$ in (71) follows that

$$r^4(y) = \left(1 + r\left(y^4\right) + \sum_{k=2}^{\infty} r\left(y^{2^{2k}}\right)\right)^4 - \left(1 - r\left(y^4\right) + \sum_{k=2}^{\infty} r\left(y^{2^{2k}}\right)\right)^4$$

Denote $S = 1 + \sum_{k=2}^{\infty} r\left(y^{2^{2k}}\right)$, then

$$\begin{aligned} r^4(y) &= \left(r\left(y^4\right) + S\right)^4 - \left(S - r\left(y^4\right)\right)^4 \\ &= \left(\left(r\left(y^4\right) + S\right)^2 - \left(S - r\left(y^4\right)\right)^2\right)\left(\left(r\left(y^4\right) + S\right)^2 + \left(S - r\left(y^4\right)\right)^2\right) \end{aligned}$$

Simplifying, we arrive at the function equation for $r(y)$,

$$\frac{1}{8} r^4(y) = r\left(y^4\right) S\left(S^2 + r^2\left(y^4\right)\right) \tag{78}$$

It remains to introduce the series expansion (77) to develop both sides into a Taylor series around $y = 0$. Equating corresponding powers in y will result into equations that determine all Taylor coefficients $\{g_k\}_{k\geq 1}$ in (77). The computation is quite involved and deferred to appendix E, where we found that

$$r(y) = 2y^{\frac{1}{4}} \sum_{k=0}^{\infty} y^{\left(k+\frac{1}{2}\right)^2 - \frac{1}{4}} = 2\sum_{k=0}^{\infty} y^{\left(k+\frac{1}{2}\right)^2} \tag{79}$$

Armed with the explicit series (79), the Taylor series of $p(y)$ and $q(y)$ follows as

$$p(y) = 1 + \sum_{k=1}^{\infty} r\left(y^{2^{2k}}\right) = 1 + 2\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} y^{2^{2k}\left(j+\frac{1}{2}\right)^2} = 1 + 2\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} y^{\left(2^k\left(j+\frac{1}{2}\right)\right)^2}$$

Let $m = \left(2^k\left(j+\frac{1}{2}\right)\right)^2$, then $j = \frac{\sqrt{m}}{2^k} - \frac{1}{2} \in \mathbb{N}$ from which $\frac{\sqrt{m}}{2^k} - \frac{1}{2} \geq 0$ and $m \geq 2^{2k-2}$ and the double sum is

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} y^{\left(2^k\left(j+\frac{1}{2}\right)\right)^2} = \sum_{k=1}^{\infty}\sum_{m=2^{2(k-1)}}^{\infty} 1_{\left\{\frac{\sqrt{m}}{2^k} - \frac{1}{2} \in \mathbb{N}\right\}} y^m = \sum_{k=0}^{\infty}\sum_{m=2^{2k}}^{\infty} 1_{\left\{\frac{\sqrt{m}}{2^{k+1}} - \frac{1}{2} \in \mathbb{N}\right\}} y^m$$

After reversal of summations

$$\sum_{k=0}^{\infty}\sum_{m=2^{2k}}^{\infty} 1_{\left\{\frac{\sqrt{m}}{2^{k+1}} - \frac{1}{2} \in \mathbb{N}\right\}} y^m = \sum_{m=1}^{\infty}\left(\sum_{k=0}^{\left[\log_2\sqrt{m}\right]} 1_{\left\{\frac{\sqrt{m}}{2^{k+1}} - \frac{1}{2} \in \mathbb{N}\right\}}\right) y^m$$

we obtain[22] the Taylor series

$$p(y) = 1 + 2\sum_{m=1}^{\infty}\left(\sum_{k=0}^{\left[\log_2\sqrt{m}\right]} 1_{\left\{\frac{1}{2}\left(\frac{\sqrt{m}}{2^k} - 1\right) \in \mathbb{N}\right\}}\right) y^m = 1 + 2\sum_{m=1}^{\infty} 1_{\{m \text{ is a square}\}} y^m$$

[22] The sequence $m = \left(2^k\left(j+\frac{1}{2}\right)\right)^2$ for $k \geq 1$ and $j \geq 0$ only consists of squares. Since 2^k is never equal to $j + \frac{1}{2}$, all squares in the sequence are different and only appear once in the double sum. Numerical computation of $\sum_{k=0}^{\left[\log_2\sqrt{m}\right]} 1_{\left\{\frac{1}{2}\left(\frac{\sqrt{m}}{2^k} - 1\right) \in \mathbb{N}\right\}} = 1_{\{m \text{ is a square}\}}$ verfies the observation.

which is equivalent to the series

$$p(y) = 1 + 2\sum_{m=1}^{\infty} y^{m^2} = 1 + 2\left(y + y^4 + y^9 + y^{16} + \cdots\right)$$

The series for $q(y)$ follows similarly from $q(y) = 1 - r\left(y^4\right) + \sum_{k=2}^{\infty} r\left(y^{2^{2k}}\right)$ and (79) as

$$q(y) = 1 + 2\sum_{m=1}^{\infty} (-1)^m y^{m^2} = 1 - 2\left(y - y^4 + y^9 - y^{16} + \cdots\right)$$

27. *The functions $p(y)$, $q(y)$ and $r(y)$ are Jacobi theta functions.* It follows from the definition (72) that $y = \exp\left(-\pi\frac{M(a,b)}{M(a,c)}\right)$. Replacing y by $\mathfrak{q} = e^{i\pi\tau}$ and $\tau = i\frac{M(a,b)}{M(a,c)}$ with $\operatorname{Im}\tau > 0$, the three series are rewritten as

$$p(\mathfrak{q}) = \sqrt{\frac{a}{M(a,b)}} = 1 + 2\sum_{n=1}^{\infty} \mathfrak{q}^{n^2} = \vartheta_3(0,\tau) \tag{80}$$

$$q(\mathfrak{q}) = \sqrt{\frac{b}{M(a,b)}} = 1 + 2\sum_{n=1}^{\infty} (-1)^n \mathfrak{q}^{n^2} = \vartheta_4(0,\tau) \tag{81}$$

$$r(\mathfrak{q}) = \sqrt{\frac{c}{M(a,b)}} = 2\sum_{n=1}^{\infty} \mathfrak{q}^{\frac{(2n-1)^2}{4}} = \vartheta_2(0,\tau) \tag{82}$$

which are now called (**art**. 50 and **art**. 46) Jacobi theta functions $\vartheta_j(z;\tau)$ with $j = 1,2,3$ and 4, whose many fascinating properties and series expansion Gauss found via his sinus lemniscatus function (**art**. 38). Tannery and Molk [27, 28, 29, 30] cover very nicely the theory of elliptic functions and theta functions. Impressively, Cox [8, Section 2, pp. 283-309] mentions[23] that Gauss found the product expansions

$$\vartheta_3(0,\tau) = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{2n}\right)\left(1+\mathfrak{q}^{2n-1}\right)^2$$

$$\vartheta_4(0,\tau) = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{2n}\right)\left(1-\mathfrak{q}^{2n-1}\right)^2$$

$$\vartheta_2(0,\tau) = 2\mathfrak{q}^{\frac{1}{4}}\prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{2n}\right)\left(1+\mathfrak{q}^{2n}\right)^2$$

and the transformations of the theta functions, as $(-i\tau)^{\frac{1}{2}}\vartheta_3(z,\tau) = \exp\left(-i\frac{z^2}{\pi\tau}\right)\vartheta_3\left(-\frac{z}{\tau},-\frac{1}{\tau}\right)$ and similar formulae for the other $\vartheta_j(z;\tau)$, see [23, 20.7.30-33], which are examples of modular transformations. From the identities, known to Gauss,

$$\begin{cases} \vartheta_3^2(0,\tau) + \vartheta_4^2(0,\tau) = 2\vartheta_3^2(2\tau,0) \\ \vartheta_3(0,\tau)\,\vartheta_4(0,\tau) = \vartheta_4^2(0,2\tau) \end{cases} \tag{83}$$

we recognize that $\vartheta_3^2(2\tau,0) = \frac{\vartheta_3^2(0,\tau)+\vartheta_4^2(0,\tau)}{2}$ is the arithmetic mean and $\vartheta_4^2(0,2\tau) = \sqrt{\vartheta_3^2(0,\tau)\,\vartheta_4^2(0,\tau)}$ is the corresponding geometric mean. Thus, the AGM algorithm in (2) enables the computation of

[23] The product expansions do not appear in Gauss Werke, band 3, pp. 361-403 that covers the Arithmetic Geometric Mean in the Nachlass, to which I have limited myself.

theta functions (see also [1, 16.32]). If $\mu = M(a,b)$, then $a_n = \mu\vartheta_3^2(0,2^n\tau)$ and $b_n = \mu\vartheta_4^2(0,2^n\tau)$ for $n \geq 0$ satisfies[24] the AGM algorithm in (2), because $e^{\pi i 2^n\tau} \to \infty$ for $n \to \infty$ so that $\lim_{n\to\infty}\vartheta_3^2(0,2^n\tau) = \lim_{n\to\infty}\vartheta_4^2(0,2^n\tau) = 1$. Hence, every solution τ of the complementary modulus $k'(\tau) = \frac{b}{a}$ gives a value $\mu = \frac{a}{\vartheta_3^2(0,\tau)}$ of the arithmetic-geometric mean $M(a,b)$. Finding all solutions of $k'(\tau) = \frac{b}{a}$, that satisfies the functional equation $k'(\tau) = -k'\left(\frac{\tau}{2\tau+1}\right)$, relies on the admissible region in the complex plane of the modular function $k'(\tau)$.

8 Legendre's elliptic integrals and the AGM

28. *Complete elliptic integrals and AGM.* We express the integral $I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$ in (29) in terms of the complete elliptic integral $K(k) = F\left(\frac{\pi}{2},k\right)$ of the first kind, defined in Appendix B. By convention, we assume that $a \geq b$. With

$$a^2\cos^2\theta + b^2\sin^2\theta = a^2 + \left(b^2 - a^2\right)\sin^2\theta = a^2\left(1 - \frac{a^2-b^2}{a^2}\sin^2\theta\right)$$

we find, with $k^2 = \frac{a^2-b^2}{a^2} = \frac{c^2}{a^2} > 0$, that

$$I(a,b) = \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{a}K(k) = \frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right)$$

Gauss's fundamental integral (28) is rewritten in terms of the complete elliptic integral $K(k)$ as

$$\frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right) = \frac{\pi}{2}\frac{1}{M(a,b)} \tag{84}$$

or, with $w = \frac{b}{a} \leq 1$,

$$\frac{1}{M(1,w)} = \frac{2}{\pi}K\left(\sqrt{1-w^2}\right) \tag{85}$$

Alternatively, with $a = 1+x$ and $b = 1-x$, we have

$$\frac{1}{M(1+x,1-x)} = \frac{2}{\pi(1+x)}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\left(\frac{2\sqrt{x}}{1+x}\right)^2\sin^2\theta}} = \frac{2}{\pi(1+x)}K\left(\frac{2\sqrt{x}}{1+x}\right)$$

The property $M(a,b) = M(a_1,b_1)$ indicates that $M(1+x,1-x) = M\left(1,\sqrt{1-x^2}\right)$ and scaling $M(ac,bc) = cM(a,b)$ in (11) that $M(1+x,1-x) = (1+x)M\left(1,\frac{1-x}{1+x}\right)$. Combining both properties of the arithmetic-geometric mean $M(a,b)$ with the above complete elliptic integral $K(k)$ leads to

$$\frac{1}{M\left(1,\frac{1-x}{1+x}\right)} = \frac{2}{\pi}K\left(\frac{2\sqrt{x}}{1+x}\right)$$

[24] Tannery and Molk [30, p. 270-273] show that, with $a_0 = \vartheta_3^2(0,\tau)$ and $b_0 = \vartheta_4^2(0,\tau)$, it holds that $c_0 = \vartheta_2^2(0,\tau)$. Iteration of the AGM algorithm indicates that

$$a_n = \vartheta_3^2(0,2^n\tau) \quad b_n = \vartheta_4^2(0,2^n\tau) \quad c_n = \vartheta_2^2(0,2^n\tau)$$

Let $y=\frac{1-x}{1+x}$, which is an instance of the univalent and conformal Möbius transform $w=\frac{az+b}{cz+d}$, then the inverse Möbius transform is $x=\frac{1-y}{1+y}$ and $\frac{2\sqrt{x}}{1+x}=\sqrt{1-y^2}$ so that $\frac{1}{M(1,y)}=\frac{2}{\pi}K\left(\sqrt{1-y^2}\right)$, which is (85), and

$$\frac{1}{M\left(1+x,1-x\right)}=\frac{1}{M\left(1,\sqrt{1-x^2}\right)}=\frac{2}{\pi}K\left(x\right)$$

Equating $M\left(1,\sqrt{1-x^2}\right)=\left(1+x\right)M\left(1,\frac{1-x}{1+x}\right)$ establishes the *Landen transformation (1775)*

$$K\left(x\right)=\frac{1}{1+x}K\left(\frac{2\sqrt{x}}{1+x}\right) \tag{86}$$

Fig. 2 draws $\frac{1}{M(1,x)}=\frac{2}{\pi}K\left(\sqrt{1-x^2}\right)$ in (85) and of $\frac{1}{M(1+x,1-x)}=\frac{2}{\pi}K\left(x\right)$ as a function of $x\in[0,1]$.

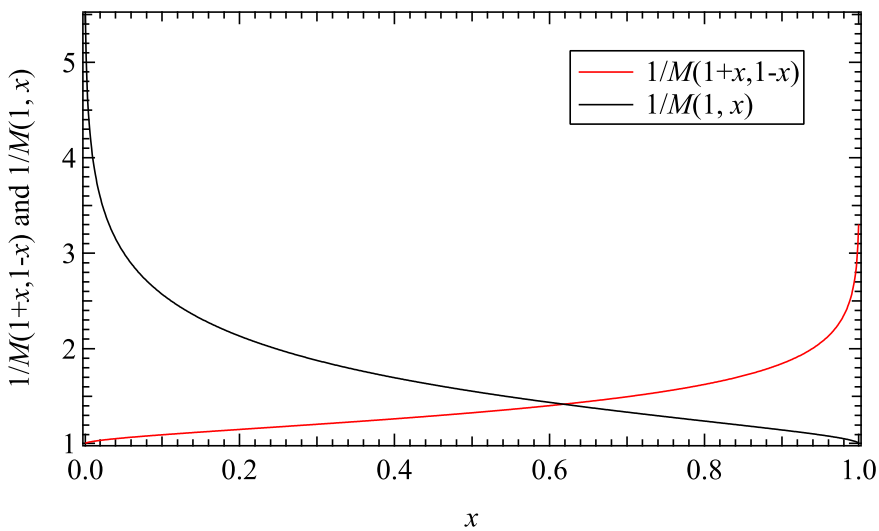

Figure 2: Plots of $\frac{1}{M(1,x)}=\frac{2}{\pi}K\left(\sqrt{1-x^2}\right)$ in (85) and of $\frac{1}{M(1+x,1-x)}=\frac{2}{\pi}K\left(x\right)$.

29. *AGM expansion of the complete elliptic integral* $K\left(x\right)$. The Mobius transformation $y=\frac{1-x}{1+x}$ has a similar form as its inverse $x=\frac{1-y}{1+y}$. We rewrite the Landen transformation (86) in terms[25] of $y=\frac{1-x}{1+x}$ as

$$\left(1+\frac{1-y}{1+y}\right)K\left(\frac{1-y}{1+y}\right)=K\left(\sqrt{1-y^2}\right)$$

[25] Gauss's fundamental relation $\frac{2}{\pi a}K\left(\frac{c}{a}\right)=\frac{1}{M(a,b)}$ in (84) with the definition $c_n=\sqrt{a_n^2-b_n^2}$ and property (6) point to

$$\frac{1}{a}K\left(\frac{c}{a}\right)=\frac{1}{a_1}K\left(\frac{c_1}{a_1}\right)=\cdots=\frac{1}{a_n}K\left(\frac{c_n}{a_n}\right)$$

and writing the first equality explicitly yields

$$\frac{1}{2}\left(1+\frac{b}{a}\right)K\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)=K\left(\frac{a-b}{a+b}\right)=K\left(\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\right)$$

illustrating that $y=\frac{b}{a}$.

Replace $w=\sqrt{1-y^2}$ or $y=\sqrt{1-w^2}$, then a rewritten form of the Landen transformation is

$$K\left(w\right)=\left(1+\frac{1-\sqrt{1-w^2}}{1+\sqrt{1-w^2}}\right)K\left(\frac{1-\sqrt{1-w^2}}{1+\sqrt{1-w^2}}\right) \tag{87}$$

We will first iterate (87) a few times. Let $w=\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$ in the left-hand side of (87), then quantities in the right-hand side of (87) are

$$\begin{aligned}\sqrt{1-w^2}&=\sqrt{1-\left(\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)^2}=\sqrt{\frac{\left(1+\sqrt{1-x^2}\right)^2-\left(1-\sqrt{1-x^2}\right)^2}{\left(1+\sqrt{1-x^2}\right)^2}}\\&=\frac{1}{1+\sqrt{1-x^2}}\sqrt{\left(1+\sqrt{1-x^2}+1-\sqrt{1-x^2}\right)\left(1+\sqrt{1-x^2}-1+\sqrt{1-x^2}\right)}\\&=\frac{2\sqrt{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}}\end{aligned}$$

and

$$\frac{1-\sqrt{1-w^2}}{1+\sqrt{1-w^2}}=\frac{1-\frac{2\sqrt{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}}}{1+\frac{2\sqrt{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}}}=\frac{1+\sqrt{1-x^2}-2\sqrt{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}+2\sqrt{\sqrt{1-x^2}}}=\frac{\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2}$$

Substituted into (87)

$$K\left(\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)=\left(1+\frac{\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2}\right)K\left(\frac{\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2}\right)$$

Thus, after replacing x by w in the above, the first iteration of (87) is

$$K\left(w\right)=\frac{2}{1+\sqrt{1-w^2}}\left(1+\frac{\left(1-\left(1-w^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-w^2\right)^{\frac{1}{4}}\right)^2}\right)K\left(\frac{\left(1-\left(1-w^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-w^2\right)^{\frac{1}{4}}\right)^2}\right)$$

For $0<A<1$, it holds that $\frac{1-A}{1+A}>\frac{1-\sqrt{A}}{1+\sqrt{A}}$, because $\sqrt{A}>A$, and thus that $\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}>\frac{\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2}$.

The next iteration of (87) is

$$\frac{1-\sqrt{1-\frac{\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^4}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^4}}}{1+\sqrt{1--\frac{\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^4}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^4}}}=\frac{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2-\sqrt{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^4-\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^4}}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2+\sqrt{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^4-\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^4}}$$

With $A^2-B^2=\left(A+B\right)\left(A-B\right)$, we have

$$\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^4-\left(1-\left(1-x^2\right)^{\frac{1}{4}}\right)^4=8\left(1+\left(1-x^2\right)^{\frac{1}{2}}\right)\left(1-x^2\right)^{\frac{1}{4}}$$

and

$$x_3 = \frac{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2 - 2\sqrt{2}\left(1-x^2\right)^{\frac{1}{8}}\sqrt{1+\left(1-x^2\right)^{\frac{1}{2}}}}{\left(1+\left(1-x^2\right)^{\frac{1}{4}}\right)^2 + 2\sqrt{2}\left(1-x^2\right)^{\frac{1}{8}}\sqrt{1+\left(1-x^2\right)^{\frac{1}{2}}}}$$

After replacing x by w in the above, the second iteration of (87) is

$$K(w) = \frac{4}{1+\sqrt{1-w^2}} \frac{\left(1+\left(1-w^2\right)^{\frac{1}{4}}\right)^2 + \left(1-\left(1-w^2\right)^{\frac{1}{4}}\right)^2}{\left(1+\left(1-w^2\right)^{\frac{1}{4}}\right)^2 + \sqrt{8}\left(1-w^2\right)^{\frac{1}{8}}\sqrt{1+\left(1-w^2\right)^{\frac{1}{2}}}}$$
$$\times K\left(\frac{\left(1+\left(1-w^2\right)^{\frac{1}{4}}\right)^2 - 2\sqrt{2}\left(1-w^2\right)^{\frac{1}{8}}\sqrt{1+\left(1-w^2\right)^{\frac{1}{2}}}}{\left(1+\left(1-w^2\right)^{\frac{1}{4}}\right)^2 + 2\sqrt{2}\left(1-w^2\right)^{\frac{1}{8}}\sqrt{1+\left(1-w^2\right)^{\frac{1}{2}}}}\right) \tag{88}$$

Instead of computing a next iteration, we observe that the iterative structure in the integer $n \geq 0 \geq$ is, for $0 < x_n < 1$,

$$x_{n+1} = \frac{1-\sqrt{1-x_n^2}}{1+\sqrt{1-x_n^2}} \qquad \text{with } x_0 = w \tag{89}$$

which is related to the AGM algorithm (2). We concentrate on iteration (89) and rewrite

$$x_{n+1} = \frac{1-\sqrt{1-x_n^2}}{1+\sqrt{1-x_n^2}} = \frac{1}{x_n^2}\left(1-\sqrt{1-x_n^2}\right)^2 = \frac{1}{x_n^2}\left(2-2\sqrt{1-x_n^2}-x_n^2\right) = \frac{2}{x_n^2}\left(1-\sqrt{1-x_n^2}\right)-1$$

The first and last equality lead to

$$\frac{x_n^2}{2}\left(1+x_{n+1}\right) = \left(1-\sqrt{1-x_n^2}\right)$$

while the first and second equality indicate that

$$x_{n+1} = \frac{1}{x_n^2}\left(1-\sqrt{1-x_n^2}\right)^2$$

Substitution of the first into the latter gives us

$$x_{n+1} = \frac{1}{x_n^2}\left(\frac{x_n^2}{2}\left(1+x_{n+1}\right)\right)^2 = \frac{x_n^2}{4}\left(1+x_{n+1}\right)^2$$

from which the lower bound $x_{n+1} \geq \frac{x_n^2}{4}$ follows and from which we find the inverse of (89)

$$\frac{2\sqrt{x_{n+1}}}{1+x_{n+1}} = x_n$$

Let $x = x_{n+1}$ in Landen's transformation (86) and using $x_n = \frac{2\sqrt{x_{n+1}}}{1+x_{n+1}}$ leads to the recursion

$$K(x_n) = \left(1+x_{n+1}\right)K\left(x_{n+1}\right) \tag{90}$$

Since $0 < x_n < 1$ for all n, it holds that $x_n = \frac{2\sqrt{x_{n+1}}}{1+x_{n+1}} > \sqrt{x_{n+1}}$ and $x_n^2 > x_{n+1}$. In summary, we have shown that $\frac{x_n^2}{4} < x_{n+1} < x_n^2 < x_n$ for $0 < x_n < 1$; in other words, the subsequent iterates x_n decrease with n. Iterating (90) p times

$$K(x_n) = \left(1+x_{n+1}\right)\left(1+x_{n+2}\right)\ldots\left(1+x_{n+p}\right)K\left(x_{n+p}\right)$$

shows, with $x_0 = w$ and choosing $n = 0$, that

$$K(w) = K(x_p) \prod_{j=1}^{p} (1 + x_j)$$

The $\lim_{p\to\infty} K(x_p) = K(0) = \frac{\pi}{2}$ finally leads to the AGM-expansion of the complete elliptic integral

$$K(w) = \frac{\pi}{2} \lim_{p\to\infty} \prod_{j=1}^{p} (1 + x_j) \tag{91}$$

which converges very fast and any truncation of p provides a lower bound. With $p = 5$ in (91), we found 45 decimals accurate for $w = 3/4$ and 33 decimals for $w = 7/8$, but more than 80 decimals for $w = 1/8$. The explicit product in (88) with $p = 3$ has 28 decimals accurate for $w = 1/8$ and 8 decimals for $w = 7/8$.

The Landen transformation is a special case [2, p. 590] of

$$(1 + x) \int_0^{\alpha} \frac{d\varphi}{\sqrt{1 - x^2 \sin^2 \varphi}} = 2 \int_0^{\beta} \frac{d\varphi}{\sqrt{1 - \frac{4x}{(1+x)^2} \sin^2 \varphi}} \qquad \text{with } x \sin \alpha = \sin(2\beta - \alpha)$$

for $\alpha = \pi$ and $\beta = \frac{\pi}{2}$. Almkvist and Berndt [2, p. 590] present the approach due to Landen, who substituted

$$\tan \varphi = \frac{\sin(2\varphi)}{x_1 + \cos(2\varphi)}$$

in the complete elliptic integral of the first kind $K(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$ to obtain, after tedious manipulations, (87). The Landen transformation is now of historical interest, because it directly follows from transformations connecting Jacobi theta functions [35, p. 476].

30. *Trigonometric form of the Landen recursion (89).* We start from the recursion $x_{n+1} = \frac{1-\sqrt{1-x_n^2}}{1+\sqrt{1-x_n^2}}$ in (89) and let $x_n = \sin \theta_n$, then

$$\sin \theta_{n+1} = \frac{1 - \sqrt{1 - \sin^2 \theta_n}}{1 + \sqrt{1 - \sin^2 \theta_n}} = \frac{1 - \cos \theta_n}{1 + \cos \theta_n} = \frac{1 - \cos^2 \frac{\theta_n}{2} + \sin^2 \frac{\theta_n}{2}}{1 + \cos^2 \frac{\theta_n}{2} - \sin^2 \frac{\theta_n}{2}} = \frac{2 \sin^2 \frac{\theta_n}{2}}{2 \cos^2 \frac{\theta_n}{2}}$$

which leads to the recursion [18, ex. 9, p. 38]

$$\sin \theta_{n+1} = \tan^2 \frac{\theta_n}{2} \tag{92}$$

Since $\frac{1}{a} K\left(\frac{c}{a}\right) = \frac{\pi}{2} \frac{1}{M(a,b)}$ in (84) and $M(a, b) = M(a_n, b_n)$, we have $\frac{1}{a_n} K\left(\frac{c_n}{a_n}\right) = \frac{\pi}{2} \frac{1}{M(a_n,b_n)}$ and $x_n = \frac{c_n}{a_n} = \sin \theta_n$, from which $\frac{c_n}{a_n} = \sqrt{1 - \left(\frac{b_n}{a_n}\right)^2}$ and $\cos \theta_n = \frac{b_n}{a_n}$. By combining $c_n = a_n \sin \theta_n$ and $a_{n+1} = \frac{a_n+b_n}{2}$, we deduce that

$$\begin{aligned} c_{n+1} = a_{n+1} \sin \theta_{n+1} &= \frac{a_n + b_n}{2} \tan^2 \frac{\theta_n}{2} \\ &= \frac{a_n}{2} (1 + \cos \theta_n) \tan^2 \frac{\theta_n}{2} = a_n \cos^2 \frac{\theta_n}{2} \frac{\sin \frac{\theta_n}{2}}{\cos \frac{\theta_n}{2}} \tan \frac{\theta_n}{2} = \frac{a_n}{2} \sin \theta_n \tan \frac{\theta_n}{2} \end{aligned}$$

and

$$c_{n+1} = \frac{c_n}{2} \tan \frac{\theta_n}{2} \tag{93}$$

After p-fold iteration, we find that $c_n = \frac{1}{2^p} c_{n-p} \prod_{j=n-p}^{n-1} \tan \frac{\theta_j}{2}$. If $p = n$, then $c_0 = c = a \sin \theta_0 = a x_0$ and $x_0 = \frac{k}{a}$ (or requiring that $a = 1$), so that

$$c_n = \frac{k}{2^n} \prod_{j=0}^{n-1} \tan \frac{\theta_j}{2}$$

By squaring the recursion (93), we obtain $c_{n+1}^2 = \frac{c_n^2}{4} \tan^2 \frac{\theta_n}{2}$, while recursion (92) indicates that $c_{n+1}^2 = \frac{c_n^2}{4} \sin \theta_{n+1}$. Iterating p times, we find $c_n^2 = \frac{1}{4^p} c_{n-p}^2 \prod_{j=n-p+1}^{n} \sin \theta_j$ and choosing $p = n$, with $c_0 = k$, we arrive at

$$c_n^2 = \frac{k^2}{4^n} \prod_{j=1}^{n} \sin \theta_j$$

Taking the logarithm of the AGM expansion (91) of the complete elliptic integral $K(k)$ yields [18, ex. 9 (ii), p. 36], with $x_n = \sin \theta_n$ and (92),

$$\log \left(\frac{2}{\pi} K(k) \right) = \sum_{j=1}^{\infty} \log (1 + \sin \theta_j) = \sum_{j=1}^{\infty} \log \left(1 + \tan^2 \frac{\theta_{j-1}}{2} \right) = 2 \sum_{j=1}^{\infty} \log \left(\sec \frac{\theta_{j-1}}{2} \right)$$

We translate $\frac{\pi}{2} \frac{M(a,b)}{M(a,c)} = \frac{1}{2^m} \log \left(\frac{4a_m}{c_m} \right) - \sum_{l=m}^{\infty} \frac{1}{2^l} \log \left(\frac{a_l}{a_{l+1}} \right)$ in (40) with $c_m = a_m \sin \theta_m$ and with (12), $\frac{a_l}{a_{l+1}} = \frac{a_{l+1} + c_{l+1}}{a_{l+1}} = 1 + \sin \theta_{l+1}$, to

$$\frac{\pi}{2} \frac{M(a,b)}{M(a,c)} = \frac{1}{2^m} \log \left(\frac{4}{\sin \theta_m} \right) - \sum_{l=m}^{\infty} \frac{1}{2^l} \log (1 + \sin \theta_{l+1})$$

which reduces, with $\frac{M(a,b)}{M(a,c)} = \frac{K(k')}{K(k)}$ and $m = 1$, to [18, ex. 9 (iv), p. 36].

31. *The complete elliptic integral* $E(k)$. The integral associated to $I(a,b)$ in (29) is

$$J(a,b) = \int_0^{\frac{\pi}{2}} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta \tag{94}$$

which is written, similarly as in **art**. 28 after observing that $J(a,b) = J(b,a)$, in terms of the complete elliptic integral $E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta$ as

$$J(a,b) = a \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{c^2}{a^2} \sin^2 \theta} d\theta = a E \left(\frac{c}{a} \right)$$

where $c = \sqrt{a^2 - b^2}$. An important relation, assuming as before that $a > b > 0$ and $c_n^2 = a_n^2 - b_n^2$, is

$$J(a,b) = \left(a^2 - \frac{1}{2} \sum_{n=0}^{\infty} 2^n c_n^2 \right) I(a,b) \tag{95}$$

which is a special case[26] of (173) in **art**. 57. We write (95) in terms of Legendre's complete elliptic integrals $K(k)$ and $E(k)$ with $k = \frac{c}{a}$ as

$$\frac{K(k) - E(k)}{K(k)} = \frac{1}{2a^2} \sum_{n=0}^{\infty} 2^n c_n^2 \tag{96}$$

[26]The proof of (173) in **art**. 57 has cost us a considerable amount of time. Almkvist and Berndt [2, Theorem 4] omit the proof and refer to King [18, pp. 7- 8], who merely states formulae, most of them without demonstrations, forcing an interesting reader to repeat his work.

The right-hand series in (96) converges extremely rapidly; a second order convergence due to the AGM algorithm (2).

The recursion $c_{n+1} = \frac{c_n^2}{4a_{n+1}}$ in (8) leads to

$$c_n^2 + 2c_{n+1}^2 = 4a_{n+1}c_{n+1} + 2c_{n+1}^2 = 2c_{n+1}\left(2a_{n+1} + c_{n+1}\right) = 2c_{n+1}\left(a_{n+1} + \frac{1}{2}\left(a_n + b_n\right) + c_{n+1}\right)$$

and with $c_{n+1} = \frac{a_n - b_n}{2}$, we arrive at $c_n^2 + 2c_{n+1}^2 = 2c_{n+1}\left(a_n + a_{n+1}\right)$. We employ that relation in

$$\sum_{n=0}^{\infty} 2^n c_n^2 = \sum_{n=0}^{\infty} 2^{2n} c_{2n}^2 + \sum_{n=0}^{\infty} 2^{2n+1} c_{2n+1}^2 = \sum_{n=0}^{\infty} 2^{2n}\left(c_{2n}^2 + 2c_{2n+1}^2\right) = 2\sum_{n=0}^{\infty} 2^{2n}\left(c_{2n+1}\left(a_{2n} + a_{2n+1}\right)\right)$$

so that (96) becomes [18, ex. 3, p. 35]

$$\frac{K(k) - E(k)}{K(k)} = \sum_{n=0}^{\infty} 2^{2n}\left(a_{2n} + a_{2n+1}\right) c_{2n+1} \tag{97}$$

32. *Legendre's formula.* Legendre [19, p. 61] has demonstrated for $0 < k < 1$ and $k' = \sqrt{1-k^2}$ that

$$K(k)\,E\left(k'\right) + K\left(k'\right)E(k) - K(k)\,K\left(k'\right) = \frac{\pi}{2} \tag{98}$$

Almkvist and Berndt [2, Theorem 3] present a "proof that appears not to have been, heretofore, given", but their proof is similar to that of Legendre, who uses differentials instead of integrals. Whittaker and Watson [35, p. 520] give a three line proof of Legendre's formula (98), based on elliptic functions; they also mention an analogous result of Legendre's formula (98) in Weierstrass's theory.

We follow the proof of Almkvist and Berndt, but generalize the derivations to the incomplete elliptic integrals $F(\varphi, k)$ and $E(\varphi, k)$. Since their proof of (98) is rather technical and less illuminating, we have placed it in Appendix D.

33. *Proof of Legendre's formula (98) via the differential $dM(a,b)$.* We derive additional relations of the differential $dM(a,b)$ in **art**. 19, that appear in [15, p. 380], from which Legendre's formula (98) follows, as first noted by King [18, ex. 18, p. 39].

The recursion $f_n = f_{n-1} + g_n$ in (51) maps, with the definition $f_n = d\log a_n b_n$ and $g_n = d\log\frac{a_n}{b_n}$ and the first equality of Δ in (58), to

$$d\log\left(a_{n-1}b_{n-1}\right) = d\log\left(a_n b_n\right) - 2^n c_n^2 \Delta$$

and (55) is written as

$$d\log\left(a_{n+p}b_{n+p}\right) = d\log\left(a_n b_n\right) + \Delta\sum_{k=n+1}^{n+p} 2^k c_k^2 \tag{99}$$

The limit $\lim_{p\to\infty} d\log\left(a_{n+p}b_{n+p}\right) = 2d\left(\log\left(M(a,b)\right)\right)$, it holds [15, p. 380] that

$$2d\left(\log\left(M(a,b)\right)\right) = d\log\left(a_n b_n\right) + \Delta\sum_{k=n+1}^{\infty} 2^k c_k^2 \tag{100}$$

On the other hand, for $n = 0$ in (99), we find for any integer p that

$$d\log\left(a_p b_p\right) = d\log\left(a_0 b_0\right) + \Delta\sum_{k=1}^{p} 2^k c_k^2$$

Since the differential computation in **art**. 19 holds for any integer $n \in \mathbb{Z}$, the complementary AGM algorithm (a'_n, b'_n, c'_n) indicates that $d\log\left(a'_{n-1}b'_{n-1}\right) = d\log\left(a'_n b'_n\right) - 2^n\left(c'_n\right)^2\Delta'$. With (15), we have $\Delta' = \frac{1}{(c')^2} d\log\frac{a'}{b'} = \frac{1}{b^2} d\log\frac{a}{c} = -\Delta$ with (58) and with $d\log\left(a'_n b'_n\right) = d\log\left(2^{-2n}a_{-n}c_{-n}\right) = d\log\left(a_{-n}c_{-n}\right)$ that

$$d\log\left(a_{-n+1}c_{-n+1}\right) = d\log\left(a_{-n}c_{-n}\right) - 2^{-n}b_{-n}^2\Delta$$

From $2d\left(\log\left(M\left(a',b'\right)\right)\right) = d\log\left(a'_n b'_n\right) + \Delta'\sum_{k=n+1}^{\infty} 2^k\left(c'_k\right)^2$ in (100), we obtain with (15)

$$2d\left(\log\left(M\left(a,c\right)\right)\right) = d\log\left(a_{-n}c_{-n}\right) - \Delta\sum_{k=n+1}^{\infty} 2^{-k}b_{-k}^2 \tag{101}$$

while the finite p variant becomes

$$d\log\left(a'_p b'_p\right) = d\log\left(a'_0 b'_0\right) - \Delta\sum_{k=1}^{p} 2^k\left(c'_k\right)^2$$

First, we subtract the finite p variants,

$$d\log\left(\frac{a_p b_p}{a'_p b'_p}\right) = d\log\left(\frac{a_0 b_0}{a'_0 b'_0}\right) + \Delta\left(\sum_{k=1}^{p} 2^k c_k^2 + \sum_{k=1}^{p} 2^k\left(c'_k\right)^2\right) \tag{102}$$

Since $a_0 = a'_0$ and $b'_0 = c_0$, it holds that $d\log\left(\frac{a_0 b_0}{a'_0 b'_0}\right) = d\log\left(\frac{b_0}{c_0}\right)$ and $\Delta = \frac{1}{2^n a_n^2} d\log\frac{c_n}{b_n}$ in (58) leads to $d\log\left(\frac{a_0 b_0}{a'_0 b'_0}\right) = -a_0^2\Delta$. If p is sufficiently large, then $a_p = b_p$ as well as $a'_p = b'_p$ so that $d\log\left(\frac{a_p b_p}{a'_p b'_p}\right) = 2d\log\left(\frac{a_p}{a'_p}\right)$. Using $\Delta = \frac{\pi}{2}\frac{1}{a_p a'_p} d\left(\log\frac{a'_p}{a_p}\right)$ in (61) yields $d\log\left(\frac{a_p b_p}{a'_p b'_p}\right) = -\frac{4}{\pi}\Delta a_p a'_p$. Hence, (102) becomes, for $p \to \infty$,

$$-\frac{4}{\pi}a_p a'_p = -a_0^2 + \left(\sum_{k=1}^{p} 2^k c_k^2 + \sum_{k=1}^{p} 2^k\left(c'_k\right)^2\right)$$

With $\frac{K(k)-E(k)}{K(k)} = \frac{1}{2a^2}\sum_{n=0}^{\infty} 2^n c_n^2$ in (96) and $a_p = \frac{\pi}{2K(k)}$ where $k = \frac{c}{a}$ for large p as well as similar expressions for the complementary functions in terms of the backward AGM in accented symbols, the above becomes with $a_0 = a = 1$

$$\frac{4}{\pi}\frac{\pi}{2K\left(k\right)}\frac{\pi}{2K\left(k'\right)} - 1 = -\left(2\frac{K\left(k\right)-E\left(k\right)}{K\left(k\right)} - c_0^2 + 2\frac{K\left(k'\right)-E\left(k'\right)}{K\left(k'\right)} - c_0'^2\right)$$

Since $c_0^2 = a^2 - b^2$ and $c_0'^2 = b^2$, we have $c_0^2 + c_0'^2 = a^2 = 1$ and the above [18, ex. 18, p. 39] simplifies to Legendre's formula (98),

$$\frac{\pi}{2K\left(k\right)K\left(k'\right)} = \frac{E\left(k\right)}{K\left(k\right)} + \frac{E\left(k'\right)}{K\left(k'\right)} - 1$$

Adding and subtracting (100) and (101) results in

$$2d\left(\log\left(M\left(a,b\right)M\left(a,c\right)\right)\right) = d\log\left(a_n b_n\right) + d\log\left(a_{-n}c_{-n}\right) + \Delta\left(\sum_{k=n+1}^{\infty} 2^k c_k^2 - \sum_{k=n+1}^{\infty} 2^{-k}b_{-k}^2\right)$$

$$2d\left(\log\left(\frac{M\left(a,b\right)}{M\left(a,c\right)}\right)\right) = d\log\left(a_n b_n\right) - d\log\left(a_{-n}c_{-n}\right) + \Delta\left(\sum_{k=n+1}^{\infty} 2^k c_k^2 + \sum_{k=n+1}^{\infty} 2^{-k}b_{-k}^2\right)$$

Since $d\log(a_n b_n) + d\log(a_{-n}c_{-n}) = d(\log(a_n b_n a_{-n} c_{-n})) = d(\log(a_n b_n a'_n b'_n))$ and if n grows large, then $a_n = b_n$ as well as $a'_n = b'_n$ and

$$d(\log(M(a,b)M(a,c))) = d\log(a_n a'_n) + \frac{\Delta}{2}\left(\sum_{k=n+1}^{\infty} 2^k c_k^2 - \sum_{k=n+1}^{\infty} 2^{-k} b_{-k}^2\right)$$

$$d\left(\log\left(\frac{M(a,b)}{M(a,c)}\right)\right) = d\log\left(\frac{a_n}{a'_n}\right) + \frac{\Delta}{2}\left(\sum_{k=n+1}^{\infty} 2^k c_k^2 + \sum_{k=n+1}^{\infty} 2^{-k} b_{-k}^2\right)$$

By elimination of the differential with the help of Δ as above, the Gauss's series [15, art. 13, p.380]

$$\frac{4}{\pi} M(a,b)M(a,c) = -\sum_{k=n+1}^{\infty} 2^{-k} b_{-k}^2 - 2^{n-1} b_{n-1}^2 + 2^n a_n^2 - \sum_{k=n+1}^{\infty} 2^k c_k^2$$

can be derived.

9 Applications of the arithmetic-geometric mean AGM

34. *The lemniscate.* We refer to Cox [8, Section 3] for the history of the lemniscate, invented by Jacob Bernoulli in 1694.

Any point with coordinates (x, y) on an oval of Cassini has a constant product b^2 of its distances to two fixed points $f_1 = (f, 0)$ and $f_2 = (-f, 0)$, called the foci, i.e.

$$\sqrt{(x-f)^2 + y^2} \times \sqrt{(x+f)^2 + y^2} = b^2$$

Squaring and simplifying yields

$$\left(x^2 + y^2\right)^2 + 2f^2\left(y^2 - x^2\right) + f^4 = b^4$$

which is transformed to Cartesian coordinates by $x = r\cos\theta$ and $y = r\sin\theta$, with $x^2 + y^2 = r^2$, as

$$r^4 - 2f^2 r^2 \cos(2\theta) = b^4 - f^4$$

The lemniscate[27] is the special case of a Cassini oval where $b = f$ and has the elegant expression in polar coordinates

$$r^2 = 2f^2\cos(2\theta) \equiv a\cos(2\theta)$$

with corresponding Cartesian representation for $a = 2f^2$,

$$\left(x^2 + y^2\right)^2 + a\left(y^2 - x^2\right) = 0$$

After combining $\frac{r^2}{a} = \cos 2\theta = 2\cos^2\theta - 1$ and $x = r\cos\theta$ and similarly $\frac{r^2}{a} = \cos 2\theta = 1 - 2\sin^2\theta$ with $y = r\sin\theta$, we obtain the parametric form of the lemniscate in r as

$$\begin{cases} x(r) = \frac{r}{\sqrt{2}}\sqrt{1 + \frac{1}{a}r^2} \\ y(r) = \pm\frac{r}{\sqrt{2}}\sqrt{1 - \frac{1}{a}r^2} \end{cases} \tag{103}$$

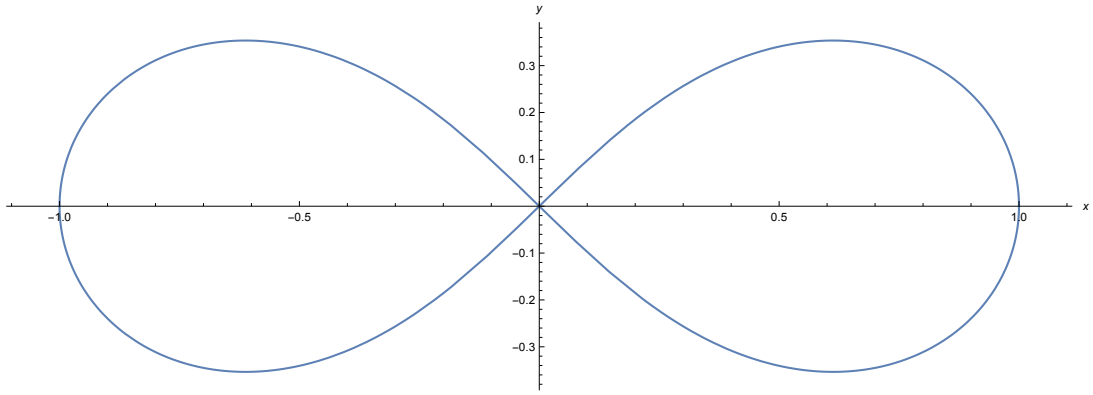

Figure 3: A lemniscate with $a = 1$

Fig. 3 draws the lemniscate for $a = 1$.

The arc length of a curve $y(x)$ with $a \leq x \leq b$ is $L_{ab} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$, as follows from Pythagoras' theorem on an infinitesimal triangle at a point (x, y) on the curve $y(x)$, or in parametric form,

$$L = \int_{t_a}^{t_b} \sqrt{dx^2(t) + dy^2(t)} = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt$$

The length $L_l(r)$ of the lemniscate[28], from the origin to a point at distance r from the origin, is with (103),

$$\begin{aligned} L_l(r) &= \int_0^r \sqrt{\left(\frac{d}{dt}\left(\frac{t}{\sqrt{2}}\sqrt{1 + \frac{1}{a}t^2}\right)\right)^2 + \left(\frac{d}{dt}\left(\frac{t}{\sqrt{2}}\sqrt{1 - \frac{1}{a}t^2}\right)\right)^2} dt \\ &= \frac{1}{\sqrt{2}} \int_0^r \sqrt{\left(\sqrt{1 + \frac{1}{a}t^2} + \frac{\frac{1}{a}t^2}{\sqrt{1 + \frac{1}{a}t^2}}\right)^2 + \left(\sqrt{1 - \frac{1}{a}t^2} - \frac{\frac{1}{a}t^2}{\sqrt{1 + \frac{1}{a}t^2}}\right)^2} dt \\ &= \frac{1}{\sqrt{2}} \int_0^r \sqrt{\left(\frac{1 + \frac{2}{a}t^2}{\sqrt{1 + \frac{1}{a}t^2}}\right)^2 + \left(\frac{1 - \frac{2}{a}t^2}{\sqrt{1 - \frac{1}{a}t^2}}\right)^2} dt = \int_0^r \frac{1}{\sqrt{1 - \left(\frac{t^2}{a}\right)^2}} dt \end{aligned}$$

and

$$L_l(r) = \sqrt{a} \int_0^{\frac{r}{\sqrt{a}}} \frac{du}{\sqrt{1 - u^4}} \tag{104}$$

Hence, the length $L_l(r)$ of the lemniscate, from the origin to a point at distance r from the origin, is written in terms of the elliptic integral $F(\varphi, k) = \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$ of the first kind in (130) as $L_l(r) = \sqrt{a} F\left(\arcsin \frac{r}{\sqrt{a}}, i\right)$. In terms of the integral $I(a, b; \varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$ in (145), we

[27]Leminscate is derived from the Greek $\lambda\eta\mu\nu\iota\sigma\kappa o\varsigma$, meaning "ribbon", e.g. a pendant ribbon fastened to a victor's garland.

[28]The length of an arc on an ellipse $x = a\sin\theta$ and $y = b\cos\theta$ is $L_e = \int_0^{\varphi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} d\theta$ and can be written in terms of the elliptic integral $E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2\sin^2\theta} d\theta$ of the second kind in (131).

have

$$L_l(r) = \sqrt{a}\int_0^{\arcsin\frac{r}{\sqrt{a}}} \frac{d\theta}{\sqrt{2\cos^2\theta + \sin^2\theta}} = \sqrt{a} I\left(\sqrt{2}, 1; \arcsin\frac{r}{\sqrt{a}}\right)$$

The total length L_l of the lemniscate with $a = 1$ is $L_l = 4L_l(1) = 4I\left(\sqrt{2}, 1\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2\cos^2\theta + \sin^2\theta}}$. The fundamental integral in (28) then shows that the total length L_l of the lemniscate with $a = 1$ equals

$$L_l = \frac{2\pi}{M\left(\sqrt{2}, 1\right)}$$

After substitution of $y = \sin\theta$, the integral $I\left(\sqrt{2}, 1\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\sin^2\theta}}$ transforms to $I\left(\sqrt{2}, 1\right) = \int_0^1 \frac{dy}{\sqrt{1-y^4}}$, where the last integral appeared in 1691 in a paper of Jacob Bernoulli. The definition $F(\varphi, k) = \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$ in (130) illustrates that $F(\varphi, i) = \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t^4}}$ and $K(i) = \int_0^1 \frac{dy}{\sqrt{1-y^4}}$; see also the imaginary transform (138) on p. 69. The Beta integral $B(\alpha, \beta) = \int_0^1 x^{a-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, after substitution of $x = y^4$, results in

$$\int_0^1 \frac{dy}{\sqrt{1-y^4}} = \frac{1}{4}\int_0^1 x^{\frac{1}{4}-1}(1-x)^{\frac{1}{2}-1}\,dx = \sqrt{\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

Consequently, a basic result in Gauss's investigations [8, p. 280-283] relates the total length of the lemniscate to the arithmetic-geometric mean

$$M\left(\sqrt{2}, 1\right) = \frac{2\pi}{4I\left(\sqrt{2}, 1\right)} = \frac{\pi}{2\int_0^1 \frac{dy}{\sqrt{1-y^4}}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \simeq 1.19814$$

Gauss denotes $\varpi = 2\int_0^1 \frac{dy}{\sqrt{1-y^4}} \simeq 2.39628$ to[29] emphasize the importance of $L_l = \frac{2\pi}{M\left(\sqrt{2},1\right)}$, which is then

$$M\left(\sqrt{2}, 1\right) = \frac{\pi}{\varpi} \tag{105}$$

In a posthumous paper[30] of 1786, Euler has proved the amazing result,

$$\int_0^1 \frac{dy}{\sqrt{1-y^4}}\int_0^1 \frac{y^2dy}{\sqrt{1-y^4}} = \frac{\pi}{4} \tag{106}$$

so that

$$M\left(\sqrt{2}, 1\right) = 2\int_0^1 \frac{y^2dy}{\sqrt{1-y^4}} \tag{107}$$

If $k = k' = \frac{1}{\sqrt{2}}$, then Legendre's formula (98) simplifies to $2K(k)E(k) - K^2(k) = \frac{\pi}{2}$. If $k = k' = \frac{1}{\sqrt{2}}$ and $\varphi = \frac{\pi}{2}$ in (139), we observe that

$$K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

Similarly, we find from (132) that $2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}$. Substituting these integrals in $2K(k)E(k) - K^2(k) = \frac{\pi}{2}$ leads [20, p. 69][2, p. 596] to Euler's lemniscate identity (106).

[29] As a comparison, the integral $\int_0^1 \frac{dy}{\sqrt{1-y^2}} = \arcsin(1) = \frac{\pi}{2} = 1.5708$.

[30] Euler, "De miris proprietatibus curvae elasticae sub equatione $y = \int \frac{xxdx}{\sqrt{1-x^4}}$ contentae", in 1786.

35. *The pendulum.* A mass m is hung on a cord that is fixed at a point 0 on a ceiling. The mass m is released at time $t = 0$ from an angle $\theta(0) = \theta_0$ and will swing around the point 0 in the well-known pendulum fashion. The length of the cord is l and we ignore its weight. Further ignoring friction and considering the mass as a point, the forces that act upon the mass m are the gravitation force $F_G = mg$ and the force F_C exerted by the cord, that restricts the movement of the mass to a circle with midpoint 0 and radius l, as depicted in Fig. 4. At time t, the angle $\theta(t)$ is measured between the cord and the line perpendicular to the ceiling. Newton's law of motion states that the sum of the forces acting on an object equals its mass multiplied by its acceleration. The only acceleration possible is due to the tangential force $F_T = F_G \sin\theta$ along the circle, because the normal force F_N, which is perpendicular to the tangent on the circle, is precisely equal to the force F_C of the cord. The tangential acceleration a_T of the mass along the circle is $a_T = l\frac{d^2\theta(t)}{dt^2}$ and its tangential velocity is $v_T = l\frac{d\theta(t)}{dt}$.

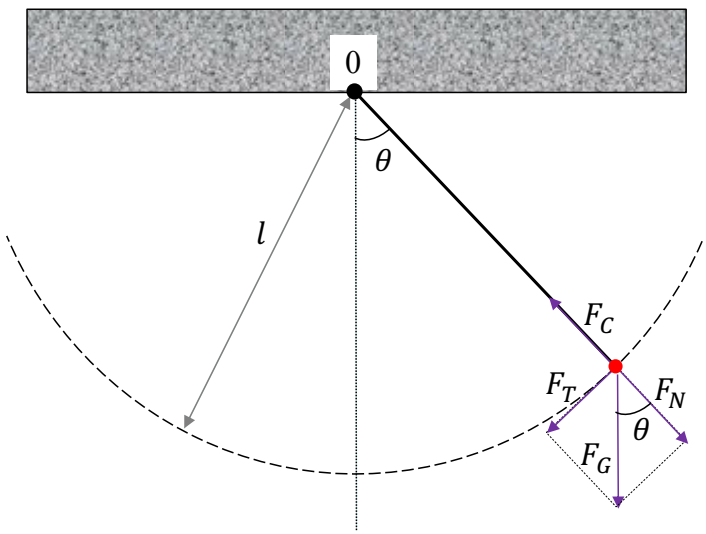

Figure 4: An ideal pendulum consists of a point mass and a weightless cord or rod, suspended from a pivot, allowing the pendulum to swing freely.

Hence, the mechanical motion of the mass obeys the differential equation

$$-F_T = ml\frac{d^2\theta(t)}{dt^2}$$

and the minus sign arises from the fact that the tangential force is opposite to the direction in which the angle $\theta(t)$ is measured. Substituting $F_T = mg\sin\theta(t)$ yields the pendulum governing equation in the angle $\theta(t)$,

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g}{l}\sin\theta(t) \tag{108}$$

This non-linear differential equation can be integration, after multiplying both sides by $\frac{d\theta(t)}{dt}$ and recognizing that $\frac{d^2\theta(t)}{dt^2}\frac{d\theta(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left(\left(\frac{d\theta(t)}{dt}\right)^2\right)$, while $\sin\theta(t)\frac{d\theta(t)}{dt} = -\frac{d}{dt}(\cos\theta(t))$. Thus, the pendulum law (108) becomes

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{d\theta(t)}{dt}\right)^2 - \frac{g}{l}\cos\theta(t)\right) = 0$$

implying that $\left(\frac{d\theta(t)}{dt}\right)^2 - \frac{2g}{l}\cos\theta(t) = c$, where c is a constant independent of time t. At time $t = 0$, the angle is $\theta(0) = \theta_0$, but the initial velocity is zero so that $c = -\frac{2g}{l}\cos\theta_0$. In terms of the velocity $v_T = l\frac{d\theta(t)}{dt}$ and the kinetic energy $E_k = \frac{mv^2}{2}$, it holds that the velocity

$$v_T(t) = \sqrt{2gl\left(\cos\theta(t) - \cos\theta_0\right)}$$

does not dependent on the mass, but on the square root $\sqrt{l}$ of the length of the rope and the difference between cosines at the starting angle and at the angle at time $t = 0$. Another integration of $\frac{d\theta(t)}{dt} = \pm\sqrt{\frac{2g}{l}\cos\theta(t) + c}$, where the sign changes at the highest angle at velocity $v_T = 0$ just when the pendulum returns and which is written in differential form as $\frac{d\theta(t)}{\pm\sqrt{\frac{2g}{l}\cos\theta(t)+c}} = dt$, is

$$t = \int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{\frac{2g}{l}\left(\cos\theta - \cos\theta_0\right)}} = \frac{1}{2}\sqrt{\frac{l}{g}}\int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} = \frac{1}{2\sin\frac{\theta_0}{2}}\sqrt{\frac{l}{g}}\int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{1 - \frac{1}{\sin^2\frac{\theta_0}{2}}\sin^2\frac{\theta}{2}}}$$

The time t can be written in terms of the elliptic integral $F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ in (130). Since $\frac{1}{\sin\frac{\theta_0}{2}} \geq 1$, we cannot simply choose $k = \frac{1}{\sin\frac{\theta_0}{2}}$ after letting $u = \frac{\theta}{2}$. The more complicated substitution $\sin u = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$ with inverse $\theta = 2\arcsin\left(\sin\frac{\theta_0}{2}\sin u\right)$ gives us

$$t = \frac{1}{2\sin\frac{\theta_0}{2}}\sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{\arcsin\frac{\sin\frac{\theta(t)}{2}}{\sin\frac{\theta_0}{2}}} \frac{du}{\sqrt{1-\sin^2 u}}\frac{2\sin\frac{\theta_0}{2}\cos u}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 u}} = \sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{\arcsin\frac{\sin\frac{\theta(t)}{2}}{\sin\frac{\theta_0}{2}}} \frac{du}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 u}}$$

With $k = \sin\frac{\theta_0}{2}$, we arrive at the time evolution of the pendulum expressed in its angle $\theta(t)$,

$$t = \sqrt{\frac{l}{g}}\left(F\left(\frac{\pi}{2}, k\right) - F\left(\arcsin\frac{\sin\frac{\theta(t)}{2}}{\sin\frac{\theta_0}{2}}, k\right)\right)$$

The period P of the pendulum is twice the time from the starting angle θ_0 to the maximum opposite angle $-\theta_0$, which is four times the time from the starting angle θ_0 and the angle $\theta = 0$ where maximal velocity v_T is reached. With $F\left(\frac{\pi}{2}, k\right) = K(k)$, the period P of the pendulum is

$$P = 4\sqrt{\frac{l}{g}}K\left(\sin\frac{\theta_0}{2}\right) \tag{109}$$

which is maximal for $\theta_0 = \pi$ and equal to $P_{\max} = 4\sqrt{\frac{l}{g}}K(1) \to \infty$. The angle $\theta_0 = \pi$ is the unstable equilibrium, while $\theta_0 = 0$ is the stable equilibrium. The AGM expansion (91) of the complete elliptic integral $K(k)$ provides a very fast converging expansion of (109).

The Taylor series (144) of $K(k)$ with $f_{2m} = \binom{-\frac{1}{2}}{m}^2 = \left(\prod_{j=1}^{m}\frac{2j-1}{2j}\right)^2$ in (22) indicates that

$$P = 2\pi\sqrt{\frac{l}{g}}\left(1 + \sum_{m=1}^{\infty}\left(\prod_{j=1}^{m}\frac{2j-1}{2j}\right)^2\sin^{2m}\frac{\theta_0}{2}\right) \tag{110}$$

If the angle θ_0 and thus all subsequent angles $\theta(t) \leq \theta_0$ are small, then the approximation $\sin\theta(t) \approx \theta(t)$ in (108) yields the linear differential equation $\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\theta(t) = 0$, with general solution $\theta(t) =$

$a\cos\left(\sqrt{\frac{g}{l}}t\right)+b\sin\left(\sqrt{\frac{g}{l}}t\right)$. If the begin velocity $v_T(0)=0$, thus $\left.\frac{d\theta(t)}{dt}\right|_{t=0}=0$ and $b=0$, then we arrive at the approximation $\theta_{\text{app}}(t)$ of the angle $\theta(t)$,

$$\theta_{\text{app}}(t)=\theta_0\cos\left(\sqrt{\frac{g}{l}}t\right) \qquad \theta_0 \text{ is small}$$

The corresponding period is $P_{\text{app}}=2\pi\sqrt{\frac{l}{g}}$ and comparison with the series (110), whose terms are non-negative, shows that $P_{\text{app}}\leq P$.

36. *Returning to the starting point in a random walk.* Random walks in several dimensions and on graphs (see e.g. [34, p. 63-65]) constitute a basic and well-studied topic in probability theory. We refer to Feller [12, Chapter III] for a clear exposition of the simple random walk on a line or in one dimension. An interesting theorem due to Polya states that a random walker in one and two dimensions returns to his initial position with probability 1, but in three dimensions, that probability is only around 0.35. Polya's theorem is proved in [12, p. 361]. We compute the probability u_n that a random walker on a 2D lattice, i.e. in two dimensions, starting at the origin, returns after n steps to the origin. If the number of steps is odd, i.e. $n=2m-1$, then returning is impossible on a 2D lattice and $u_{2m-1}=0$. A return to the initial position at the origin is only possible if the number of steps in positive x- and y-directions on the lattice are equal to those in the negative x- and y-directions, respectively, implying, using a multinomial distribution [12, p. 361], that

$$u_{2m}=\frac{1}{4^{2m}}\sum_{k=0}^{m}\frac{(2m)!}{k!k!\,(n-k)!\,(n-k)!}=\frac{1}{4^{2m}}\binom{2m}{m}\sum_{l=0}^{m}\binom{m}{l}^2=\frac{\binom{2m}{m}^2}{4^{2m}}=f_{2m}$$

where $\sum_{l=0}^{m}\binom{m}{l}^2=\binom{2m}{m}$ is an instance of Vandermonde's identity (140). Hence, the Taylor coefficient f_{2m} of $\frac{1}{M(1+x,1-x)}$ in (22) equals the probability that a random walker on a 2D lattice, starting at the origin, returns after $n=2m$ steps to the origin. The probability generating function of the random variable R of returning to the origin is

$$\varphi_R(z)=\sum_{n=0}^{\infty}u_nz^n=\sum_{n=0}^{\infty}f_nz^n=\frac{1}{M(1+z,1-z)}=\frac{1}{M\left(1,\sqrt{1-z^2}\right)}=\frac{2}{\pi}K(z)$$

10 From the arithmetic-geometric mean to elliptic functions

37. *Extension of $M(a,b)$ to complex a and b.* Based on incomplete sketches in Gauss's Nachlass, Cox [8, Section 2, pp. 283-309] has reconstructed the entire complex theory of $M(a,b)$ that Gauss has discovered. Although the fundamental integral in (28) was already a tour de force of Gauss, his extension to the complex plane is even more astonishing. Whereas Section 3 reviews the entire historical evolution towards elliptic functions and theta functions, we summarize Cox's Section 2:

1. If $a,b\in\mathbb{C}$, then the AGM algorithm (2) creates two solutions for $b_{n+1}=\sqrt{a_nb_n}$ for all $n\geq 0$, so that there are uncountably many sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ for a given pair (a,b). Moreover, it is even unclear whether the AGM algorithm (2) still converges! All these sequences converge, but only countably many have a non-zero limit.

2. A deep result[31] of which Gauss predicted in his mathematical diary that "the demonstration of $M\left(\sqrt{2},1\right)=\frac{2\pi}{L_l}$ will surely open an entirely new field of analysis" is

   Theorem 2 (Werke, Band 3, p. 378) *If one chooses the negative value of the squareroot of $b_n=\sqrt{a_{n-1}b_{n-1}}$ in the AGM algorithm (2), all values $\mathbb{M}(a,b)$ of the arithmetic-geometric mean for $a,b\in\mathbb{C}$ are comprised in*

$$\frac{1}{\mathbb{M}(a,b)}=\frac{1}{M(a,b)}+\frac{4il}{M(a,c)}$$

   where $l\in\mathbb{Z}$ and $c=\sqrt{a^2-b^2}$ as defined in (7).

   The proof of Theorem 2 is involved.

3. Amazingly, Gauss found properties of modular functions, the univalent Möbius transformation $w=\frac{az+b}{cz+d}$ in the complex plane, group properties of SL(2,$\mathbb{Z}$), reduction theory, as explained in Cox [8, Section 2, pp. 283-309].

Let us return to Cox's reformulation in Theorem 1. Suppose that τ_0 is a solution of $k'(\tau)=\frac{b}{a}$, then $a=\mu p^2(\tau_0)$ and $b=\mu q^2(\tau_0)$ are the simplest values of $M(a,b)$. Cox shows that any solution of $\widetilde{\mu}$ of $\mathbb{M}(a,b)$ can be written as $\widetilde{\mu}=\frac{\mu}{c\tau_0+d}$, where (see **art**. 27)

$$\tau_0=i\frac{M(a,b)}{M(a,c)} \tag{111}$$

which is especially useful when $a>b>0$. For example, if $a=\sqrt{2}$ and $b=1$, then $c=1$ from which (111) implies that $\tau_0=i$ and that $M\left(\sqrt{2},1\right)=\frac{\sqrt{2}}{p^2(i)}=\frac{1}{q^2(i)}$. Invoking $\frac{\pi}{\varpi}=M\left(\sqrt{2},1\right)\simeq 1.19814$ in (105) and the rapidly converging theta-function expansions (80) and (81) leads to

$$\frac{\varpi}{\pi}=\frac{1}{\sqrt{2}}\left(1+2\sum_{n=1}^{\infty}e^{-n^2\pi}\right)^2$$

and

$$\frac{\varpi}{\pi}=\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi}\right)^2$$

[31] Cox [8, p. 287-288] has proved a more accurate rephrasing of Gauss's Theorem 2:

Theorem 1 (Gauss (1800)) *Given $a,b\in\mathbb{C}$ that satisfy $a\neq\pm b$ and $|a|\geq|b|$ and let $\mu=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$ in the AG algorithm (2) for $M(a,b)$ and $\lambda=\lim_{n\to\infty}(a+b)_n=\lim_{n\to\infty}(a-b)_n$ in the AG algorithm (2) for $M(a+b,a-b)$. Then all values $\widetilde{\mu}$ of $M(a,b)$ obey*

$$\frac{1}{\widetilde{\mu}}=\frac{d}{\mu}+i\frac{c}{\lambda}$$

where d and c are arbitrary relatively prime integers satisfying $d\equiv 1\bmod 4$ and $c\equiv 0\bmod 4$.

We add that $M(a_n,c_n)=M\left(\frac{a_{n-1}+b_{n-1}}{2},\frac{a_{n-1}-b_{n-1}}{2}\right)=\frac{1}{2}M(a_{n-1}+b_{n-1},a_{n-1}-b_{n-1})=\frac{1}{2}M(a+b,a-b)$

Finally, with $k(\tau)=\frac{r^2(\tau)}{p^2(\tau)}$ and the complementary modulus $k'(\tau)=\sqrt{1-k^2(\tau)}$, the generalization of Gauss's basic integral (28) is written in terms of Jacobi's theta function

$$\frac{1}{M(1,k'(\tau))}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-(k(\tau))^2\sin^2\theta}}=p^2(\tau)=\vartheta_3^2(\tau,0)$$

38. *Sinus Lemniscatus.* Theorem 2 on p. 50 was Gauss's avenue towards double periodicity in the complex plane and towards elliptic functions. Analogous to trigonometric functions, where $y=\arcsin x=\int_0^x\frac{dt}{\sqrt{1-t^2}}$ is the inverse function of $x=\sin y$, Gauss realized that $\frac{1}{M(a,b)}$ is the inverse function of elliptic functions in the complex plane! There is a similarity between $\frac{\pi}{2}=\int_0^1\frac{dy}{\sqrt{1-y^2}}$ and $\frac{\varpi}{2}=\int_0^1\frac{dy}{\sqrt{1-y^4}}$. Gauss[32] defined the lemniscatic functions as

$$\begin{aligned}\operatorname{sinlem}\left(\int_0^x\frac{dy}{\sqrt{1-y^4}}\right)&=x\\ \operatorname{coslem}\left(\frac{\varpi}{2}-\int_0^x\frac{dy}{\sqrt{1-y^4}}\right)&=x\end{aligned}\tag{112}$$

After writing $sl(x)=\operatorname{sinlem}(x)$ and $cl(x)=\operatorname{coslem}(x)$, Gauss derived the basic identities

$$\begin{aligned}sl^2(x)+cl^2(x)+sl^2(x)\,cl^2(x)&=1\\ \left(1+sl^2(x)\right)\left(1+cl^2(x)\right)&=\left(\frac{1}{sl^2(x)}-1\right)\left(\frac{1}{cl^2(x)}-1\right)=2\\ \operatorname{sinlem}(a\pm b)&=\frac{sl(a)\,cl(b)\pm sl(b)\,cl(a)}{1\mp sl(a)\,cl(b)\,sl(b)\,cl(a)}\\ \operatorname{coslem}(a\pm b)&=\frac{cl(a)\,cl(b)\mp sl(b)\,sl(a)}{1\pm sl(a)\,cl(b)\,sl(b)\,cl(a)}\\ sl(x)&=\sqrt{\frac{1-cl^2(x)}{1+cl^2(x)}}\text{ and }cl(x)=\sqrt{\frac{1-sl^2(x)}{1+sl^2(x)}}\end{aligned}$$

and deduced from these many beautiful series expansion and theta-function like expansions, that were never published by Gauss, only in his Nachlass. Gauss generalized the sinus lemniscatus to complex values,

$$sl(iy)=isl(y)\text{ and }cl(iy)=\frac{1}{cl(y)}$$

Indeed, formally let $y=it$ in $\int_0^x\frac{dy}{\sqrt{1-y^4}}=i\int_0^{ix}\frac{dt}{\sqrt{1-t^4}}$ in (112), then $\operatorname{sl}\left(i\int_0^{ix}\frac{dy}{\sqrt{1-y^4}}\right)=ix$ and replace $y=ix$. Using $sl(iy)=isl(y)$, the formula $cl(iy)=\frac{1}{cl(y)}$ follows from $cl(x)=\sqrt{\frac{1-sl^2(x)}{1+sl^2(x)}}$. Then, $\operatorname{sinlem}(a\pm b)=\frac{sl(a)cl(b)\pm sl(b)cl(a)}{1\mp sl(a)cl(b)sl(b)cl(a)}$ shows that

$$sl(x+iy)=\frac{sl(x)+isl(y)\,cl(x)\,cl(y)}{cl(y)-isl(x)\,sl(y)\,cl(x)}$$

[32] Gauss Werke, Band 3, p. 404 on "Elegantiores integralis $\int_0^x\frac{dy}{\sqrt{1-y^4}}$ proprietatis", in which he defines the sinuslemniscatus and derives many of its functional properties, much more than the sinus possesses.

which illustrates that $sl(z)$ is doubly periodic with periods 2ϖ and $2i\varpi$, because $sl(0) = 0$ and $cl\left(\frac{\varpi}{2}\right) = 0$ as follows from the definition (112). The zeros $sl(z)$ occur at $z = (m + in)\varpi$ and the poles at $z = ((2m-1) + i(2n-1))\frac{\varpi}{2}$, where $m, n \in \mathbb{Z}$.

39. *Sinus Amplitudinis.* Jacobi's sinus amplitudinis $sn(z;k)$, cosinus amplitudinis $cn(z;k)$ and delta amplitudinis $cn(z;k)$ are defined [23, Chapter 22] as

$$\begin{aligned} sn(z(\varphi);k) &= \sin\varphi \\ cn(z(\varphi);k) &= \cos\varphi \\ dn(z(\varphi);k) &= \sqrt{1-k^2\sin^2\varphi} \end{aligned}$$

where

$$z(\varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi,k)$$

and its inverse $\varphi = am(z,k)$ is Jacobi's amplitude [25, p. 286]. If $k = 0$, then $z(\varphi) = \varphi$ and we obtain $sn(\varphi;0) = \sin\varphi$, $cn(\varphi;0) = \cos\varphi$ and $dn(\varphi;0) = 1$. If $k = 1$, then $z(\varphi) = \int_0^{\varphi} \frac{d\theta}{\cos\theta} = \ln\left(\frac{1+\sin\varphi}{\cos\varphi}\right)$. We invert the latter, i.e. solve $e^z = \frac{1+y}{\sqrt{1-y^2}}$ for $y = \sin\varphi$ and find that $y = \frac{-1\pm e^{2z}}{1+e^{2z}} = \frac{\pm e^z - e^{-z}}{e^z + e^{-z}}$, where the plus must be chosen because $y > 0$ for small φ, which results in $\sin\varphi = \tanh z$ and

$$\begin{aligned} sn(z(\varphi);1) &= \tanh z \\ cn(z(\varphi);1) = dn(z(\varphi);1) &= \operatorname{sech} z \end{aligned}$$

If $\varphi = \frac{\pi}{2}$, then $z\left(\frac{\pi}{2}\right) = K(k)$ in **art**. 47 and

$$\begin{aligned} sn(K(k);k) &= 1 \\ cn(K(k);k) &= 0 \\ dn(K(k);k) &= \sqrt{1-k^2} \end{aligned}$$

The relation between Gauss's and Jacobi's elliptic functions is

$$sl(z) = \frac{1}{\sqrt{2}} \frac{sn\left(z\sqrt{2};\frac{1}{\sqrt{2}}\right)}{dn\left(z\sqrt{2};\frac{1}{\sqrt{2}}\right)} \quad \text{and} \quad cl(z) = cn\left(z\sqrt{2};\frac{1}{\sqrt{2}}\right)$$

which appears if $k^2 = \frac{1}{2}$. Hence, Gauss's sinus and cosinus lemniscatus are a special case of Jacobi's elliptic amplitudinis functions. Finally, Cox [8, Section 3] demonstrates that Gauss had a complete theory of elliptic functions!

11 Computations of π

The number π has fascinated humans for over 4000 years since the Babylonians and old-Egyptians and is still captivating current mathematicians. The Borwein brothers and Bailey [6] overview the history of the computing π and also mention that π is used to test the hardware of supercomputers today.

40. *Leibniz's series.* Perhaps the simplest or most classic series to compute π are derived from inverse trigonometric functions. We confine ourselves to series for $\arctan z$.

The inverse function $z = f^{-1}(w)$ of the function $w = f(z) = \tan z$ is $f^{-1}(z) = \arctan z$. The definition of the inverse function implies that $f\left(f^{-1}(z)\right) = z$ and differentiation gives

$$\left(f^{-1}(z)\right)' = \frac{1}{f'\left(f^{-1}(z)\right)} \tag{113}$$

Applied to $f(z) = \tan z$ for which $f'(z) = \sec^2 z = 1 + \tan^2 z = 1 + f^2(z)$, we arrive at

$$\frac{d}{dz}(\arctan z) = \frac{1}{1+z^2}$$

After integration, we find

$$\arctan z = \int_0^z \frac{du}{1+u^2} \tag{114}$$

Let $u = -t$, then $\arctan z = -\int_0^{-z} \frac{dt}{1+t^2}$, from which $\arctan(-z) = -\arctan z$ is an odd function around $z = 0$. Substitution of the geometric series $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, convergent for $|z| < 1$, leads to the Taylor series around $z = 0$

$$\arctan z = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} \qquad \text{for } |z| < 1 \tag{115}$$

which converges for $z = 1$, because Leibniz' series

$$\frac{\pi}{4} = \lim_{K\to\infty} \sum_{k=0}^{K} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots \tag{116}$$

is an alternating sum with decreasing terms. Most likely, Leibniz' series (116) is one of the simplest, but also slowest convergent series for π. If $K = 10^m$ terms are computed in (116), then about m decimal digits are correct. In [4, Section 1.8.1], the remarkable observation that Leibniz' series, computed up to $K = 5\ 10^6$ terms, contains many correct digits of π and only a few incorrect digits, is explained as due to Euler numbers that appear in the Euler-MacLaurin summation [4, Theorem 1.8]. While the alternating Leibniz' series (116) is very slowly converging, the errors are highly predictable. Leibniz[33] (1646-1716) did not obtain (116) as derived above, but from his general method of "transmutation", which is nicely explained by Edwards [11, p. 245-252]. The companion series of (116), due to Newton (1643-1727) and equally slowly converging, is

$$\frac{\pi}{2\sqrt{2}} = \lim_{K\to\infty} \sum_{k=0}^{K} (-1)^k \left(\frac{1}{4k+1} + \frac{1}{4k+3}\right) = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \cdots \tag{117}$$

Newton's π-series (117) follows from $\int_0^1 \frac{1+z^2}{1+z^4} dz$. First, expanding the integrand in a geometric series $\frac{1}{1+z^4} = \sum_{k=0}^{\infty} (-1)^k x^{4k}$ and term-wise integrating results in the right-hand side series in (117). Since

[33] Edwards [11, p. 222] mentions that Newton in 1676 inquired Henry Oldenburg, secretary of the Royal Society in London, to send a first letter (epistola prima) to Leibniz in which Newton announced his bionomial series $(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$ for any (complex) α, but did not mention the validity range $|z| < 1$. Leibniz replied Newton's epistola prima by his alternating series (116). In the second (and last) letter to Leibniz, the epistola posterior, Newton gave a general integral of which (117) was a special case.

$1+z^4=\left(1+z^2+\sqrt{2}z\right)\left(1+z^2-\sqrt{2}z\right)$, partial fraction expansion yields $\frac{1+z^2}{1+z^4}=\frac{\frac{1}{2}}{1+z^2+\sqrt{2}z}+\frac{\frac{1}{2}}{1+z^2-\sqrt{2}z}$ and

$$\int_0^1 \frac{1+z^2}{1+z^4}dz=\frac{1}{2}\int_{-1}^1 \frac{dz}{1+z^2+\sqrt{2}z}=\int_{-1}^1 \frac{dz}{\left(\sqrt{2}z+1\right)^2+1}=\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}z+1\right)\Big|_{-1}^1$$
$$=\frac{1}{\sqrt{2}}\left(\arctan\left(1+\sqrt{2}\right)-\arctan\left(1-\sqrt{2}\right)\right)=\frac{1}{\sqrt{2}}\left(\frac{3\pi}{8}-\left(-\frac{\pi}{8}\right)\right)=\frac{\pi}{2\sqrt{2}}$$

By splitting the series (116) in odd and even terms in k, Leibniz deduces

$$\frac{\pi}{8}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}-\frac{1}{4k+3}\right)=\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+3)}=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\cdots$$

Leibniz [11, p. 249] was intrigued by the comparison with Mercator's series $\log 2=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, derived from $\log(1+z)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^k}{k}$; thus,

$$\log 2=\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\sum_{k=1}^{\infty}\frac{1}{(2k-1)2k}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2k+2)}=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}$$

but,

$$\frac{1}{2}\log 2=\sum_{k=0}^{\infty}\frac{1}{(4k+2)(4k+4)}\simeq 0.346574 \text{ and } \frac{\pi}{8}=\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+3)}\simeq 0.392699$$

The mathematician and astronomer Madhava of Sangamagrama (ca. 1350 - ca. 1425) or his followers in the Kerala school of astronomy and mathematics in India found the series[34]

$$\pi=\sqrt{12}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\frac{1}{3^k} \tag{118}$$

which is an instance of the Taylor series (115) for $z=\frac{1}{\sqrt{3}}$, because $\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. The series (118) converges signficantly faster than Leibniz' series (116) and was discovered about 250 years earlier, but Madhava's method was considerably less powerful than Newton's and Leibniz' calculus.

Many other variations on $\arctan z$ exists [5, p. 352]. John Machin (1680-1752) found[35] that

$$\pi=16\arctan\left(\frac{1}{5}\right)-4\arctan\left(\frac{1}{239}\right)$$

while Leonhard Euler (1707-1783) started from

$$\pi=20\arctan\left(\frac{1}{7}\right)+8\arctan\left(\frac{3}{79}\right)$$

[34] Information found on Wikipedia.

[35] Gilbert [16, p. 114] gives the details. In order to compute π, we consider $u=kx-\frac{\pi}{4}$ and $\tan u=\tan\left(kx-\frac{\pi}{4}\right)=\frac{\tan kx-1}{\tan kx+1}$. We try to find integers k and n with $\tan x=\frac{1}{n}$, so that $\tan k\arctan\frac{1}{n}-1$ small and n is large. Numerically, $k=4$ and $n=5$ achieves a minimum for $\left|\tan k\arctan\frac{1}{n}-1\right|$ for all $1\le k\le 77$ and $1\le n\le 13$. For $k=4$, Bernoulli's formula $\tan kx=\frac{\sum_{n=1}^{\left[\frac{k+1}{2}\right]}\binom{k}{2n-1}(-1)^{n-1}(\tan x)^{2n-1}}{\sum_{n=0}^{\left[\frac{k}{2}\right]}\binom{k}{2n}(-1)^n(\tan x)^{2n}}$ in [26, p. 282] reduces to $\tan 4x=\frac{4\tan x-4\tan^3 x}{1-6\tan^2 x+\tan^4 x}$, from which $\tan 4\arctan\frac{1}{5}=\frac{120}{119}$ and $\tan u=\frac{\tan 4\arctan\frac{1}{5}-1}{\tan 4\arctan\frac{1}{5}+1}=-\frac{1}{239}$. Since $\frac{\pi}{4}=kx-u$, we finally arrive at Machin's expression, which has a much smaller u value than Euler's form, at the expense of a larger $x=\frac{1}{n}$ value.

Euler introduced his famous Euler transform and derived [32] the series in 1755

$$\begin{aligned}\arctan z &= \frac{z}{1+z^2}\lim_{K\to\infty}\sum_{k=0}^{K}\frac{(k!)^2\,2^{2k}}{(2k+1)!}\left(\frac{z^2}{1+z^2}\right)^k \\ &= \frac{z}{1+z^2}\left(1+\frac{2}{3}y+\frac{2.4}{3.5}y^2+\frac{2.4.6}{3.5.7}y^3+\cdots\right)_{y=\frac{z^2}{1+z^2}}\end{aligned} \tag{119}$$

Borwein & Borwein [5] mention that Euler computed π to 20 decimal places in an hour, because powers of $\frac{z^2}{1+z^2}\Big|_{z=\frac{1}{7}} = \frac{1}{50}$ and $\frac{z^2}{1+z^2}\Big|_{z=\frac{3}{79}} = \frac{9}{6250}$ are small. Since all terms in (119) are positive, a truncation at K terms is a lower bound for π. The table below shows the relatively fast convergence.

K	Eq.(119)	numeric	error : $\pi -$ Eq.(119)
1	$\frac{92021032}{29296875}$	3.1409845589333333333	0.000608094
2	$\frac{1438102216076}{457763671875}$	3.1415822277585578667	0.000010425831
3	$\frac{31458588519871552}{10013580322265625}$	3.1415924681726507152	$1.8541714252\ 10^{-7}$
4	$\frac{42132040637669901232}{13411045074462890625}$	3.1415926502176251985	$3.3721680399385\ 10^{-9}$
5	$\frac{2027604457824094913367616}{645406544208526611328125}$	3.1415926535275248823	$6.2268356170484\ 10^{-11}$
6	$\frac{748831191824600261193678102 4}{238360371440649032592773437 5}$	3.1415926535886307107	$1.1625277423450\ 10^{-12}$
7	$\frac{1300054152473736362971900138 9056}{4138200893066823482513427734375}$	3.1415926535897713528	$2.1885627663394\ 10^{-14}$
8	$\frac{6837472308166603914046844855729559 1168}{2176435032197332475334405899047851 5625}$	3.1415926535897928237	$4.1471729103989\ 10^{-16}$
9	$\frac{4059749182973921599678281054996440790084608}{1292258300367166157229803502559661865234375}$	3.1415926535897932306	$7.9000419090266\ 10^{-18}$
10	$\frac{1122221689234277313603455710212883556391424}{3572142581731441168813034892082214355468 75}$	3.1415926535897932383	$1.5114192562194\ 10^{-19}$

41. *Archimedes' computation of* π. Surprisingly similar to Gauss's AGM recursion (2), the Borwein brothers [5] (see also [11, p. 31-35]) mention Archimedes' recursion

$$A_n = \tfrac{A_{n-1}+B_n}{2} \quad \text{and} \quad B_n = \sqrt{A_{n-1}B_{n-1}} \tag{120}$$

where $\frac{1}{A_n}$ is the area of the circumscribed regular 2^n-gon and $\frac{1}{B_n}$ denotes the area of an inscribed regular 2^n-gon around a circle with radius 1. The recursion of Archimedes (ca. 287-212 BC) in (120) seems due to Gauss's teacher Pfaff [6, p. 205]. Comparing with the circle, we find the inequalities $\frac{1}{B_n} < \pi < \frac{1}{A_n}$ and the recursion (120), starting at $n = 2$ with $B_2 = \frac{1}{2}$ and $A_2 = \frac{1}{4}$ as shown below, converges to π. The angle $\theta = \frac{\pi}{2^n}$ for $n \geq 2$; thus, for a square $\theta_2 = \frac{\pi}{4}$, for an octagon $\theta_3 = \frac{\pi}{8}$, etc. Fig. 5 shows both square (dotted lines) and octagon (full lines). The area $\alpha_{c;2}$ of circumscribed square (red) is $\alpha_{c;2} = 4 \times 1 = 4$, but also 8 times the area of the triangle $r0p$, which is $\frac{|0p||rp|}{2} = \frac{1}{2}\tan\theta_2 = \frac{1}{2}$; thus $\alpha_{c;2} = 8 \times \frac{1}{2} = 4$. The corresponding perimeter $p_{c;2}$ is 8 times $|rp| = 1$; thus $p_{c;2} = 8 = 2\alpha_{c;2}$. The area $\alpha_{i;2}$ of inscribed square (blue) is $\alpha_{i;2} = \left(\sqrt{2}\right)^2 = 2$, but also 4 times a triangle with area $\frac{1}{2}$. The corresponding perimeter $p_{i;2} = 4\sqrt{2}$. Comparing with the circle, we find the inequalities $\alpha_{i;2} = \frac{1}{B_2} < \pi < \alpha_{c;2} = \frac{1}{A_2}$ and

$$\frac{1}{B_2} = 2 < \pi < \frac{1}{A_2} = 4$$

and $p_{i;2} < 2\pi < p_{c;2}$. We repeat the exercise for the octagon. The area $\alpha_{c;3}$ of circumscribed octagon (green) is 8 times the area of the triangle $q0q'$. The area of the triangle $q0q'$ is two times the area of the triangle $p0q = \frac{1}{2}\tan\frac{\pi}{8}$, hence $\alpha_{c;3} = 2^3\tan\frac{\pi}{2^3} = 3.31371$. The perimeter $p_{c;3}$ is $16|pq| = 2\alpha_{c;3}$. The area $\alpha_{i;3}$ of inscribed octagon (blue) is 8 times the area of the triangle $p0p'$, whose area is

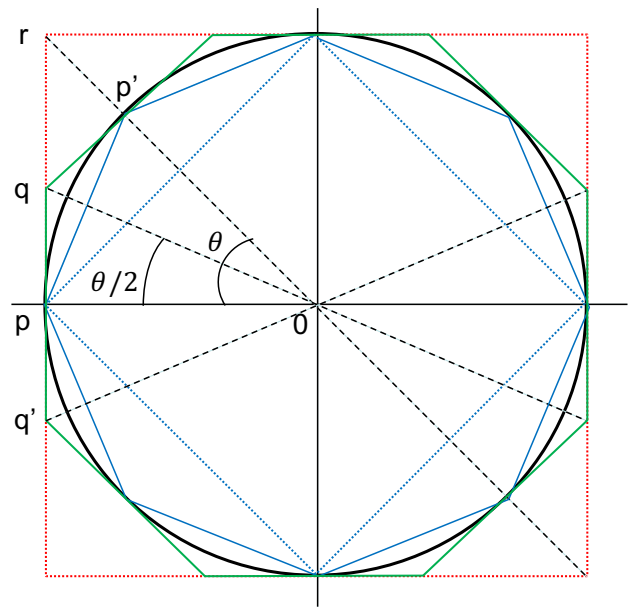

Figure 5: Inscribed and circumscribed 2^n-gon of a circle with radius 1.

$\frac{1}{2}\sin 2\theta_3 = \frac{1}{2}\sin\theta_2 = \frac{\sqrt{2}}{4}$; hence, $\alpha_{i;3} = 2\sqrt{2} = 2.82843$. The corresponding perimeter $p_{i;3}$ is 8 times $|pp'| = \sqrt{2 - 2\cos(2\theta_3)} = \sqrt{2-\sqrt{2}}$; hence, $p_{i;3} = 8\sqrt{2-\sqrt{2}} = 6.12293$. In summary, we obtain the inequalities

$$\frac{1}{B_3} = 2.82843 < \pi < \frac{1}{A_3} = 3.31371$$

and $p_{i;3} = 6.12293 < 2\pi < p_{c;3} = 6,62742$.

The area and perimeter of the circumscribed 2^n-gon are $\alpha_{c;n} = \frac{1}{A_n} = 2^n \tan\left(\frac{\pi}{2^n}\right)$ and $p_{c;n} = 2^{n+1}\tan\left(\frac{\pi}{2^n}\right)$, respectively. The area of the inscribed 2^n-gon is $\alpha_{i;n} = \frac{1}{B_n} = \frac{2^n}{2}\sin 2\theta_n = 2^{n-1}\sin\frac{\pi}{2^{n-1}}$, while its perimeter is $p_{i;n} = 2^n\sqrt{2 - 2\cos\left(2\frac{\pi}{2^n}\right)} = 2^{n+1}\sin\left(\frac{\pi}{2^n}\right)$. Invoking $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$ yields

$$\begin{aligned} A_n &= \frac{1}{2^n\tan\left(\frac{\pi}{2^n}\right)} = \frac{1}{2^n\tan\left(\frac{1}{2}\frac{\pi}{2^{n-1}}\right)} = \frac{1+\cos\left(\frac{\pi}{2^{n-1}}\right)}{2^n\sin\left(\frac{\pi}{2^{n-1}}\right)} = \frac{1}{2}\left(\frac{1}{2^{n-1}\sin\left(\frac{\pi}{2^{n-1}}\right)} + \frac{1}{2^{n-1}\tan\left(\frac{\pi}{2^{n-1}}\right)}\right) \\ &= \frac{1}{2}\left(B_n + A_{n-1}\right) \end{aligned}$$

while $\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos x}{2}}$ leads to

$$\begin{aligned} B_n &= \frac{1}{2^{n-1}\sin\frac{\pi}{2^{n-1}}} = \frac{1}{2^{n-1}\sin\frac{1}{2}\frac{\pi}{2^{n-2}}} = \frac{1}{2^{n-1}}\sqrt{\frac{2}{1-\cos\frac{\pi}{2^{n-2}}}} \\ &= \frac{1}{2^{n-1}}\sqrt{\frac{2\left(1+\cos\frac{\pi}{2^{n-2}}\right)}{1-\cos^2\frac{\pi}{2^{n-2}}}} = \sqrt{\frac{2\left(1+\cos\frac{\pi}{2^{n-2}}\right)}{2^{2n-2}\sin^2\frac{\pi}{2^{n-2}}}} = \sqrt{\frac{1}{2^{n-2}\sin\frac{\pi}{2^{n-2}}}\left(\frac{1+\cos\frac{\pi}{2^{n-2}}}{2^{n-1}\sin\frac{\pi}{2^{n-2}}}\right)} \\ &= \sqrt{B_{n-1}A_{n-1}} \end{aligned}$$

which demonstrates the Archimedes recursion (120). The computation shows, as mentioned in Borwein and Borwein [5], that $A_n(\theta) = \frac{1}{2^n\tan\left(\frac{\theta}{2^n}\right)}$ and $B_n(\theta) = \frac{1}{2^{n-1}\sin\frac{\theta}{2^{n-1}}}$ satisfy the Archimedes recursion

(120) for any complex θ. Since

$$\lim_{n\to\infty} A_n(\theta) = \lim_{n\to\infty} B_n(\theta) = \frac{1}{\theta}$$

Archimedes' recursion (120) can be used to calculate the inverse trigonometric and inverse hyperbolic functions. The difference

$$\begin{aligned} A_n - B_n &= \frac{1}{2}\left(A_{n-1} + \sqrt{B_{n-1}A_{n-1}}\right) - \sqrt{B_{n-1}A_{n-1}} \\ &= \frac{1}{2}\left(A_{n-1} - \sqrt{B_{n-1}A_{n-1}}\right) = \frac{\sqrt{A_{n-1}}}{2}\left(\sqrt{A_{n-1}} - \sqrt{B_{n-1}}\right) = \frac{\sqrt{A_{n-1}}\left(A_{n-1} - B_{n-1}\right)}{2\left(\sqrt{A_{n-1}} + \sqrt{B_{n-1}}\right)} \end{aligned}$$

illustrates that $\frac{1}{4}\left(A_{n-1} - B_{n-1}\right) < A_n - B_n < \frac{1}{2}\left(A_{n-1} - B_{n-1}\right)$. Iterated

$$A_n - B_n < \frac{1}{2}\left(A_{n-1} - B_{n-1}\right) < \frac{1}{2^2}\left(A_{n-2} - B_{n-2}\right) < \cdots < \frac{1}{2^p}\left(A_{n-p} - B_{n-p}\right)$$

shows that Archimedes' recursion (120) converges as $A_n - B_n < \frac{1}{2^{n-1}}\left(A_{n-1} - B_{n-1}\right)$, implying that the error decreases at each iteration with a factor a little less than 4, because as $B_n \le A_n$, the factor $\frac{\sqrt{A_{n-1}}}{2\left(\sqrt{A_{n-1}}+\sqrt{B_{n-1}}\right)} = \frac{1}{2\left(1+\frac{\sqrt{B_{n-1}}}{\sqrt{A_{n-1}}}\right)} \le \frac{1}{4}$ and equality only holds when $n \to \infty$. In other words, the computation of n decimal digits of π (or inverses of trigonometric or hyperbolic functions) requires $O(n)$ iterations. The table below computes Archimedes' recursion (120) for π up to $n = 15$:

n	$\frac{1}{A_n}$	$\frac{1}{B_n}$	$\frac{1}{A_n} - \frac{1}{B_n}$
1	4	2	2
2	3.3137084989	2.82842712474	0.485281
3	3.1825978780	3.06146745892	0.12113
4	3.1517249074	3.12144515225	0.0302798
5	3.1441183852	3.13654849054	0.00756989
6	3.1422236299	3.14033115695	0.00189247
7	3.1417503691	3.14127725093	0.000473118
8	3.1416320807	3.14151380114	0.00011828
9	3.1416025102	3.14157294036	0.0000295699
10	3.1415951177	3.14158772527	$7.39247\ 10^{-6}$
11	3.1415932696	3.14159142151	$1.84812\ 10^{-6}$
12	3.1415928075	3.14159234557	$4.6203\ 10^{-7}$
13	3.1415926920	3.14159257658	$1.15507\ 10^{-7}$
14	3.1415926632	3.14159263433	$2.88768\ 10^{-8}$
15	3.1415926559	3.14159264877	$7.21921\ 10^{-9}$

42. *AGM computation of* π. Following Almkvist and Berndt [2, Theorem 5], we start from Legendre's formula (98) for $k = k' = \frac{1}{\sqrt{2}}$,

$$2K\left(\frac{1}{\sqrt{2}}\right)E\left(\frac{1}{\sqrt{2}}\right) - K^2\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}$$

With $I(a,b) = \frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right)$ in **art**. 28 and choosing $a=1$ and $b=\frac{1}{\sqrt{2}}$, we have $I\left(1,\frac{1}{\sqrt{2}}\right) = K\left(\frac{1}{\sqrt{2}}\right)$ and, similar for the integral $J(a,b) = aE\left(\frac{c}{a}\right)$ in **art**. 31, it holds that $J\left(1,\frac{1}{\sqrt{2}}\right) = E\left(\frac{1}{\sqrt{2}}\right)$. As a second relation between $K\left(\frac{1}{\sqrt{2}}\right)$ and $E\left(\frac{1}{\sqrt{2}}\right)$, (95) is used,

$$E\left(\frac{1}{\sqrt{2}}\right) = \left(1-\frac{1}{2}\sum_{n=0}^{\infty}2^n c_n^2\right)K\left(\frac{1}{\sqrt{2}}\right)$$

which we substitute in Legendre's formula

$$\left(2\left(1-\frac{1}{2}\sum_{n=0}^{\infty}2^n c_n^2\right)-1\right)K^2\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}$$

Finally, we invoke Gauss's basic integral $\frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right) = \frac{\pi}{2}\frac{1}{M(a,b)}$ in (84), indicating that $K\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}\frac{1}{M\left(1,\frac{1}{\sqrt{2}}\right)}$, which leads to $\pi = \frac{2M^2\left(1,\frac{1}{\sqrt{2}}\right)}{1-\sum_{n=0}^{\infty}2^n c_n^2}$ or

$$\pi = \frac{4M^2\left(1,\frac{1}{\sqrt{2}}\right)}{1-\sum_{n=1}^{\infty}2^{n+1}c_n^2} \tag{121}$$

Borwein & Borwein [5, Section 5] present another algorithm for π with second order convergence.

The corresponding computation for π in (121), based on Gauss's arithmetic-geometric mean algorithm (2), is considerably fast. Only 10 iterations , with $a=1$ and $b=\frac{1}{\sqrt{2}}$, lead to an astonishingly small error at $n=9$ of less than $10^{-2^9} \approx 10^{-512}$, as illustrated in the table, where 30 decimal digits are given:

n	a_n	b_n	$2c_{n+1} = a_n - b_n$	formula (121)
0	1	0.707106781186547524400844362105	0.29289	4
1	0.853553390593273762200422181052	0.840896415253714543031125476233	0.012656	3.18767264271210862720192997053
2	0.847224902923494152615773828643	0.847201266746891460403631453693	0.000023636	3.14168029329765329391807042456
3	0.847213084835192806509702641168	0.847213084752765366704298051780	$8.24274\ 10^{-11}$	3.14159265389544649600291475882
4	0.847213084793979086607000346474	0.847213084793979086605997900490	$1.00244\ 10^{-21}$	3.14159265358979323846636060271
5	0.847213084793979086606499123482	0.847213084793979086606499123482	$1.48265\ 10^{-43}$	3.14159265358979323846264338328
6			$3.24336\ 10^{-87}$	
7			$1.55206\ 10^{-174}$	
8			$3.55415\ 10^{-349}$	
9			$1.86375\ 10^{-698}$	

We again observe that, at each iteration in n, the number of decimal digits approximately doubles!!

We add another AGM computation of π due to Gauss [15, p. 377]. We deduce from (40) and (42) the inequality

$$\frac{1}{2^n}\log\left(\frac{4b_{n+1}}{c_n}\right) \le \frac{\pi}{2}\frac{M(a,b)}{M(a,c)} \le \frac{1}{2^n}\log\left(\frac{4a_n}{c_n}\right) \tag{122}$$

Thus, if $a = b\sqrt{2} = c\sqrt{2}$, then $\frac{M(a,b)}{M(a,c)} = 1$ and, confining to $n=0$ in (122), we have that $\log\left(\frac{4a_0}{c_0}\right) = \log\left(4\sqrt{2}\right) = \frac{5}{2}\log(2)$, while $\log\left(\frac{4b_1}{c_0}\right) = \log\left(\frac{4\sqrt{a_0b_0}}{c_0}\right) = \log\left(2^{2+\frac{1}{4}}\right) = \frac{9}{4}\log 2$ from which Gauss concludes that $\frac{9}{4}\log 2 < \frac{\pi}{2} < \frac{10}{4}\log 2$. If n increases in (122), we find lower and upper bounds for π where each step in n results in the famous approximate doubling of decimal digits. Choosing $a=1$ and $b=\frac{1}{\sqrt{2}}$ as above, the first four iterations of (122) are

n	$\frac{1}{2^{n-1}}\log\left(\frac{4b_{n+1}}{c_n}\right)$	$\frac{1}{2^{n-1}}\log\left(\frac{4a_n}{c_n}\right)$
0	3.11916231251975389237754454656	3.46573590279972654708616060729
1	3.14157172949917097506914830630	3.14904153515897666929968289288
2	3.14159265355330856833419360289	3.14159962823802109942254203509
3	3.14159265358979323846242152809	3.14159265360195479517183083648
4	3.14159265358979323846264338328	3.14159265358979323846271733501

while for $n = 5$ all 30 digits are correct. The lower bound for $n = 4$ has already 30 decimals correct, while the upper bound only has 22 correct decimals. Hence, we observe that Gauss's upper and lower bounds (122) converge a little faster in n towards π than (121), in spite of the computation of the logarithm, which is numerically more demanding.

In summary[36], Leibniz method requires about 10^n terms for n correct decimals in π, Archimedes' recursion (120) needs n iterations for about $\frac{n}{p}$ with $2 < p < 4$ correct decimals, Euler's series (119) gives about $2n$ correct digits after n terms, while Gauss's AGM (121) returns about 2^n correct digits at iteration n!

43. *Newton-Raphson method.* Borwein and Borwein [5] mention that, apart from Newton-Raphson's iteration, no generally convergence method of quadratic order is known, except for Gauss's AGM algorithm (2).

Let us briefly explain Newton's method. Assume that the Taylor series $\sum_{k=0}^{\infty} f_k(z_0)(z-z_0)^k$ of a function $f(z)$ is known around z_0, where $f_k(z_0) = \frac{1}{k!}\left.\frac{d^k f(u)}{du^k}\right|_{u=z_0}$, and assume also that z_0 is a reasonably good approximation of a zero of $f(z)$. Let $z - z_0 = h$ and if h is sufficiently small, then

$$f(z_0+h) = f(z_0) + f_1(z_0)h + O\left(h^2\right)$$

Newton observes that by requiring that $f(z_0+h) = 0$, a good approximation of h up to $O\left(h^2\right)$ can be computed by solving the linear equation in h,

$$h_{(1)} = -\frac{f(z_0)}{f_1(z_0)}$$

If the first derivative can be computed in a range around z_0, then Newton's iteration scheme for the zero is

$$z_k = z_{k-1} - \frac{f(z_{k-1})}{f_1(z_{k-1})} \tag{123}$$

and the sequence $z_0, z_1, z_2, \ldots, z_m$ must converge to the correct[37] zero ζ of $f(z)$, which is close enough to z_0. Indeed and limiting ourselves to the first order Newton scheme,

$$\begin{aligned} f(z_k) &= f\left(z_{k-1} - \frac{f(z_{k-1})}{f_1(z_{k-1})}\right) \\ &= f(z_{k-1}) - f_1(z_{k-1})\frac{f(z_{k-1})}{f_1(z_{k-1})} + f_2(z_{k-1})\left(\frac{f(z_{k-1})}{f_1(z_{k-1})}\right)^2 + O\left(\left(\frac{f(z_{k-1})}{f_1(z_{k-1})}\right)^3\right) \\ &= f_2(z_{k-1})\left(\frac{f(z_{k-1})}{f_1(z_{k-1})}\right)^2 + O\left(\frac{f(z_{k-1})}{f_1(z_{k-1})}\right) \end{aligned}$$

[36] There exist many more computations of π for which we refer to Wikipedia. The latest computations are based on series deduced from modular forms, first exploited by Ramanujan, but further developed by others.

[37] If $f(z)$ is real on the real z-axis, then Newton's iteration (123) will only converge to a possibly real zero. In case $f(z)$ has a complex zero, then the initial starting value must be complex and not purely real.

which shows, approximately provided that $h_{(1)} = -\frac{f(z_{k-1})}{f_1(z_{k-1})}$ is small enough to ignore terms of order 3 and higher, that

$$f(z_k) \simeq \frac{f_2(z_{k-1})}{f_1^2(z_{k-1})} (f(z_{k-1}))^2$$

In other words, the sequence $\{f(z_k)\}_{k\geq 0}$ converges *quadratically*: if $f(z_{k-1}) = 10^{-a}$ is already close to zero, then $f(z_k) \simeq 10^{-2a}$, provided that the derivatives $f_1(z_{k-1})$ are not too small, nor $f_2(z_{k-1})$ too large.

A slightly more accurate variant of Newton's iteration scheme (123) uses the Taylor series up to order $O(h^3)$

$$f(z_0 + h) = f(z_0) + f_1(z_0) h + f_2(z_0) h^2 + O(h^3)$$

Solving the quadratic equation $f(z_0)+f_1(z_0) h+f_2(z_0) h^2 = 0$ in h gives $h_{(2)} = \frac{-f_1(z_0)\pm\sqrt{f_1^2(z_0)-4f(z_0)f_2(z_0)}}{2f_2(z_0)}$, where the proper sign must be chosen. If the first and second derivatives can be computed, then Newton's more accurate iteration scheme is

$$z_k = z_{k-1} + \frac{-f_1(z_{k-1}) \pm \sqrt{f_1^2(z_{k-1}) - 4f(z_{k-1}) f_2(z_{k-1})}}{2f_2(z_{k-1})} \tag{124}$$

Similarly, Newton's accurate iteration scheme (124) yields

$$\begin{aligned} f(z_k) &= f\left(z_{k-1} + \frac{-f_1(z_{k-1}) \pm \sqrt{f_1^2(z_{k-1}) - 4f(z_{k-1}) f_2(z_{k-1})}}{2f_2(z_{k-1})}\right) \\ &= f_3(z_{k-1}) \left(\frac{-f_1(z_{k-1}) \pm \sqrt{f_1^2(z_{k-1}) - 4f(z_{k-1}) f_2(z_{k-1})}}{2f_2(z_{k-1})}\right)^3 + O(h^4) \end{aligned}$$

illustrating that the sequence $\{f(z_k)\}_{k\geq 0}$ converges *cubically*: if $f(z_{k-1}) = 10^{-a}$ is already close to zero, then $f(z_k) \simeq 10^{-3a}$, provided that the step $h_{(2)}$ is small enough to enter the cubic convergence regime.

The Borwein brothers [5] mention that inverse functions can be effectively computed by Newton's recursion (123). Indeed, replace $f(x)$ by $g(x) - y$, then Newton's recursion returns the zero of $g(x) - y = 0$, which equals $x = g^{-1}(y)$. For example, if $g(x) = x^p$, then Newton's first order recursion (123) becomes $z_k = \frac{z_{k-1}}{p}\left\{(p-1) + \frac{y}{z_{k-1}^p}\right\}$. For $p = 2$, Newton's recursion simplifies to

$$z_k = \frac{1}{2}\left\{z_{k-1} + \frac{y}{z_{k-1}}\right\} \tag{125}$$

and converges to $y^{\frac{1}{2}}$. The recursion for $\sqrt{y}$ in (125) with initial start $z_0 = y$ was already known [5, p. 353] by the Babylonians! Numerical computations of the Babylonian algorithm, starting with $z_0 = 9$, gives $z_1 = 5$, $z_2 = 3.4$, $z_3 = 3.023529$, $z_4 = 3.0000915$, $z_5 = 3.000000001396$ and $z_6 = 3.000000000000000000032526$ has 19 decimals correct, while z_7 about 38 decimals, z_8 about 77 and further about a doubling each iteration. Of course, the smaller y, the closer it is to $\sqrt{y}$ and the faster the second order convergence kicks in. For $y = 2$, the third iteration z_3 has 6 correct decimals, 12 correct decimals for z_4, 24 for z_5 and so on. The extremely fast converges for $\sqrt{y}$ may simplify the computation of the geometric mean Gauss's AGM algorithm (2). Newton's iterative scheme (123) was likely inspired by François Viète's method for the computation of $\sqrt{y}$ as outlined in Edward [11, p. 184].

Housedorf's method $z_k = z_{k-1} + m\frac{\left(\frac{1}{f}\right)^{(m-1)}(z_{k-1})}{\left(\frac{1}{f}\right)^{(m)}(z_{k-1})}$, with initial guess z_0, is a generalization of Newton's recursion with convergence of m-th order. If $m = 1$, Housedorf's method reduces to Newton's recursion (123), because

$$m\frac{\left(\frac{1}{f}\right)^{(m-1)}(z_{k-1})}{\left(\frac{1}{f}\right)^{(m)}(z_{k-1})} = \frac{\frac{1}{f(z_{k-1})}}{\left(\frac{1}{f}\right)^{(1)}(z_{k-1})} = \frac{1}{f(z_{k-1})}\left(-\frac{1}{\frac{1}{f^2(z_{k-1})}f'(z_{k-1})}\right) = -\frac{f(z_{k-1})}{f_1(z_{k-1})}$$

Due to the higher efforts and possible computational complications of the higher order derivatives of $\frac{1}{f(z)}$, the effective convergence of Household's method can be of lower order than m. A similar observation holds for Newton's more accurate iteration scheme (124) compared to (123).

12 Summary

We have tried to follow the steps in Gauss's Nachlass from the AGM algorithm towards his series of what are now called the Jacobi theta functions. Almost all statements or formulae of Gauss along the track are re-derived. Sections 9-11 are applications of the AGM.

While the first part [14] (in Latin) is well organized and relatively easy to follow and verify, the second part [15, art. 13, p.380] (in German) in the Nachlass of Gauss's results about the AGM is challenging and mainly outlines end results with little guidance how these formulae or results were obtained. On the other hand, that second part illustrates that Gauss must have made many more computations and studies that are not recorded, thus lost. It is also unclear to me what Gauss wrote himself in [15, art. 13, p.380] and what others, who have published this posthumous work, have written or interpreted. Apart from one minus sign in the differential equation (66) – where I and not Gauss must have been mistaken – I did not find a single mistake, nor in his numerical computations. Being close to error-free is again extremely unusual for a normal human. But, Gauss was definitely not "normal", as measured by his own Gaussian or normal distribution, in which the level of his mind-blowing achievements lies several standard deviations away from the human mean!

Critics or negatively oriented comments about Gauss's work have appeared. In those cases, where I understood Gauss's work, I find that these comments, written at a much later time with the current insights, degrade the master's statue.

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A Elliptic functions

We briefly review elliptic functions.

44. *Elliptic functions.* The general theory of elliptic functions starts today from the Weierstrass elliptic functions [27, 25, 35]. Elliptic functions $\mathcal{E}(z)$ are completely defined in a parallelogram P in the complex plane. Liouville's theorem [35], stating that a complex function which is bounded in the entire complex plane is a constant, indicates that elliptic functions (with a minimum amount of singularities in their parallelogram P) must[38] have either two simple poles or a double pole within each parallelogram. The basic parallelogram P_0 consists of the linear combination $x\omega_1+y\omega_2$ of the two periods ω_1 and ω_2, where ω_1 and ω_2 are complex numbers so that $\tau=\frac{\operatorname{Im}\omega_2}{\operatorname{Im}\omega_1}>0$ and the real numbers $0\leq x<1$ and $0\leq y<1$. By translations of the basic parallelogram P_0, the entire complex plane can be covered. Any set of independent complex numbers ω_1' and ω_2' can represent a parallelogram [7, Chapter I],[27, p. 145-148; art.86; p. 205-221; p. 238-246], but each of such set can be produced by a linear, unimodular transformation $\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$ from the two periods ω_1 and ω_2. The ratio $\frac{\omega_1'}{\omega_2'}=\frac{a\frac{\omega_1}{\omega_2}+b}{c\frac{\omega_1}{\omega_2}+d}$ is a Möbius or unimodular transform $u=\frac{az+b}{cz+d}$. The geometry of the basic parallelogram leads to the Möbius transform, that provides rapidly converging series for elliptic functions. Liouville's theorem [35] is a powerful tool to deduce many properties of elliptic functions and theta functions.

45. *Weierstrass elliptic functions.* A direct application of Weierstrass's beautiful theory of entire functions [31] in the complex plane generates the Weierstrass elliptic functions. The Weierstrass's elliptic functions $\sigma(u)$ is

$$\sigma(u)=\sigma(u|\omega_1,\omega_2)=u\prod_{n=-\infty;n\neq 0}^{\infty}\prod_{m=-\infty;m\neq 0}^{\infty}\left(1-\frac{u}{n\omega_1+m\omega_2}\right)e^{\left(\frac{u}{n\omega_1+m\omega_2}+\frac{1}{2}\left(\frac{u}{n\omega_1+m\omega_2}\right)^2\right)}$$

whose zeros corresponds to $\Omega=n\omega_1+m\omega_2$, i.e. all complex corner points of all period parallelograms. The Weierstrass zeta-function is $\zeta(u)=\frac{d}{du}\log\sigma(u)$ and Weierstrass P-function is $\mathcal{P}(u)=-\frac{d\zeta(u)}{du}=$

[38] Indeed, the contour integral along the parallelogram is $\int_{\partial P}\mathcal{E}(z)\,dz=0$, due to periodicity. Cauchy's integral theorem tells us that $\int_{\partial P}\mathcal{E}(z)\,dz=2\pi i\sum_{k=1}^{m}\operatorname{res}(p_k)$, where p_k is a pole of $\mathcal{E}(z)$ in P. If $m=1$ and $\operatorname{res}(p_1)=0$, then $\mathcal{E}(z)$ is analytic inside P and on its perimeter ∂P and thus bounded, which Liouville's theorem prevents. Hence, m must be at least 2, implying either two single poles with opposite residue or a double pole with residue zero, i.e. $\lim_{z\to p_2}2\pi i\frac{d}{dz}\left((z-p_2)^2\mathcal{E}(z)\right)=0$ from which it must hold that

$$\lim_{z\to p_2}(z-p_2)^2\frac{d}{dz}\mathcal{E}(z)=2\lim_{z\to p_2}(z-p_2)\mathcal{E}(z)$$

$-\frac{d^2}{du^2}\log\sigma(u) = \frac{1}{u^2} + \sum_\Omega' \left(\frac{1}{(u-\Omega)^2} - \frac{1}{\Omega^2}\right)$. Further variants are defined as $\sigma_a(u) = \frac{e^{-\eta_a u}\sigma(u+\omega_a)}{\sigma(\omega_a)}$ and $\eta_a = \zeta(\omega_a)$ for $a = 1, 2, 3$, where ω_3 obeys $\omega_1 + \omega_2 + \omega_3 = 0$. Also, it holds that $\eta_1 + \eta_2 + \eta_3 = 0$, $\mathcal{P}(\omega_1) + \mathcal{P}(\omega_2) + \mathcal{P}(\omega_3) = 0$ and a variant of Legendre's relation is $\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi}{2}$ for $\operatorname{Im}\frac{\omega_2}{\omega_1} = \operatorname{Im}\tau > 0$.

46. *Theta functions.* The basic properties of theta functions are treated in many books (see e.g. [35, chap. XXI], [7, pp. 69], [24, pp. 172], [25, Chapter 5]). A summary of elegant properties of theta functions, illustrating their mathematical beauty as well as their power, is written by Bellman [3].

Tannery and Molk [28, p. 1-14] start from Weierstrass's elliptic functions $\sigma(u)$ and $\sigma_\alpha(u)$ for $\alpha = 1, 2, 3$, in which they make the substitutions $\mathfrak{q} = e^{i\pi\tau}$ for $\operatorname{Im}\tau > 0$, $\tau = \frac{\omega_2}{\omega_1}$ and $z = \frac{u}{\omega_1}$, where ω_1 and ω_2 are the complex periods[39]. After those substitutions and deducing nice algebraic transformations of infinite products, they [28, p. 14] define[40], as Jacobi did in [17], the theta functions as

$$\vartheta_1(z|\tau) = \frac{1}{i}\sum_{m=-\infty}^{\infty}(-1)^m\,\mathfrak{q}^{\left(m+\frac{1}{2}\right)^2}e^{(2m+1)i\pi z}$$

$$\vartheta_2(z|\tau) = \sum_{m=-\infty}^{\infty}\mathfrak{q}^{\left(m+\frac{1}{2}\right)^2}e^{(2m+1)i\pi z}$$

$$\vartheta_3(z|\tau) = \sum_{m=-\infty}^{\infty}\mathfrak{q}^{m^2}e^{2mi\pi z}$$

$$\vartheta_4(z|\tau) = \sum_{m=-\infty}^{\infty}(-1)^m\,\mathfrak{q}^{m^2}e^{2mi\pi z}$$

which are rewritten [28, art. 161, p. 16] as

$$\vartheta_1(z|\tau) = 2\sum_{m=0}^{\infty}(-1)^m\,\mathfrak{q}^{\left(m+\frac{1}{2}\right)^2}\sin((2m+1)\pi z) \tag{126}$$

$$\vartheta_2(z|\tau) = 2\sum_{m=0}^{\infty}\mathfrak{q}^{\left(m+\frac{1}{2}\right)^2}\cos((2m+1)\pi z) \tag{127}$$

$$\vartheta_3(z|\tau) = 1 + 2\sum_{m=1}^{\infty}\mathfrak{q}^{m^2}\cos(2m\pi z) \tag{128}$$

$$\vartheta_4(z|\tau) = 1 + 2\sum_{m=1}^{\infty}(-1)^m\,\mathfrak{q}^{m^2}\cos(2m\pi z) \tag{129}$$

with $\mathfrak{q} = e^{\pi i\tau}$ and $\operatorname{Im}(\tau) > 0$. We further refer for a wealth of theta-function properties to Olver et al. [23, Chapter 20].

[39] Tannery and Molk define the period as $2\omega_1$ and $2\omega_3$. They define ω_2 such that $\omega_1 + \omega_2 + \omega_3 = 0$.

[40] Unfortunately, there exist many slightly different notations for theta functions, but that of Tannery and Molk is now standard [24, pp. 172]. A general form defining all four theta functions [24, sec. 76],

$$\theta_{\mu\,\nu}(v \mid \tau) = \sum_{k=-\infty}^{\infty}(-1)^{\nu\,k}e^{\left(k+\frac{\mu}{2}\right)^2\pi i\tau}\,e^{2\pi i\left(k+\frac{\mu}{2}\right)v}$$

where μ and ν are 0 or 1.

In terms of the products

$$\begin{array}{ll} q_0 = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{2n}\right) & q_1 = \prod_{n=1}^{\infty}\left(1+\mathfrak{q}^{2n}\right) \\ q_2 = \prod_{n=1}^{\infty}\left(1+\mathfrak{q}^{2n-1}\right) & q_3 = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{2n-1}\right) \end{array}$$

that[41] obey $q_1q_2q_3 = 1$, the relations between theta and Weierstrass's sigma functions are [28, p. 17]

$$\begin{aligned} \frac{\pi}{\omega_1}\mathfrak{q}^{\frac{1}{4}}q_0^3\sigma\left(u\right) &= e^{2\eta_1\omega_1 z^2}\vartheta_1\left(z|\tau\right) \\ 2\mathfrak{q}^{\frac{1}{4}}q_0q_1^2\sigma_1\left(u\right) &= e^{2\eta_1\omega_1 z^2}\vartheta_2\left(z|\tau\right) \\ q_0q_2^2\sigma_2\left(u\right) &= e^{2\eta_1\omega_1 z^2}\vartheta_3\left(z|\tau\right) \\ q_0q_3^2\sigma_3\left(u\right) &= e^{2\eta_1\omega_1 z^2}\vartheta_4\left(z|\tau\right) \end{aligned}$$

B The incomplete elliptic integrals $F\left(\varphi,k\right)$ and $E\left(\varphi,k\right)$

47. *Elliptic integrals.* Legendre in [19] developed and studied elliptic integrals. The Legendre's elliptic integral [23, Section 19.2] of the first kind is[42]

$$F\left(\varphi,k\right) = \int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\varphi}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \tag{130}$$

and of the second kind

$$E\left(\varphi,k\right) = \int_0^{\varphi}\sqrt{1-k^2\sin^2\theta}d\theta = \int_0^{\sin\varphi}\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt \tag{131}$$

The cases with $\varphi = \frac{\pi}{2}$ are called the *complete elliptic integrals*

$$K\left(k\right) = F\left(\frac{\pi}{2},k\right) \text{ and } E\left(k\right) = E\left(\frac{\pi}{2},k\right)$$

If $k = 0$, then

$$F\left(\varphi,0\right) = E\left(\varphi,0\right) = \varphi$$

from which

$$K\left(0\right) = E\left(0\right) = \frac{\pi}{2}$$

If $k = 1$, then $E\left(\varphi,1\right) = \int_0^{\varphi}\cos\theta d\theta = \sin\varphi$ and

$$\begin{aligned} F\left(\varphi,1\right) &= \int_0^{\varphi}\frac{d\theta}{\cos\theta} = \int_0^{\sin\varphi}\frac{dt}{1-t^2} = \frac{1}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) = \log\left(\frac{1+\sin\varphi}{\cos\varphi}\right) = \log\left(\frac{\cos\varphi}{1-\sin\varphi}\right) = \\ &= \log\left(\frac{\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}}{1-2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}\right) = \log\left(\frac{\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}-2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}\right) = \log\left(\frac{\cos\frac{\varphi}{2}+\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}-\sin\frac{\varphi}{2}}\right) \end{aligned}$$

[41] Indeed, the product q_0 is

$$q_0 = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{2n}\right) = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{n}\right)\left(1+\mathfrak{q}^{n}\right) = \prod_{n=1}^{\infty}\left(1-\mathfrak{q}^{n}\right)\prod_{n=1}^{\infty}\left(1+\mathfrak{q}^{n}\right)$$

Writing each product over all positive integers n as products over even and odd integers, i.e. $\prod_{n=1}^{\infty}\left(1\pm\mathfrak{q}^{n}\right) = \prod_{n=1}^{\infty}\left(1\pm\mathfrak{q}^{2n}\right)\prod_{n=1}^{\infty}\left(1\pm\mathfrak{q}^{2n-1}\right)$, results in $q_0 = q_0q_1q_2q_3$.

[42] Unfortunately, in the literature, there are slightly different definitions of the elliptic integrals. There is also an elliptic integral of the third kind, for which we refer to [23, Section 19.2].

Hence, if $\varphi \to \frac{\pi}{2}$, then with $\cos\varphi = \sin\left(\frac{\pi}{2}-\varphi\right) = \left(\frac{\pi}{2}-\varphi\right)\left(1+O\left(\left(\frac{\pi}{2}-\varphi\right)^2\right)\right)$ and $F(\varphi,1) = \log\left(\frac{1+\sin\varphi}{\cos\varphi}\right)$, we find that $F(\varphi,1) = \log 2 - \log\left(\frac{\pi}{2}-\varphi\right) + O\left(\left(\frac{\pi}{2}-\varphi\right)^2\right)$, which illustrates a logarithmic singularity at $F\left(\frac{\pi}{2},1\right) = K(1)$.

Since $\sqrt{1-k^2\sin^2\theta} = \frac{1-k^2\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}$, the elliptic integral of the second kind is written by Legendre [19] as

$$E(\varphi,k) = F(\varphi,k) - k^2\int_0^{\varphi}\frac{\sin^2\theta d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{132}$$

48. *Behavior when* $k \to 1$. A different rewriting than Legendre, inspired by [5, p. 355] who fixed $\varphi = \frac{\pi}{2}$, is

$$F(\varphi,k) = \int_0^{\varphi}\frac{k\sin\theta\, d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_0^{\varphi}\frac{(1-k\sin\theta)\, d\theta}{\sqrt{1-k^2\sin^2\theta}} = -\int_0^{\varphi}\frac{d(k\cos\theta)}{\sqrt{1-k^2+k^2\cos^2\theta}} + \int_0^{\varphi}\sqrt{\frac{1-k\sin\theta}{1+k\sin\theta}}d\theta$$

Substitution of $u = \frac{k}{\sqrt{1-k^2}}\cos\theta$ transforms the first integral in the last equality to

$$\int_0^{\varphi}\frac{d(k\cos\theta)}{\sqrt{1-k^2+k^2\cos^2\theta}} = \int_0^{\varphi}\frac{d\left(\frac{k}{\sqrt{1-k^2}}\cos\theta\right)}{\sqrt{1+\frac{k^2}{1-k^2}\cos^2\theta}} = \int_{\frac{k}{\sqrt{1-k^2}}}^{\frac{k}{\sqrt{1-k^2}}\cos\varphi}\frac{du}{\sqrt{1+u^2}} = \operatorname{arcsinh}(u)\Big|_{\frac{k}{\sqrt{1-k^2}}}^{\frac{k}{\sqrt{1-k^2}}\cos\varphi}$$

The logarithmic representation [1, 4.6.20] of $\operatorname{arcsinh}(u) = \log\left(u+\sqrt{1+u^2}\right)$ indicates that

$$\int_0^{\varphi}\frac{d(k\cos\theta)}{\sqrt{1-k^2+k^2\cos^2\theta}} = \log\left(\frac{\frac{k}{\sqrt{1-k^2}}\cos\varphi + \sqrt{\frac{1-k^2+k^2\cos^2\varphi}{1-k^2}}}{\frac{k}{\sqrt{1-k^2}} + \sqrt{1+\frac{k^2}{1-k^2}}}\right) = \log\left(\frac{k\cos\varphi + \sqrt{1-k^2\sin^2\varphi}}{k+1}\right)$$

The second integral[43] exist for $k \to 1$ and for all angles $-\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$ and equals

$$\begin{aligned}\lim_{k\to 1}\int_0^{\varphi}\frac{(1-k\sin\theta)\, d\theta}{\sqrt{1-k^2\sin^2\theta}} &= \int_0^{\varphi}\frac{(1-\sin\theta)}{\cos\theta}d\theta = \int_0^{\varphi}\frac{d\theta}{\cos\theta} - \int_0^{\varphi}\tan\theta\, d\theta \\ &= \log\left(\frac{1+\sin\varphi}{\cos\varphi}\right) + \log(\cos\varphi) = \log(1+\sin\varphi)\end{aligned}$$

where $F(\varphi,1) = \int_0^{\varphi}\frac{d\theta}{\cos\theta}$ is evaluated above. In summary, if k tends to one, then the first elliptic integral behaves as

$$F(\varphi,k) = \log\left(\frac{k+1}{k\cos\varphi + \sqrt{1-k^2\sin^2\varphi}}\right) + \log(1+\sin\varphi) \qquad \text{if } k \to 1 \tag{133}$$

For $\varphi = \frac{\pi}{2}$, (133) simplifies to

$$K(k) = \frac{1}{2}\log\left(4\frac{1+k}{1-k}\right) \qquad \text{if } k \to 1$$

which is equivalent to $k \to 0$ in the complementary modulus $k' = \sqrt{1-k^2}$,

$$K(k') = \frac{1}{2}\log\left(4\frac{1+\sqrt{1-k^2}}{1-\sqrt{1-k^2}}\right) = \log\left(\frac{2\left(1+\sqrt{1-k^2}\right)}{k}\right) = O\left(\log\left(\frac{4}{k}\right)\right) \tag{134}$$

[43]The integrand in $\sqrt{\frac{1-k\sin\theta}{1+k\sin\theta}} \le 1$ for $\theta \in [0,\pi]$ can be expanded in a Taylor series in $k\sin\theta$, in which the remaining integral is evaluated as in **art**. 52 to provide an exact result for all k.

49. *Fundamental angle of* $\varphi \in \left[0, \frac{\pi}{2}\right]$. Parts of **art**. 10 are generalized to an arbitrary angle $\varphi \in \left[0, \frac{\pi}{2}\right]$ instead of just confining to $\varphi = \frac{\pi}{2}$. The substitution of $\theta = m\pi - w$ or $w = m\pi - \theta$, where m is an integer, in the integral (130)

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

leading to

$$F(m\pi - \varphi, k) = \int_0^{m\pi - \varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = -\int_{m\pi}^{\varphi} \frac{dw}{\sqrt{1 - k^2 \sin^2 (m\pi - w)}} = \int_{\varphi}^{m\pi} \frac{dw}{\sqrt{1 - k^2 \sin^2 (w)}}$$

and, similarly for the integral (131)

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

show that

$$F(m\pi - \varphi, k) = F(m\pi, k) - F(\varphi, k) \tag{135}$$

and, similarly,

$$E(m\pi - \varphi, k) = E(m\pi, k) - E(\varphi, k)$$

If $m = 0$, then $F(0, k) = E(0, k) = 0$ and we find that both elliptic integrals are odd functions of φ,

$$\begin{aligned} F(-\varphi, k) &= -F(\varphi, k) \\ E(-\varphi, k) &= -E(\varphi, k) \end{aligned} \tag{136}$$

If $\varphi = \pi$, then (135) simplifies to the recursion $F(m\pi, k) = F((m-1)\pi, k) + F(\pi, k)$, from which follows

$$\begin{aligned} F(m\pi, k) &= mF(\pi, k) \\ E(m\pi, k) &= mE(\pi, k) \end{aligned}$$

If $\varphi = \frac{\pi}{2}$, then (135) becomes, with $F(m\pi, k) = mF(\pi, k)$,

$$F\left((2m-1)\frac{\pi}{2}, k\right) = mF(\pi, k) - F\left(\frac{\pi}{2}, k\right)$$

which reduces for $m = 1$ to $F(\pi, k) = 2F\left(\frac{\pi}{2}, k\right) = 2K(k)$. Hence, with

$$\begin{aligned} F\left((2m-1)\frac{\pi}{2}, k\right) &= (2m-1)K(k) \\ E\left((2m-1)\frac{\pi}{2}, k\right) &= (2m-1)E(k) \end{aligned}$$

and (136), relation (135), and similarly for $E(\varphi, k)$, becomes

$$\begin{aligned} F(m\pi \pm \varphi, k) &= 2mK(k) \pm F(\varphi, k) \\ E(m\pi \pm \varphi, k) &= 2mE(k) \pm E(\varphi, k) \end{aligned} \tag{137}$$

The case $m = 1$ in (137), $F(\pi - \varphi, k) + F(\varphi, k) = 2K(k)$, illustrates for a real angle φ that its fundamental range is $\varphi \in \left[0, \frac{\pi}{2}\right]$.

50. *Inversion of elliptic integrals.* Let us define

$$z=\int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^{\sin\varphi}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}=F(\varphi,k)$$

then its inverse $\varphi=F^{-1}(z,k)=am(z,k)$ is Jacobi's amplitude [25, p. 286]. We start by writing the definition of the incomplete elliptic integral in (130) with $w=\sin\varphi$ as

$$F(\arcsin w,k)=\int_0^{w}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

For $k=0$, we obtain the trigonometric case, where $F(\arcsin w,0)=\arcsin(w)=z$ and its inverse function is $w=\sin(F(\arcsin w,0))=\sin z$. The novel observation for $k\neq 0$ and $k\neq 1$ is that the inversion of the Legendre's incomplete elliptic integral $F(\varphi,k)$ leads to an elliptic function, which is a single-valued complex function with not one but two independent complex periods. This fundamental discovery was first made by Gauss in 1799 and later, independently, by Abel in 1827.

Since $F^{-1}(z,k)=F^{-1}(F(\varphi,k),k)=\varphi$, we have $am(F(\varphi,k),k)=\varphi=F(am(z,k),k)$. The function $am(z,k)$ is odd, due to (136), and quasi-periodic, because it holds that

$$am(z+2K(k),k)=am(z,k)+\pi$$

which follows from $F(\pi+\varphi,k)=2K(k)+F(\varphi,k)$ in (137) by taking the inverse function $am(z,k)$ of both sides. If $k=0$, then $z=F(\varphi,k)=\varphi$, so that the inverse is $\varphi=am(z,0)=z$. The derivative of the basic inverse relation $f\left(f^{-1}(x)\right)=x$, being $\left.\frac{df}{du}\right|_{u=f^{-1}(x)}\frac{df^{-1}(x)}{dx}=1$, shows here that $\frac{dz}{d\varphi}=\frac{1}{\sqrt{1-k^2\sin^2\varphi}}$ and $\left(\frac{dz}{d\varphi}\right)^{-1}=\frac{d\varphi}{dz}=\sqrt{1-k^2\sin^2\varphi}$, thus the derivative, called delta amplitudinis, is

$$dn(z)=\frac{d(am(z,k))}{dz}=\sqrt{1-k^2\sin^2 am(z,k)}\geq 0$$

and

$$\frac{d^2(am(z,k))}{dz^2}=\frac{-k^2\sin(2am(z,k))}{\sqrt{1-k^2\sin^2 am(z,k)}}\frac{d(am(z,k))}{dz}=-k^2\sin(2am(z,k))$$

Hence, the function $am(z,k)$ is increasing and concave for $0\leq z\leq K(k)$ and convex for $K\leq z\leq 2K$. Jacobi further defines the sinus amplitudinis $sn(z,k)=\sin(am(z,k))$ and his cosinus amplitudinis $cs(z,k)=\cos(am(z,k))$. Both are periodic functions with period $4K(k)$. The zeros of $sn(z,k)$ are at $z=2nK(k)$ and the zeros of $cs(z,k)$ lies at $z=(2n+1)K(k)$. Since $am(-z,k)=-am(z,k)$ is odd, so is $sn(z,k)$, but $cs(z,k)$ is even. The Jacobian functions satisfy

$$sn^2(z,k)+cs^2(z,k)=1$$

and

$$dn^2(z,k)+k^2sn^2(z,k)=1$$

We further refer to [35, Chapter XXII] for the complex function theoretical treatment of Jacobian elliptic functions [17]. The integral $u=\int_0^y\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$ is equivalent to the differential equation

$$\left(\frac{dy}{du}\right)^2=\left(1-y^2\right)\left(1-k^2y^2\right)$$

with initial condition $\left.\frac{dy}{du}\right|_{y=u=0}=1$. It is shown in Whittaker and Watson [35, Chapter XXI] that the solution $y=\frac{\vartheta_3(0)}{\vartheta_2(0)}\frac{\vartheta_1\left(\frac{u}{\vartheta_3^2(0)}\right)}{\vartheta_4\left(\frac{u}{\vartheta_3^2(0)}\right)}$ exists, where the Jacobian theta functions $\vartheta_1,\vartheta_2,\vartheta_3$ and ϑ_4 possess the form $\vartheta_j(z|\tau)$ with the parameter τ that satisfies $k^2=\frac{\vartheta_2^4(0|\tau)}{\vartheta_3^4(0|\tau)}$. In other words, the solution $y=sn(u,k)$ can be written as a quotient of Jacobian theta functions, a fact that Gauss has discovered first, but never published.

51. *Elliptic integral transformations.* The integral $F(\varphi,k)=\int_0^{\sin\varphi}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$ in (130) becomes for $k\to ik$,

$$F(\varphi,ik)=\int_0^{\sin\varphi}\frac{dt}{\sqrt{1-t^2}\sqrt{1+k^2t^2}}=\frac{1}{\sqrt{1+k^2}}\int_0^{\sin\varphi}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\frac{k^2}{1+k^2}(1-t^2)}}$$

Substitute $u^2=1-t^2$, then $t=\sqrt{1-u^2}$ and $dt=\frac{-udu}{\sqrt{1-u^2}}$,

$$F(\varphi,ik)=\frac{1}{\sqrt{1+k^2}}\int_{\cos\varphi}^1\frac{dt}{\sqrt{1-\frac{k^2}{1+k^2}u^2}}\frac{du}{\sqrt{1-u^2}}$$

to obtain

$$F(\varphi,ik)=\frac{1}{\sqrt{1+k^2}}\left(F\left(\frac{\pi}{2},\frac{k}{\sqrt{1+k^2}}\right)-F\left(\cos\varphi,\frac{k}{\sqrt{1+k^2}}\right)\right)$$

Hence, for $\varphi=\frac{\pi}{2}$, we find for the imaginary modulus that

$$K(ik)=\frac{1}{\sqrt{1+k^2}}K\left(\frac{k}{\sqrt{1+k^2}}\right) \tag{138}$$

If $x=\cos\theta$, then (130) becomes

$$\begin{aligned}F(\varphi,k)&=\int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2+k^2\cos^2\theta}}=-\int_1^{\cos\varphi}\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2+k^2x^2}}\\&=-\frac{1}{\sqrt{1-k^2}}\int_1^{\cos\varphi}\frac{dx}{\sqrt{1-x^2}\sqrt{1+\frac{k^2}{1-k^2}x^2}}\end{aligned}$$

which is written, in terms of the modulus k and complementary modulus $k'=\sqrt{1-k^2}$ as

$$F(\varphi,k)=\frac{1}{k'}\int_{\cos\varphi}^1\frac{dx}{\sqrt{(1-x^2)\left(1+\left(\frac{k}{k'}\right)^2x^2\right)}} \tag{139}$$

If $u=\frac{1}{t}$, then

$$F(\varphi,k)=\int_{\frac{1}{\sin\varphi}}^{\infty}\frac{du}{\sqrt{u^2-1}\sqrt{u^2-k^2}}$$

The conformal map of the half-plane into a rectangle is treated in [20, p. 55] and [25, Vol. 2, p. 127-138].

52. *Taylor series for $F(\varphi,k)$ and $E(\varphi,k)$.* The Taylor series $(1+z)^\alpha=\sum_{m=0}^{\infty}\binom{\alpha}{m}z^m$, convergent for $|z|<1$ and for any complex α, shows that

$$(1+z)^\alpha(1+z)^\beta=\sum_{m=0}^{\infty}\binom{\alpha}{m}z^m\sum_{m=0}^{\infty}\binom{\beta}{m}z^m=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\binom{\alpha}{n}\binom{\beta}{m-n}\right)z^m$$

Equating corresponding powers in z in the above Taylor series and $(1+z)^{\alpha+\beta}=\sum_{m=0}^{\infty}\binom{\alpha+\beta}{m}z^m$ leads to Vandermonde's formula

$$\binom{\alpha+\beta}{m}=\sum_{n=0}^{m}\binom{\alpha}{n}\binom{\beta}{m-n} \tag{140}$$

If $\beta=1$, then $\binom{\beta}{m-n}=\binom{1}{m-n}=1_{\{\{m-n=0\}\text{or}\{m-n=1\}\}}$ and (140) reduces to

$$\binom{\alpha+1}{m}=\binom{\alpha}{m}+\binom{\alpha}{m-1}$$

which generalizes the binomial recursion $\binom{n+1}{m}=\binom{n}{m}+\binom{n}{m-1}$ in e.g. [1, 24.1.1.II.A] to complex α. If $\beta=-\alpha$, then Vandermonde's formula (140) gives

$$\sum_{n=0}^{m}\binom{\alpha}{n}\binom{-\alpha}{m-n}=1$$

which can be written as a recursion $\binom{-\alpha}{m}=1-\sum_{n=1}^{m}\binom{\alpha}{n}\binom{-\alpha}{k-n}$, from which $\binom{-\alpha}{m}$ can be determined in terms of $\binom{\alpha}{n}$. On the other hand, the definition of the complex binomial coefficient $\binom{\alpha}{n}=\frac{\alpha!}{n!(\alpha-n)!}=\frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}$ encourages to employ the properties of the Gamma function $\Gamma(z)$. The binomial coefficients of the Taylor series $(1-z)^{-\frac{1}{2}}=\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}z^m$ and $(1-z)^{\frac{1}{2}}=\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}z^m$ are investigated in **art**. 9.

Introducing the Taylor series $(1+z)^{\alpha}=\sum_{m=0}^{\infty}\binom{\alpha}{k}z^m$ into the integral

$$\int_0^{\varphi}\left(1-k^2\sin^2\theta\right)^{\alpha}d\theta=\sum_{m=0}^{\infty}\binom{\alpha}{m}(-1)^m k^{2m}\int_0^{\varphi}\sin^{2m}\theta d\theta \tag{141}$$

establishes its Taylor series in k and requires the evaluation of the integral $\int_0^{\varphi}\sin^{2m}\theta d\theta$. There are different methods[44]. We start from Newton's binomium

$$\begin{aligned}\sin^{2m}\theta&=\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^{2m}=\frac{(-1)^m}{2^{2m}}\sum_{l=0}^{2m}\binom{2m}{l}(-1)^l e^{i\theta l}e^{-i\theta(2m-l)}\\&=\frac{(-1)^m}{2^{2m}}\sum_{l=0}^{2m}\binom{2m}{l}(-1)^l e^{-2i\theta(m-l)}\end{aligned}$$

[44] In general, we define

$$\begin{aligned}T(\beta)&=\int_0^{\varphi}\sin^{\beta}\theta d\theta=\int_0^{\varphi}\sin^{\beta-1}\theta\sin\theta d\theta=-\int_0^{\varphi}\sin^{\beta-1}\theta d\cos\theta\\&=-\int_0^{\varphi}\left(1-\cos^2\theta\right)^{\frac{\beta-1}{2}}d\cos\theta=\int_{\cos\varphi}^{1}\left(1-u^2\right)^{\frac{\beta-1}{2}}du=\sum_{m=0}^{\infty}\binom{\frac{\beta-1}{2}}{k}(-1)^m\frac{1-\cos^{2m+1}\varphi}{2m+1}\end{aligned}$$

Another method starts from partial integration of $\int_0^{\varphi}\sin^{2m}\theta d\theta=\int_0^{\varphi}\sin^{2m-1}\theta\sin\theta d\theta$,

$$\begin{aligned}\int_0^{\varphi}\sin^{2m}\theta d\theta&=-\cos\varphi\sin^{2m-1}\varphi+(2m-1)\int_0^{\varphi}\sin^{2m-2}\theta\cos^2\theta d\theta\\&=-\cos\varphi\sin^{2m-1}\varphi+(2m-1)\int_0^{\varphi}\sin^{2m-2}\theta d\theta-(2m-1)\int_0^{\varphi}\sin^{2m}\theta d\theta\end{aligned}$$

from which follows the recursion

$$2mT(2m)=-\cos\varphi\sin^{2m-1}\varphi+(2m-1)T(2m-2)$$

with $T(0)=\varphi$ or $T(2)=\int_0^{\varphi}\sin^2\theta d\theta=\int_0^{\varphi}\frac{1-\cos 2\theta}{2}d\theta=\frac{\varphi-\frac{1}{2}\sin 2\varphi}{2}$.

and obtain

$$\int_0^{\varphi} \sin^{2m}\theta d\theta = \frac{(-1)^m}{2^{2m}} \sum_{l=0}^{2m} \binom{2m}{l} (-1)^l \frac{1-e^{-2i\varphi(m-l)}}{2i\,(m-l)}$$
$$= \frac{(-1)^m}{2^{2m}} \sum_{l=0}^{2m} \binom{2m}{l} (-1)^l e^{-i\varphi(m-l)} \frac{\sin\varphi\,(m-l)}{(m-l)}$$

Since the integral is real, we rewrite the sum $S = \sum_{l=0}^{2m} \binom{2m}{l} (-1)^l e^{-i\varphi(m-l)} \frac{\sin\varphi(m-l)}{(m-l)}$ by letting $j = 2m-l$ as

$$S = \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j e^{i\varphi(m-j)} \frac{\sin\varphi\,(m-j)}{(m-j)}$$

so that adding both sums leads to

$$S = \sum_{l=0}^{2m} \binom{2m}{l} (-1)^l \cos(\varphi\,(m-l)) \frac{\sin\varphi\,(m-l)}{(m-l)} = \frac{1}{2} \sum_{l=0}^{2m} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{(m-l)}$$

Employing $\lim_{l\to m} \frac{\sin\varphi(m-l)}{m-l} = \varphi$, we split the sum as

$$2S = 2\varphi \binom{2m}{m} (-1)^m + \sum_{l=0;l\neq m}^{2m} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l}$$

We simplify the last sum as

$$\sum_{l=0;l\neq m}^{2m} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{(m-l)} = \sum_{l=0}^{m-1} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l} + \sum_{l=m+1}^{2m} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l}$$
$$= \sum_{l=0}^{m-1} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l} + \sum_{j=0}^{m-1} \binom{2m}{j} (-1)^j \frac{\sin 2\varphi\,(m-j)}{m-j}$$
$$= 2\sum_{l=0}^{m-1} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l}$$

and find

$$S = \varphi \binom{2m}{m} (-1)^m + \sum_{l=0}^{m-1} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l}$$

and[45]

$$\int_0^{\varphi} \sin^{2m}\theta d\theta = \frac{(-1)^m}{2^{2m}} \left\{ \varphi \binom{2m}{m} (-1)^m + \sum_{l=0}^{m-1} \binom{2m}{l} (-1)^l \frac{\sin 2\varphi\,(m-l)}{m-l} \right\} \tag{142}$$

If $\varphi = \frac{\pi}{2}$, then $\frac{\sin\pi(m-l)}{m-l} = 0$ for $0 \le l < m$ and the sum in (142) vanishes. We verify $\int_0^{\frac{\pi}{2}} \sin^{2m}\theta d\theta = \frac{\binom{2m}{m}\pi}{2^{2m+1}}$ in (31).

[45] Differentiation of (142) with respect to φ leads to

$$\sin^{2m}\varphi = \frac{1}{2^{2m}} \left\{ \binom{2m}{m} + 2\sum_{l=0}^{m-1} \binom{2m}{l} (-1)^{m-l} \cos 2\varphi\,(m-l) \right\}$$

Substituting (142) into the Taylor series (141) gives us

$$\int_0^{\varphi}\left(1-k^2\sin^2\theta\right)^{\alpha}d\theta=\sum_{m=0}^{\infty}\binom{\alpha}{m}\frac{k^{2m}}{2^{2m}}\left\{\varphi\binom{2m}{m}(-1)^m+\sum_{l=0}^{m-1}\binom{2m}{l}(-1)^l\frac{\sin\left(2\varphi\left(m-l\right)\right)}{m-l}\right\} \quad (143)$$

We recall that $\binom{\frac{1}{2}}{m}=\frac{1}{(1-2m)}\binom{-\frac{1}{2}}{m}$ and $\binom{-\frac{1}{2}}{m}=\frac{(-1)^m}{2^{2m}}\binom{2m}{m}$ in (25). For $\alpha=-\frac{1}{2}$ and $\varphi=\frac{\pi}{2}$, we have that $\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}(-1)^m\binom{2m}{m}\frac{k^{2m}}{2^{2m}}=\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2k^{2m}=\frac{1}{M(1+k,1-k)}$ in (26) and that the expansion (143) simplifies to Gauss's form (32) for $\frac{1}{M(1+k,1-k)}=\frac{2}{\pi}K\left(k\right)$, where the latter is deduced in **art**. 28. Hence, the Taylor expansion of the complete elliptic integral $K\left(k\right)$ for $|k|<1$ is

$$K\left(k\right)=\frac{\pi}{2}\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}^2k^{2m}=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1;k^2\right) \quad (144)$$

where the last equality follows from (27). We can also write (143) for $\alpha=-\frac{1}{2}$ as

$$F\left(\varphi,k\right)=\int_0^{\varphi}\frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta=\frac{2\varphi}{\pi}K\left(k\right)+\sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}\frac{k^{2m}}{2^{2m}}\sum_{l=0}^{m-1}\binom{2m}{l}(-1)^l\frac{\sin\left(2\varphi\left(m-l\right)\right)}{m-l}$$

C Gauss and Landen transformation of elliptic integrals

53. *Elliptic integrals and AGM.* Gauss's basic integral $I\left(a,b\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}=\frac{\pi}{2}\frac{1}{M(a,b)}$ in (28) obeys the property $I\left(a,b\right)=I\left(a_1,b_1\right)$ as a consequence of the convergence property of the arithmetic-geometric mean $M\left(a,b\right)=M\left(a_n,b_n\right)$ for any integer n, as demonstrated in **art.** 1, as well as the property $I\left(a,b\right)=I\left(b,a\right)$ due to $M\left(a,b\right)=M\left(b,a\right)$. Conversely, these properties imply that the integral $I\left(a,b\right)$ can be transformed into $I\left(a_1,b_1\right)$ by integral manipulations.

Here, we would like to transform the more general integral

$$I\left(a,b;\varphi\right)=I\left(b,a;\varphi\right)=\int_0^{\varphi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{1}{a}F\left(\varphi,k\right)\qquad\text{with } k=\frac{c}{a} \quad (145)$$

in a similar manner as $I\left(a,b;\varphi\right)=I\left(a_1,b_1;\varphi_1\right)$ to deduce how the angle φ transforms to the angle φ_1. It turns out that there are several integral substitutions that map $I\left(a,b\right)=I\left(a_1,b_1\right)$ and one is due to Gauss. Gauss substitutes $\sin\theta=\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}$ for which the value $0\leq\theta\leq\frac{\pi}{2}$ transforms to the new variable $0\leq\psi\leq\frac{\pi}{2}$. He then asserts "after the development has been made correctly, it will be seen" that $\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{d\psi}{\sqrt{a_1^2\cos^2\psi+b_1^2\sin^2\psi}}$. Cox [8, p. 278] comments that Jacobi[46] has provided more details which Gauss has omitted. In **art.** 54, we give the entire derivation. In **art.** 55, we concentrate on the Landen transformation.

54. *The Gauss transformation.* We explore Gauss's substitution

$$\sin\theta=\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi} \quad (146)$$

that expresses the angle θ as a function of the new angle ψ. First, we determine the inverse that expresses ψ as a function θ. Let $w=\sin\psi$, then $\sin\theta=\frac{2aw}{(a+b)+(a-b)w^2}$, which is equivalent to the

[46] Fundamenta nova theoriae functionum ellipticorum, on p. 151 in C. C. J. Jacobi, Gesammelte Werke, G. Reimer, Berlin, 1881.

quadratic equation $(a-b)\sin\theta\, w^2 - 2aw + (a+b)\sin\theta = 0$, whose solution is

$$w = \frac{2a \pm 2\sqrt{a^2 - (a^2-b^2)\sin^2\theta}}{2\,(a-b)\sin\theta}$$

If $\theta = 0$, then $\sin\theta = \frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}$ tells us that $\sin\psi = 0$, because $a \neq -b$ and $a \neq 0$. Hence, we must choose the minus sign. With $c^2 = a^2 - b^2$ and $k = \frac{c}{a}$, we find that

$$\sin\psi = a\frac{1-\sqrt{1-k^2\sin^2\theta}}{(a-b)\sin\theta}$$

Next, we perform the substitution $\theta = \arcsin\left(\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}\right)$ based on (146) into the integral $I\,(a,b;\varphi)$ in (145) and compute

$$\begin{aligned}
d\theta &= \frac{1}{\sqrt{1-\left(\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}\right)^2}}\frac{d}{d\psi}\left(\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}\right)d\psi \\
&= \frac{1}{\sqrt{1-\left(\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}\right)^2}}\frac{\left((a+b)+(a-b)\sin^2\psi\right)2a\cos\psi - 2a\sin\psi\left(2\,(a-b)\sin\psi\cos\psi\right)}{\left((a+b)+(a-b)\sin^2\psi\right)^2}d\psi \\
&= \frac{2a\cos\psi}{\sqrt{1-\left(\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}\right)^2}}\frac{(a+b)-(a-b)\sin^2\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2}d\psi
\end{aligned}$$

as well as $\cos^2\theta$ in $\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$. From $\sin\theta = \frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}$ in (146), we have

$$\begin{aligned}
\cos^2\theta = 1 - \left(\frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}\right)^2 &= \frac{(a+b)^2 + 2\left(a^2-b^2\right)\sin^2\psi + (a-b)^2\sin^4\psi - 4a^2\sin^2\psi}{(a+b)^2 + 2\,(a+b)\,(a-b)\sin^2\psi + (a-b)^2\sin^4\psi} \\
&= \frac{(a+b)^2 - 2\left(a^2+b^2\right)\sin^2\psi + (a-b)^2\sin^4\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2}
\end{aligned}$$

which we use to compute

$$\begin{aligned}
a^2\cos^2\theta + b^2\sin^2\theta &= b^2\frac{4a^2\sin^2\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2} + a^2\frac{(a+b)^2 - 2\left(a^2+b^2\right)\sin^2\psi + (a-b)^2\sin^4\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2} \\
&= a^2\frac{(a+b)^2 + 2\left(2b^2 - \left(a^2+b^2\right)\right)\sin^2\psi + (a-b)^2\sin^4\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2} \\
&= a^2\frac{(a+b)^2 - 2\,(a+b)\,(a-b)\sin^2\psi + (a-b)^2\sin^4\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2} \\
&= a^2\frac{\left((a+b)-(a-b)\sin^2\psi\right)^2}{\left((a+b)+(a-b)\sin^2\psi\right)^2}
\end{aligned}$$

Hence, we arrive at

$$\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = a\frac{\left((a+b)-(a-b)\sin^2\psi\right)}{\left((a+b)+(a-b)\sin^2\psi\right)}$$

After this preparation, we obtain, denoting the angle $\varphi_1 = \arcsin\left(a\frac{1-\sqrt{1-k^2\sin^2\varphi}}{(a-b)\sin\varphi}\right)$,

$$\int_0^{\varphi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} = \int_0^{\varphi_1}\frac{1}{a\frac{\left((a+b)-(a-b)\sin^2\psi\right)}{\left((a+b)+(a-b)\sin^2\psi\right)^2}}\frac{2a\cos\psi\frac{(a+b)-(a-b)\sin^2\psi}{\left((a+b)+(a-b)\sin^2\psi\right)^2}}{\sqrt{(a+b)^2-2\left(a^2+b^2\right)\sin^2\psi+(a-b)^2\sin^4\psi}}d\psi$$
$$= \int_0^{\varphi_1}\frac{\cos\psi}{\sqrt{\left(\frac{a+b}{2}\right)^2-\frac{1}{2}\left(a^2+b^2\right)\sin^2\psi+\left(\frac{a-b}{2}\right)^2\sin^4\psi}}d\psi$$

It remains to write the integrand as $\sqrt{a_1^2\cos^2\psi+b_1^2\sin^2\psi}$, because $I(a,b;\varphi) = I(a_1,b_1;\varphi_1)$ must hold. Now,

$$\sqrt{a_1^2\cos^2\psi+b_1^2\sin^2\psi} = \sqrt{\left(\frac{a+b}{2}\right)^2\cos^2\psi+ab\sin^2\psi}$$
$$= \sqrt{\left(\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2\sin^2\psi+ab\sin^2\psi}$$
$$= \sqrt{\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\sin^2\psi}$$

which suggests that

$$\frac{\left(\frac{a+b}{2}\right)^2-\frac{1}{2}\left(a^2+b^2\right)\sin^2\psi+\left(\frac{a-b}{2}\right)^2\sin^4\psi}{\cos^2\psi} = \left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\sin^2\psi$$

We verify this guess and compute

$$\left(\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\sin^2\psi\right)\cos^2\psi = \left(\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\sin^2\psi\right)1-\sin^2\psi$$
$$= \left(\frac{a+b}{2}\right)^2-\frac{1}{2}\left(a^2+b^2\right)\sin^2\psi-\left(\frac{a-b}{2}\right)^2\sin^4\psi$$

illustrating consistency. Finally, we arrive at

$$\int_0^{\psi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} = \int_0^{\psi_1}\frac{d\psi}{\sqrt{a_1^2\cos^2\psi+b_1^2\sin^2\psi}} \tag{147}$$

where the angle $\psi_1 = \arcsin\left(a\frac{1-\sqrt{1-k^2\sin^2\psi}}{(a-b)\sin\psi}\right)$. With $k=\frac{c}{a}$, the angle ψ_1 obeys

$$\sin\psi_1 = \frac{a-\sqrt{a^2-\left(a^2-b^2\right)\sin^2\psi}}{(a-b)\sin\psi} = \frac{a-\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}{(a-b)\sin\psi}$$

but it is more convenient to proceed with the inverted relation between the angle ψ and ψ_1,

$$\sin\psi = \frac{2a\sin\psi_1}{(a+b)+(a-b)\sin_1^2\psi}$$

Backwards substitution with $a=a_1+c_1$ and $b=a_1-c_1$ in (12) shows that

$$\sin\psi = \frac{2\left(a_1+c_1\right)\sin\psi_1}{\left(a_1+c_1+a_1-c_1\right)+\left(a_1+c_1-a_1+c_1\right)\sin_1^2\psi} = \frac{\left(a_1+c_1\right)\sin\psi_1}{a_1+c_1\sin_1^2\psi}$$

from which

$$\begin{aligned}
\cos^2\psi &= \frac{\left(a_1 + c_1\sin_1^2\psi\right)^2 - \left(a_1+c_1\right)^2\sin_1^2\psi}{\left(a_1 + c_1\sin_1^2\psi\right)^2} \\
&= \frac{a_1^2 + 2a_1c_1\sin^2\psi_1 + c_1^2\sin^4\psi_1 - \left(a_1+c_1\right)^2\sin^2\psi_1}{\left(a_1 + c_1\sin^2\psi_1\right)^2} \\
&= \frac{a_1^2\sin^2\psi_1 + a_1^2\cos^2\psi_1 + 2a_1c_1\sin^2\psi_1 + c_1^2\sin^4\psi_1 - \left(a_1+c_1\right)^2\sin^2\psi_1}{\left(a_1 + c_1\sin^2\psi_1\right)^2} \\
&= \frac{a_1^2\cos^2\psi_1 + \left(a_1^2 + 2a_1c_1 - \left(a_1+c_1\right)^2\right)\sin^2\psi_1 + c_1^2\sin^4\psi_1}{\left(a_1 + c_1\sin^2\psi_1\right)^2} \\
&= \frac{a_1^2\cos^2\psi_1 - c_1^2\sin^2\psi_1 + c_1^2\sin^4\psi_1}{\left(a_1 + c_1\sin^2\psi_1\right)^2} = \frac{a_1^2\cos^2\psi_1 - c_1^2\sin^2\psi_1\left(1-\sin^2\psi_1\right)}{\left(a_1 + c_1\sin^2\psi_1\right)^2} \\
&= \frac{a_1^2\cos^2\psi_1 - c_1^2\sin^2\psi_1\cos^2\psi_1}{\left(a_1 + c_1\sin^2\psi_1\right)^2} = \cos^2\psi_1\frac{a_1^2 - c_1^2\sin^2\psi_1}{\left(a_1 + c_1\sin^2\psi_1\right)^2}
\end{aligned}$$

and

$$\cos\psi = \cos\psi_1\frac{\sqrt{a_1^2 - c_1^2\sin^2\psi_1}}{\left(a_1 + c_1\sin^2\psi_1\right)}$$

so that

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{\left(a_1+c_1\right)\sin\psi_1}{a_1 + c_1\sin^2\psi_1}\frac{\left(a_1 + c_1\sin^2\psi_1\right)}{\cos\psi_1\sqrt{a_1^2 - c_1^2\sin^2\psi_1}} = \tan\psi_1\frac{\left(a_1+c_1\right)}{\sqrt{a_1^2\cos^2\psi_1 + b_1^2\sin^2\psi_1}}$$

Finally with $a = a_1 + c_1$, we arrive at Gauss's recurrence [18, p. 4]

$$\tan\psi = \tan\psi_1\frac{a}{\sqrt{a_1^2\cos^2\psi_1 + b_1^2\sin^2\psi_1}}$$

More generally and defining $\nabla_n = \sqrt{a_n^2\cos^2\psi_n + b_n^2\sin^2\psi_n}$ for an integer n results in

$$\tan\psi_{n+1} = \frac{\nabla_{n+1}}{a_n}\tan\psi_n \tag{148}$$

Since $0 \le \frac{\nabla_{n+1}}{a_n} = \sqrt{\frac{a_{n+1}^2 - c_{n+1}^2\sin^2\psi_{n+1}}{a_n^2}} \le \frac{a_{n+1}}{a_n} = \frac{1}{2}\left(1 + \frac{b_n}{a_n}\right) \le 1$, the limit $\lim_{n\to\infty}\frac{\nabla_{n+1}}{a_n} = 1$ and (148) shows that $\lim_{n\to\infty}\tan\psi_n$ exists. In particular, for sufficiently large n, the Gauss angle recursion (148) indicates that $\psi_{n+1} \simeq \psi_n$, which contrasts the Landen angle doubling evolution $\varphi_{n+1} \simeq 2\varphi_n$ in (156). Another variant of the Gauss angle recursion follows from (146) with $c_{n+1} = \frac{1}{2}\left(a_n - b_n\right)$ and $a_n = a_{n+1} + c_{n+1}$ in (12) as

$$\sin\psi_n = \frac{\left(a_{n+1} + c_{n+1}\right)\sin\psi_{n+1}}{a_{n+1} + c_{n+1}\sin^2\psi_{n+1}} \tag{149}$$

which is a backward recursion, while the corresponding forward recursion is

$$\sin\psi_{n+1} = \frac{a_n - \sqrt{a_n^2\cos^2\psi_n + b_n^2\sin^2\psi_n}}{\left(a_n - b_n\right)\sin\psi_n} = \frac{a_n - \nabla_n}{\left(a_n - b_n\right)\sin\psi_n} \tag{150}$$

In summary, given Gauss's transform $\sin\theta = \frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}$, the demonstration of $I(a,b;\varphi) = I(a_1,b_1;\varphi_1)$ is already quite involved. But finding the transform must have been considerably more difficult and we may question how Gauss has discovered the non-trivial transform $\sin\theta = \frac{2a\sin\psi}{(a+b)+(a-b)\sin^2\psi}$.

Once we have the recursion (148) for the angle φ_n at our disposal, its solution by iteration is

$$\tan\varphi_n = \frac{\nabla_n}{a_{n-1}}\tan\varphi_{n-1} = \frac{\nabla_n}{a_{n-1}}\frac{\nabla_{n-1}}{a_{n-1}}\tan\varphi_{n-2} = \cdots = \prod_{j=0}^{n-1}\frac{\nabla_{j+1}}{a_j}\tan\varphi_0$$

from which we deduce, with $\varphi_0 = \varphi$, that

$$\varphi_n = \arctan\left(\prod_{j=0}^{n-1}\frac{\nabla_{j+1}}{a_j}\tan\varphi\right)$$

For large n, we have that $\nabla_n = \sqrt{a_n^2\cos^2\varphi_n + b_n^2\sin^2\varphi_n} = \sqrt{a_n^2 - c_n^2\sin^2\varphi_n} \to a_n$ and that φ_n converges towards a limit Φ, so that

$$\lim_{n\to\infty}\int_0^{\varphi_n}\frac{d\theta}{\sqrt{a_n^2\cos^2\theta + b_n^2\sin^2\theta}} = \lim_{n\to\infty}\int_0^{\varphi_n}\frac{d\psi}{\nabla_n} = \lim_{n\to\infty}\frac{\varphi_n}{a_n} = \lim_{n\to\infty}\frac{1}{a_n}\arctan\left(\prod_{j=0}^{n-1}\frac{\nabla_{j+1}}{a_j}\tan\varphi\right)$$

It follows from Gauss's AGM algorithm that

$$I(a,b;\varphi) = I(a_1,b_1;\varphi_1) = \cdots = I(a_n,b_n;\varphi_n) = \cdots$$

and we arrive at

$$\int_0^{\varphi}\frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \lim_{n\to\infty}\frac{1}{a_n}\arctan\left(\prod_{j=0}^{n-1}\frac{\nabla_{j+1}}{a_j}\tan\varphi\right) \tag{151}$$

where the right-hand side converges extremely fast. If $\varphi = \frac{\pi}{2}$, then $\nabla_n = b_n$, $\lim_{n\to\infty}\varphi_n = \frac{\pi}{2}$ and the right-hand side of (151) becomes $\frac{\pi}{2M(a,b)}$, so that (151) reduces to Gauss's basic integral (28).

55. *The Landen transformation.* Almkvist and Berndt [2, p. 590] give the Landen transformation

$$\tan\psi = \frac{\sin(2\theta)}{k_1 + \cos(2\theta)} \qquad \text{with } k_1 = \frac{c_1}{a_1} = \frac{a-b}{a+b} = \frac{a-\sqrt{a^2-c^2}}{a+\sqrt{a^2-c^2}} = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} \tag{152}$$

We repeat the work of **art.** 54 by substituting[47] $\psi = \arctan\left(\frac{\sin(2\theta)}{k_1+\cos(2\theta)}\right)$ in (152) into the integral $I(a,b;\varphi) = \int_0^{\varphi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$. Differentiation of the Landen transformation (152) gives

$$\sec^2\psi\, d\psi = \frac{d}{d\theta}\left(\frac{\sin(2\theta)}{k_1+\cos(2\theta)}\right)d\theta = 2\frac{(k_1+\cos(2\theta))\cos(2\theta)+\sin^2(2\theta)}{(k_1+\cos(2\theta))^2}d\theta = 2\frac{1+k_1\cos(2\theta)}{(k_1+\cos(2\theta))^2}d\theta$$

[47] Using $\arctan z = \arcsin\frac{z}{\sqrt{1+z^2}}$ translates ψ into

$$\psi = \arcsin\frac{\frac{\sin(2\theta)}{k_1+\cos(2\theta)}}{\sqrt{1+\left(\frac{\sin(2\theta)}{k_1+\cos(2\theta)}\right)^2}} = \arcsin\frac{\sin(2\theta)}{\sqrt{k_1^2+2k_1\cos(2\theta)+1}} = \arcsin\frac{2\sin(\theta)\cos\theta}{(k_1+1)\sqrt{1-\frac{4k_1}{(k_1+1)^2}\sin^2\theta}}$$

With $k_1+1 = \frac{2}{1+\sqrt{1-k^2}} = \frac{2}{1+k'}$ and $\frac{4k_1}{(k_1+1)^2} = (1+k')^2\frac{1-k'}{1+k'} = 1-(k')^2 = k^2$, we have [23, 19.8.11]

$$\psi = \arcsin\frac{(1+k')\sin(\theta)\cos\theta}{\sqrt{1-k^2\sin^2\theta}}$$

Since $\sec^2\psi = 1+\tan^2\psi = 1+\left(\frac{\sin(2\theta)}{k_1+\cos(2\theta)}\right)^2 = \frac{k_1^2+2k_1\cos(2\theta)+1}{(k_1+\cos(2\theta))^2}$, we find that

$$d\psi = 2\frac{1+k_1\cos(2\theta)}{1+k_1^2+2k_1\cos(2\theta)}d\theta$$

The inversion of the Landen transformation (152) follows from $\sec^2\psi = \frac{k_1^2+2k_1\cos(2\theta)+1}{(k_1+\cos(2\theta))^2}$ with $w = \cos(2\theta)$ as

$$\sec^2\psi = \frac{k_1^2+2k_1w+1}{(k_1+w)^2} = \frac{(k_1+w)^2+1-w^2}{(k_1+w)^2} = 1+\frac{1-w^2}{(k_1+w)^2}$$

or

$$\tan^2\psi = \frac{1-w^2}{(k_1+w)^2}$$

The quadratic equation, $\sec^2\psi\, w^2 + 2\tan^2\psi\, k_1 w + \tan^2\psi\, k_1^2 - 1 = 0$, rewritten as

$$w^2 + 2\sin^2\psi\, k_1 w + \sin^2\psi\, k_1^2 - \cos^2\psi = 0$$

has as solution

$$\begin{aligned} w &= -\sin^2\psi\, k_1 \pm \sqrt{\sin^4\psi\, k_1^2 - \sin^2\psi\, k_1^2 + \cos^2\psi} \\ &= -\sin^2\psi\, k_1 \pm \cos\psi\sqrt{1-k_1^2\sin^2\psi} \end{aligned}$$

After simplification and choosing the plus sign, because $\theta = 0$ corresponds to $\psi = 0$ in the Landen transformation (152), we find the inverse Landen transformation

$$\cos(2\theta) = -k_1\sin^2\psi + \cos\psi\sqrt{1-k_1^2\sin^2\psi} \tag{153}$$

The integrand of $I(a,b;\varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$ contains

$$\begin{aligned} \sqrt{a^2\cos^2\theta+b^2\sin^2\theta} &= \sqrt{c^2\cos^2\theta+b^2} = \sqrt{c^2\frac{1+\cos 2\theta}{2}+b^2} = \sqrt{\frac{c^2+2b^2+c^2\cos 2\theta}{2}} \\ &= \sqrt{\frac{a^2+b^2+(a^2-b^2)\cos 2\theta}{2}} \end{aligned}$$

Backwards substitution with $a = a_1 + c_1$ and $b = a_1 - c_1$ in (12) shows that,

$$\begin{aligned} \sqrt{a^2\cos^2\theta+b^2\sin^2\theta} &= \sqrt{\frac{(a_1+c_1)^2+(a_1-c_1)^2+\left((a_1+c_1)^2-(a_1-c_1)^2\right)\cos 2\theta}{2}} \\ &= \sqrt{a_1^2+c_1^2+2a_1c_1\cos 2\theta} \\ &= a_1\sqrt{1+k_1^2+2k_1\cos 2\theta} \end{aligned} \tag{154}$$

Combining the above with $d\theta = \frac{1}{2}\frac{1+k_1^2+2k_1\cos(2\theta)}{1+k_1\cos(2\theta)}d\psi$ yields

$$\begin{aligned} I(a,b;\varphi) &= \int_0^\varphi \frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \\ &= \frac{1}{2}\int_0^{\varphi_1} \frac{1}{a_1\sqrt{1+k_1^2+2k_1\cos 2\theta}}\frac{1+k_1^2+2k_1\cos(2\theta)}{1+k_1\cos(2\theta)}d\psi \\ &= \frac{1}{2a_1}\int_0^{\varphi_1} \frac{\sqrt{1+k_1^2+2k_1\cos 2\theta}}{1+k_1\cos(2\theta)}d\psi \end{aligned}$$

where $\varphi_1 = \arctan\left(\frac{\sin(2\varphi)}{k_1+\cos(2\varphi)}\right) = \arcsin\frac{(1+k')\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}$. Finally, we evaluate the integrand with (153) as a function ψ,

$$
\begin{aligned}
In &= \frac{\sqrt{1+k_1^2+2k_1\cos 2\theta}}{1+k_1\cos(2\theta)} = \frac{\sqrt{1+k_1^2-2k_1^2\sin^2\psi+2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}}}{1-k_1^2\sin^2\psi+k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}} \\
&= \frac{\sqrt{1+k_1^2-2k_1^2\sin^2\psi+2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}}}{\left(1-k_1^2\sin^2\psi\right)^2-\left(k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)^2}\left(1-k_1^2\sin^2\psi-k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)
\end{aligned}
$$

The denominator is

$$
\begin{aligned}
\left(1-k_1^2\sin^2\psi\right)^2-\left(k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)^2 &= 1-2k_1^2\sin^2\psi+k_1^4\sin^4\psi-k_1^2\cos^2\psi\left(1-k_1^2\sin^2\psi\right) \\
&= 1-2k_1^2\sin^2\psi+k_1^4\sin^2\psi\left(\sin^2\psi+\cos^2\psi\right)-k_1^2\cos^2\psi \\
&= 1-2k_1^2\sin^2\psi+k_1^4\sin^2\psi-k_1^2+k_1^2\sin^2\psi \\
&= 1-k_1^2+\left(k_1^2-1\right)k_1^2\sin^2\psi \\
&= \left(1-k_1^2\right)\left(1-k_1^2\sin^2\psi\right)
\end{aligned}
$$

The numerator N is simplified by first computing the square

$$
\begin{aligned}
Y &= \left(1-k_1^2\sin^2\psi-k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)^2 \\
&= \left(1-k_1^2\sin^2\psi\right)^2-2k_1\left(1-k_1^2\sin^2\psi\right)\cos\psi\sqrt{1-k_1^2\sin^2\psi}+k_1^2\cos^2\psi\left(1-k_1^2\sin^2\psi\right) \\
&= 1-2k_1^2\sin^2\psi+k_1^4\sin^4\psi+k_1^2\cos^2\psi-k_1^4\sin^2\psi\cos^2\psi \\
&- 2k_1\left(1-k_1^2\sin^2\psi\right)\cos\psi\sqrt{1-k_1^2\sin^2\psi} \\
&= 1-k_1^2\sin^2\psi+k_1^4\sin^2\psi\left(\sin^2\psi-\cos^2\psi\right)+k_1^2\left(\cos^2\psi-\sin^2\psi\right) \\
&- 2k_1\left(1-k_1^2\sin^2\psi\right)\cos\psi\sqrt{1-k_1^2\sin^2\psi} \\
&= \left(1-k_1^2\sin^2\psi\right)\left\{1+k_1^2\left(\cos^2\psi-\sin^2\psi\right)-2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right\}
\end{aligned}
$$

Then, the numerator N is

$$
\begin{aligned}
N &= \sqrt{\left(1+k_1^2-2k_1^2\sin^2\psi+2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)\left(1-k_1^2\sin^2\psi-k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)^2} \\
&= \sqrt{1-k_1^2\sin^2\psi}\sqrt{\left(1+k_1^2\cos 2\psi+2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)\left\{1+k_1^2\cos 2\psi-2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right\}} \\
&= \sqrt{1-k_1^2\sin^2\psi}\sqrt{\left(\left(1+k_1^2\cos 2\psi\right)^2-\left(2k_1\cos\psi\sqrt{1-k_1^2\sin^2\psi}\right)^2\right)} \\
&= \sqrt{1-k_1^2\sin^2\psi}\sqrt{\left(1+2k_1^2\left(\cos 2\psi\right)+k_1^4\cos^2 2\psi-4k_1^2\cos^2\psi+k_1^4\sin^2 2\psi\right)} \\
&= \sqrt{1-k_1^2\sin^2\psi}\sqrt{\left(1+2k_1^2\left(2\cos^2\psi-1-2\cos^2\psi\right)+k_1^4\right)} \\
&= \sqrt{1-k_1^2\sin^2\psi}\sqrt{\left(1-2k_1^2+k_1^4\right)} = \left(1-k_1^2\right)\sqrt{1-k_1^2\sin^2\psi}
\end{aligned}
$$

Finally, we arrive at

$$\begin{aligned} I(a,b;\varphi) &= \frac{1}{2a_1}\int_0^{\varphi_1} \frac{\left(1-k_1^2\right)\sqrt{1-k_1^2\sin^2\psi}}{\left(1-k_1^2\right)\left(1-k_1^2\sin^2\psi\right)}d\psi \\ &= \frac{1}{2a_1}\int_0^{\varphi_1}\frac{d\psi}{\sqrt{1-k_1^2\sin^2\psi}} = \frac{1}{2a_1}\int_0^{\varphi_1}\frac{d\psi}{\sqrt{1-\frac{c_1^2}{a_1}\sin^2\psi}} = \frac{1}{2}\int_0^{\varphi_1}\frac{d\psi}{\sqrt{a_1^2\cos^2\psi+b_1^2\sin^2\psi}} \end{aligned}$$

illustrating that $I(a,b;\varphi)=\frac{1}{2}I(a_1,b_1;\varphi_1)$, but the Landen angle φ_1 is different from the Gauss angle recursion!

Landen's transformation for the angle is

$$\varphi_1 = \arctan\left(\frac{\sin(2\varphi)}{k_1+\cos(2\varphi)}\right) = \arcsin\frac{(1+k')\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

from which we deduce the recursion

$$\tan\varphi_{n+1} = \frac{\sin(2\varphi_n)}{\frac{c_{n+1}}{a_{n+1}}+\cos(2\varphi_n)} \tag{155}$$

If n grows large, then $c_n\to 0$ and the Landen angle recursion (155) shows that

$$\varphi_{n+1}\approx 2\varphi_n \tag{156}$$

Iterated from a certain $n=m$ on, for which the term $\frac{c_{n+1}}{a_{n+1}}$ in (155) can be ignored, indicates that $\varphi_n\simeq 2^{n-m}\varphi_m$ for $n\geq m$ and that $\lim_{n\to\infty}2^{-n}\varphi_n\simeq 2^{-m}\varphi_m$, which is some finite angle implying that the limit exists. Following King [18, p. 7], we rewrite the Landen recursion (155) in three different, but equivalent ways. The first is

$$\frac{c_{n+1}}{a_{n+1}}\sin\varphi_{n+1} = \sin(2\varphi_n)\cos\varphi_{n+1}-\sin\varphi_{n+1}\cos(2\varphi_n)$$

and

$$\sin(2\varphi_n-\varphi_{n+1}) = \frac{c_{n+1}}{a_{n+1}}\sin\varphi_{n+1} \tag{157}$$

The second follows from (157) as

$$\cos^2(2\varphi_n-\varphi_{n+1}) = 1-\frac{c_{n+1}^2}{a_{n+1}^2}\sin^2\varphi_{n+1} = \frac{a_{n+1}^2\cos^2\varphi_{n+1}+b_{n+1}^2\sin\varphi_{n+1}}{a_{n+1}^2} = \left(\frac{\nabla_{n+1}}{a_{n+1}}\right)^2$$

and, thus,

$$\cos(2\varphi_n-\varphi_{n+1}) = \frac{\nabla_{n+1}}{a_{n+1}} \tag{158}$$

The third variant starts from (155) and moves to undoubling the angle,

$$\tan\varphi_{n+1} = \frac{2\sin(\varphi_n)\cos(\varphi_n)}{\frac{c_{n+1}}{a_{n+1}}+2\cos^2(\varphi_n)-1} = \frac{2\sin(\varphi_n)\cos(\varphi_n)}{\frac{-a_{n+1}+c_{n+1}}{a_{n+1}}+2\cos^2(\varphi_n)}$$

We divide the right-hand side numerator and denominator side by $\cos^2(\varphi_n)$ and invoke $b_n=a_{n+1}-c_{n+1}$ in (12)

$$\begin{aligned} \tan\varphi_{n+1} &= \frac{2\tan(\varphi_n)}{\frac{-b_n}{a_{n+1}\cos^2(\varphi_n)}+2} = \frac{2\tan(\varphi_n)}{\frac{-2b_n}{a_n+b_n}\left(1+\tan^2(\varphi_n)\right)+2} = \frac{(a_n+b_n)\tan(\varphi_n)}{-b_n\left(1+\tan^2(\varphi_n)\right)+a_n+b_n} \\ &= \frac{(a_n+b_n)\tan(\varphi_n)}{a_n-b_n\tan^2(\varphi_n)} = \frac{\left(1+\frac{b_n}{a_n}\right)\tan\varphi_n}{1-\frac{b_n}{a_n}\tan^2(\varphi_n)} = \frac{\tan(\varphi_n)+\frac{b_n}{a_n}\tan\varphi_n}{1-\frac{b_n}{a_n}\tan^2(\varphi_n)} \end{aligned}$$

We rewrite the fraction as

$$\begin{aligned}\frac{b_n}{a_n}\tan\varphi_n &= \tan\varphi_{n+1}\left(1-\frac{b_n}{a_n}\tan^2(\varphi_n)\right)-\tan\varphi_n\\ &= \tan\varphi_{n+1}-\tan\varphi_n-\frac{b_n}{a_n}\tan\varphi_{n+1}\tan^2\varphi_n\end{aligned}$$

and collect terms in $\frac{b_n}{a_n}\tan(\varphi_n)$

$$\frac{b_n}{a_n}\tan\varphi_n\left(1+\tan\varphi_{n+1}\tan\varphi_n\right)=\tan\varphi_{n+1}-\tan\varphi_n$$

to arrive at the third variant of Landen's recursion for the angle,

$$\frac{b_n}{a_n}\tan\varphi_n=\frac{\tan\varphi_{n+1}-\tan(\varphi_n)}{1+\tan\varphi_{n+1}\tan(\varphi_n)}=\tan(\varphi_{n+1}-\varphi_n) \tag{159}$$

56. *Further deductions from the Landen transformation.* From the Landen recursions (155), (157), (158) and (159) of the angle forms, King [18, p. 7] states without proof nor any hint that

$$\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}=\nabla_n \tag{160}$$

$$\nabla_{n+1}-c_{n+1}\cos\varphi_{n+1}=\frac{a_nb_n}{\nabla_n} \tag{161}$$

and that

$$\nabla_{n+1}^2+c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}=\frac{1}{2}\nabla_n^2+\frac{1}{2}a_nb_n \tag{162}$$

The remainder consists of proving (160), (161) and (162). In part **A**, we give a direct proof of (160) and (161), followed in part **B** by a verification type of proof. In part **C**, we demonstrate (162) as a consequence of (160) and (161).

A. The inverse Landen transformation (153) yields

$$\begin{aligned}\cos(2\varphi_n) &= -\frac{c_{n+1}}{a_{n+1}}\sin^2\varphi_{n+1}+\cos\varphi_{n+1}\sqrt{1-\frac{c_{n+1}^2}{a_{n+1}^2}\sin^2\varphi_{n+1}}\\ &= -\frac{c_{n+1}}{a_{n+1}}\sin^2\varphi_{n+1}+\frac{\cos\varphi_{n+1}}{a_{n+1}}\sqrt{a_{n+1}^2\cos^2\varphi_{n+1}+b_{n+1}^2\sin^2\varphi_{n+1}}\\ &= -\frac{c_{n+1}}{a_{n+1}}\sin^2\varphi_{n+1}+\frac{\cos\varphi_{n+1}}{a_{n+1}}\nabla_{n+1}\end{aligned}$$

from which

$$\nabla_{n+1}\cos\varphi_{n+1}=a_{n+1}\cos(2\varphi_n)+c_{n+1}\sin^2\varphi_{n+1} \tag{163}$$

We rewrite (163) as

$$c_{n+1}\cos^2\varphi_{n+1}=a_{n+1}\cos(2\varphi_n)+c_{n+1}-\nabla_{n+1}\cos\varphi_{n+1}$$

and multiplying by c_{n+1},

$$c_{n+1}^2\cos^2\varphi_{n+1}=a_{n+1}c_{n+1}\cos(2\varphi_n)+c_{n+1}^2-c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}$$

First, we add $2c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}$ at both sides,

$$c_{n+1}^2\cos^2\varphi_{n+1}+2c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}=a_{n+1}c_{n+1}\cos(2\varphi_n)+c_{n+1}^2+c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}$$

and deduce that

$$(c_{n+1}\cos\varphi_{n+1}+\nabla_{n+1})^2=\nabla_{n+1}^2+a_{n+1}c_{n+1}\cos(2\varphi_n)+c_{n+1}^2+c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}$$

We use (163) again in the right-hand side,

$$\begin{aligned}(c_{n+1}\cos\varphi_{n+1}+\nabla_{n+1})^2&=\nabla_{n+1}^2+2a_{n+1}c_{n+1}\cos(2\varphi_n)+c_{n+1}^2+c_{n+1}^2\sin^2\varphi_{n+1}\\&=a_{n+1}^2\cos^2\varphi_{n+1}+\left(b_{n+1}^2+c_{n+1}^2\right)\sin^2\varphi_{n+1}+2a_{n+1}c_{n+1}\cos(2\varphi_n)+c_{n+1}^2\\&=a_{n+1}^2+2a_{n+1}c_{n+1}\cos(2\varphi_n)+c_{n+1}^2\end{aligned}$$

Invoking $\nabla_n=\sqrt{a_{n+1}^2+c_{n+1}^2+2a_{n+1}c_{n+1}\cos 2\varphi_n}$ in (154) proves (160). Formula (161) directly follows from the first verification in part **B** below, once (160) is known.

B. We give a verification for (160) and (161). First, multiplying (160) and (161) yields

$$\begin{aligned}(\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1})(\nabla_{n+1}-c_{n+1}\cos\varphi_{n+1})&=\nabla_{n+1}^2-c_{n+1}^2\cos^2\varphi_{n+1}\\&=a_{n+1}^2\cos^2\varphi_{n+1}+b_{n+1}^2\sin^2\varphi_{n+1}-c_{n+1}^2\cos^2\varphi_{n+1}\\&=b_{n+1}^2=a_nb_n\end{aligned}$$

Second, dividing (160) by (161) yields

$$\begin{aligned}\frac{\nabla_n^2}{a_nb_n}&=\frac{\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}}{\nabla_{n+1}-c_{n+1}\cos\varphi_{n+1}}=\frac{(\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1})^2}{\nabla_{n+1}^2-c_{n+1}^2\cos^2\varphi_{n+1}}\\&=\frac{\nabla_{n+1}^2+2c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}+c_{n+1}^2\cos^2\varphi_{n+1}}{a_nb_n}\end{aligned}$$

and we must shows that

$$\begin{aligned}\nabla_n^2&=\nabla_{n+1}^2+2c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}+c_{n+1}^2\cos^2\varphi_{n+1}\\&=\left(a_{n+1}^2+c_{n+1}^2\right)\cos^2\varphi_{n+1}+b_{n+1}^2\sin^2\varphi_{n+1}+2c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}\end{aligned}$$

Substituting (163) in the above,

$$\begin{aligned}\nabla_n^2&=a_{n+1}^2\cos^2\varphi_{n+1}+c_{n+1}^2\cos^2\varphi_{n+1}+b_{n+1}^2\sin^2\varphi_{n+1}+2c_{n+1}a_{n+1}\cos(2\varphi_n)+2c_{n+1}^2\sin^2\varphi_{n+1}\\&=a_{n+1}^2\cos^2\varphi_{n+1}+c_{n+1}^2\left(\cos^2\varphi_{n+1}+\sin^2\varphi_{n+1}\right)+\left(b_{n+1}^2+c_{n+1}^2\right)\sin^2\varphi_{n+1}+2c_{n+1}a_{n+1}\cos(2\varphi_n)\\&=a_{n+1}^2\cos^2\varphi_{n+1}+c_{n+1}^2+a_{n+1}^2\sin^2\varphi_{n+1}+2c_{n+1}a_{n+1}\cos(2\varphi_n)\\&=a_{n+1}^2+c_{n+1}^2+2c_{n+1}a_{n+1}\cos(2\varphi_n)\end{aligned}$$

which is an identity by (154). In summary, from the product of the factors $(\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1})$ and $(\nabla_{n+1}-c_{n+1}\cos\varphi_{n+1})$ and their ratio, we deduce (160) and (161).

C. Finally, multiplying (160) with ∇_{n+1} yields

$$\nabla_{n+1}^2+c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}=\nabla_n\nabla_{n+1}$$

On the other hand, multiplying (160) with ∇_n,

$$\nabla_n\nabla_{n+1}+\nabla_nc_{n+1}\cos\varphi_{n+1}=\nabla_n^2$$

and multiplying (161) with ∇_n,

$$\nabla_n \nabla_{n+1} - \nabla_n c_{n+1} \cos\varphi_{n+1} = a_n b_n$$

indicates that their addition is

$$2\nabla_n \nabla_{n+1} = \nabla_n^2 + a_n b_n \tag{164}$$

Substituted into the above proves (162).

57. *Differentiation of forms of the Landen transformation.* We compute the differentials of $\nabla_{n+1} + c_{n+1}\cos\varphi_{n+1} = \nabla_n$ in (160) and $\nabla_{n+1} - c_{n+1}\cos\varphi_{n+1} = \frac{a_n b_n}{\nabla_n}$ in (161). First, the differential of the right-hand side $\nabla_n = \sqrt{a_n^2\cos^2\varphi_n + b_n^2\sin^2\varphi_n}$ of (160) is

$$d\nabla_n = \frac{d\nabla_n}{d\varphi_n} d\varphi_n = \frac{1}{2\nabla_n}\left(-a_n^2 \sin(2\varphi_n) + b_n^2 \sin(2\varphi_n)\right) d\varphi_n$$

and

$$d\nabla_n = -\frac{c_n^2 \sin(2\varphi_n)}{2\nabla_n} d\varphi_n \tag{165}$$

The differential of the left-hand side of (160) follows as

$$\begin{aligned} d\left(\nabla_{n+1} + c_{n+1}\cos\varphi_{n+1}\right) &= d\nabla_{n+1} - c_{n+1}\sin\varphi_{n+1} d\varphi_{n+1} \\ &= -\left(\frac{c_{n+1}^2 \sin\varphi_{n+1}\cos\varphi_{n+1}}{\nabla_{n+1}} + c_{n+1}\sin\varphi_{n+1}\right) d\varphi_{n+1} \\ &= -\left(c_{n+1}\cos\varphi_{n+1} + \nabla_{n+1}\right)\frac{c_{n+1}\sin\varphi_{n+1}}{\nabla_{n+1}} d\varphi_{n+1} \\ &= -\nabla_n \frac{c_{n+1}\sin\varphi_{n+1}}{\nabla_{n+1}} d\varphi_{n+1} \end{aligned}$$

Combining both sides[48],

$$\frac{c_n^2 \sin(2\varphi_n)}{2\nabla_n^2} d\varphi_n = \frac{c_{n+1}\sin\varphi_{n+1}}{\nabla_{n+1}} d\varphi_{n+1} \tag{166}$$

We rewrite (166) as

$$\frac{d\varphi_n}{\nabla_n} = \frac{2\nabla_n c_{n+1}\sin\varphi_{n+1}}{c_n^2 \sin(2\varphi_n)} \frac{d\varphi_{n+1}}{\nabla_{n+1}}$$

With $\tan\varphi_{n+1} = \frac{\sin(2\varphi_n)}{\frac{c_{n+1}}{a_{n+1}} + \cos(2\varphi_n)}$,

$$\begin{aligned} \frac{2\nabla_n c_{n+1}\sin\varphi_{n+1}}{c_n^2 \sin(2\varphi_n)} &= \frac{2\nabla_n c_{n+1}\sin\varphi_{n+1}}{c_n^2 \tan\varphi_{n+1}\left(\frac{c_{n+1}}{a_{n+1}} + \cos(2\varphi_n)\right)} = \frac{2\nabla_n a_{n+1} c_{n+1}\cos\varphi_{n+1}}{c_n^2\left(c_{n+1} + a_{n+1}\cos(2\varphi_n)\right)} \\ &= \frac{1}{2}\frac{\nabla_n \cos\varphi_{n+1}}{c_{n+1} + a_{n+1}\cos(2\varphi_n)} \end{aligned}$$

[48] The differential of the left-hand side of (161) is

$$\begin{aligned} d\left(\nabla_{n+1} - c_{n+1}\cos\varphi_{n+1}\right) &= d\nabla_{n+1} + c_{n+1}\sin\varphi_{n+1} d\varphi_{n+1} \\ &= \left(-\frac{c_{n+1}^2 \sin\varphi_{n+1}\cos\varphi_{n+1}}{\nabla_{n+1}} + c_{n+1}\sin\varphi_{n+1}\right) d\varphi_{n+1} \\ &= \left(-c_{n+1}\cos\varphi_{n+1} + \nabla_{n+1}\right)\frac{c_{n+1}\sin\varphi_{n+1}}{\nabla_{n+1}} d\varphi = \frac{a_n b_n}{\nabla_n}\frac{c_{n+1}\sin\varphi_{n+1}}{\nabla_{n+1}} d\varphi_{n+1} \end{aligned}$$

Combining both sides yields, with $d\left(\frac{a_n b_n}{\nabla_n}\right) = -\frac{a_n b_n}{\nabla_n^2} d\nabla_n = \frac{a_n b_n}{\nabla_n^2}\frac{c_n^2 \sin(2\varphi_n)}{2\nabla_n} d\varphi_n$ leads to the same result (166).

Multiplying numerator and denominator with c_{n+1} yields

$$\frac{2\nabla_n c_{n+1}\sin\varphi_{n+1}}{c_n^2\sin(2\varphi_n)}=\frac{1}{2}\frac{\nabla_n c_{n+1}\cos\varphi_{n+1}}{c_{n+1}^2+c_{n+1}a_{n+1}\cos(2\varphi_n)}=\frac{1}{2}\frac{\nabla_n c_{n+1}\cos\varphi_{n+1}}{c_{n+1}^2+c_{n+1}a_{n+1}\cos(2\varphi_n)}$$

With $\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}=\nabla_n$ in (160) and (164), it holds that

$$\nabla_n c_{n+1}\cos\varphi_{n+1}=\nabla_n^2-\nabla_{n+1}\nabla_n=\nabla_n^2-\frac{1}{2}\left(\nabla_n^2+a_nb_n\right)=\frac{1}{2}\nabla_n^2-\frac{1}{2}a_nb_n$$

such that, replacing $a_nb_n=b_{n+1}^2$

$$\frac{2\nabla_n c_{n+1}\sin\varphi_{n+1}}{c_n^2\sin(2\varphi_n)}=\frac{1}{2}\frac{\nabla_n^2-b_{n+1}^2}{2c_{n+1}^2+2c_{n+1}a_{n+1}\cos(2\varphi_n)}$$

It follows from $\nabla_n=\sqrt{a_{n+1}^2+c_{n+1}^2+2a_{n+1}c_{n+1}\cos 2\varphi_n}$ in (154) that

$$\nabla_n^2-b_{n+1}^2=a_{n+1}^2+c_{n+1}^2+2a_{n+1}c_{n+1}\cos 2\varphi_n-b_{n+1}^2=2c_{n+1}^2+2c_{n+1}a_{n+1}\cos(2\varphi_n)$$

resulting in

$$\frac{2\nabla_n c_{n+1}\sin\varphi_{n+1}}{c_n^2\sin(2\varphi_n)}=\frac{1}{2} \tag{167}$$

In summary, we arrive at the remarkable differential recursion

$$\frac{d\varphi_n}{\nabla_n}=\frac{1}{2}\frac{d\varphi_{n+1}}{\nabla_{n+1}} \tag{168}$$

from which, after iterations, we find, for any positive integer n,

$$\frac{d\varphi_0}{\nabla_0}=\frac{d\varphi}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}=\frac{1}{2^n}\frac{d\varphi_n}{\nabla_n} \tag{169}$$

If we take the differential of $\nabla_{n+1}^2+c_{n+1}\nabla_{n+1}\cos\varphi_{n+1}=\frac{1}{2}\nabla_n^2+\frac{1}{2}a_nb_n$ in (162), then

$$2\nabla_{n+1}d\nabla_{n+1}-\nabla_{n+1}c_{n+1}\sin\varphi_{n+1}d\varphi_{n+1}+c_{n+1}\cos\varphi_{n+1}d\nabla_{n+1}=\nabla_n d\nabla_n$$

and

$$\left(2\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}\right)d\nabla_{n+1}-\nabla_{n+1}c_{n+1}\sin\varphi_{n+1}d\varphi_{n+1}=\nabla_n d\nabla_n$$

Employing $d\nabla_n=-\frac{c_n^2\sin(2\varphi_n)}{2\nabla_n}d\varphi_n$ in (165), we execute

$$\left(2\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}\right)\left(-\frac{c_{n+1}^2\sin(2\varphi_{n+1})}{2\nabla_{n+1}}d\varphi_{n+1}\right)-\nabla_{n+1}c_{n+1}\sin\varphi_{n+1}d\varphi_{n+1}=-\nabla_n\frac{c_n^2\sin(2\varphi_n)}{2\nabla_n}d\varphi_n$$

At the right-hand side, we substitute $\frac{c_n^2\sin(2\varphi_n)}{2\nabla_n}=2c_{n+1}\sin\varphi_{n+1}$ in (167), divide both sides by $c_{n+1}\sin\varphi_{n+1}$ and obtain

$$\left(1+\left(\frac{\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}}{\nabla_{n+1}}\right)\right)c_{n+1}\cos\varphi_{n+1}d\varphi_{n+1}+\nabla_{n+1}d\varphi_{n+1}=2\nabla_n d\varphi_n$$

We substitute $\nabla_{n+1}+c_{n+1}\cos\varphi_{n+1}=\nabla_n$ in (160),

$$c_{n+1}\cos\varphi_{n+1}d\varphi_{n+1}+\left(\frac{\nabla_n}{\nabla_{n+1}}c_{n+1}\cos\varphi_{n+1}\right)d\varphi_{n+1}+\nabla_{n+1}d\varphi_{n+1}=\nabla_n d\varphi_n+\nabla_n d\varphi_n$$

and arrange to construct the difference

$$\nabla_n d\varphi_n - \nabla_{n+1} d\varphi_{n+1} = c_{n+1}\cos\varphi_{n+1} d\varphi_{n+1} + \left(c_{n+1}\cos\varphi_{n+1}\right)\frac{\nabla_n}{\nabla_{n+1}} d\varphi_{n+1} - \nabla_n d\varphi_n$$

We invoke $d\varphi_n = \frac{1}{2}\frac{\nabla_n}{\nabla_{n+1}} d\varphi_{n+1}$ in (168)

$$\nabla_n d\varphi_n - \nabla_{n+1} d\varphi_{n+1} = c_{n+1}\cos\varphi_{n+1} d\varphi_{n+1} + \left(2c_{n+1}\cos\varphi_{n+1} - \nabla_n\right) d\varphi_n$$

We rewrite $2c_{n+1}\cos\varphi_{n+1} - \nabla_n$ with $\nabla_{n+1} + c_{n+1}\cos\varphi_{n+1} = \nabla_n$ in (160) as

$$2c_{n+1}\cos\varphi_{n+1} - \nabla_n = c_{n+1}\cos\varphi_{n+1} - \nabla_{n+1} = -\frac{a_n b_n}{\nabla_n}$$

where the last equality follows from (161). Finally, we arrive at

$$\nabla_n d\varphi_n - \nabla_{n+1} d\varphi_{n+1} = c_{n+1}\cos\varphi_{n+1} d\varphi_{n+1} - \frac{a_n b_n}{\nabla_n} d\varphi_n \tag{170}$$

There is still a useful identity, following from $\frac{d\varphi_n}{\nabla_n} = \frac{1}{2}\frac{d\varphi_{n+1}}{\nabla_{n+1}}$ in (168)

$$a_n^2 \frac{d\varphi_n}{\nabla_n} - a_{n+1}^2 \frac{d\varphi_{n+1}}{\nabla_{n+1}} = \left(a_n^2 - 2a_{n+1}^2\right)\frac{d\varphi_n}{\nabla_n} = \left(\frac{1}{2}c_n^2 - a_n b_n\right)\frac{d\varphi_n}{\nabla_n} \tag{171}$$

because $\left(a_n^2 - 2a_{n+1}^2\right) = a_n^2 - \frac{1}{2}\left(a_n^2 + 2a_n b_n + b_n^2\right) = \frac{1}{2}\left(a_n^2 - b_n^2\right) - a_n b_n = \frac{1}{2}c_n^2 - a_n b_n$. After subtracting (171) from (170), we have

$$\left(\nabla_n - \frac{a_n^2}{\nabla_n}\right) d\varphi_n - \left(\nabla_{n+1} - \frac{a_{n+1}^2}{\nabla_{n+1}}\right) d\varphi_{n+1} = c_{n+1}\cos\varphi_{n+1} d\varphi_{n+1} - \frac{1}{2}\frac{c_n^2}{\nabla_n} d\varphi_n \tag{172}$$

The main reason of the subtraction towards (172) is that $\lim_{n\to\infty} \nabla_n - \frac{a_n^2}{\nabla_n} = 0$. Summing (172) over n from $n = m$ to $n = p$

$$\sum_{n=m}^{p}\left(\nabla_n - \frac{a_n^2}{\nabla_n}\right) d\varphi_n - \sum_{n=m}^{p}\left(\nabla_{n+1} - \frac{a_{n+1}^2}{\nabla_{n+1}}\right) d\varphi_{n+1} = \sum_{n=m}^{p} c_{n+1}\cos\varphi_{n+1} d\varphi_{n+1} - \frac{1}{2}\sum_{n=m}^{p}\frac{c_n^2}{\nabla_n} d\varphi_n$$

and recognizing the telescoping series at the left-hand side results in

$$\left(\nabla_m - \frac{a_m^2}{\nabla_m}\right) d\varphi_m - \left(\nabla_{p+1} - \frac{a_{p+1}^2}{\nabla_{p+1}}\right) d\varphi_{p+1} = \sum_{n=m}^{p} c_{n+1}\cos\varphi_{n+1} d\varphi_{n+1} - \frac{1}{2}\sum_{n=m}^{p}\frac{c_n^2}{\nabla_n} d\varphi_n$$

With $\frac{d\varphi_n}{\nabla_n} = 2^n \frac{d\varphi_0}{\nabla_0}$ in (169), the last sum is

$$\frac{1}{2}\sum_{n=m}^{p}\frac{c_n^2}{\nabla_n} d\varphi_n = \frac{1}{2}\frac{d\varphi_0}{\nabla_0}\sum_{n=m}^{p} 2^n c_n^2$$

We let $p \to \infty$ and choose $m = 0$ with $\nabla_0 = \nabla = \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}$, then after integration, we find

$$\int_0^{\varphi}\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi = \sum_{n=0}^{\infty} c_{n+1}\sin\varphi_{n+1} + \int_0^{\varphi}\frac{d\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\left(a^2 - \sum_{n=0}^{\infty} 2^{n-1}c_n^2\right)$$

With $I(a,b;\varphi)=\int_0^{\varphi}\frac{d\varphi}{\sqrt{a^2\cos\varphi+b^2\sin^2\varphi}}$ in (145) and $J(a,b;\varphi)=\int_0^{\varphi}\sqrt{a^2\cos\varphi+b^2\sin^2\varphi}d\varphi$ leads to

$$J(a,b;\varphi)=\sum_{n=1}^{\infty}c_n\sin\varphi_n+\left(a^2-\sum_{n=0}^{\infty}2^{n-1}c_n^2\right)I(a,b;\varphi) \tag{173}$$

With $k=\frac{c}{a}$, rephrased in terms of the Jacobi elliptic integrals in $F(\varphi,k)=\int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ in (130) and $E(\varphi,k)=\int_0^{\varphi}\sqrt{1-k^2\sin^2\theta}d\theta$ in (131) is

$$aE(\varphi,k)=\sum_{n=0}^{\infty}c_{n+1}\sin\varphi_{n+1}+\frac{1}{a}F(\varphi,k)\left(a^2-\sum_{n=0}^{\infty}2^{n-1}c_n^2\right)$$

If n increases, then $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=M(a,b)$ and the angle $\frac{\varphi_n}{2^n}\to\Phi$ as shown by (156). Hence, $I(a,b;\varphi)=\lim_{n\to\infty}\frac{\varphi_n}{2^na_n}$ and $F(\varphi,k)=a\lim_{n\to\infty}\frac{\varphi_n}{2^na_n}$. If $\varphi=\frac{\pi}{2}$, then Landen's angle recursion (156) reduces to $\frac{\varphi_n}{2^n}=\frac{\pi}{2}$ for $n>0$, so that $\sin\varphi_n=0$ and (173) reduces to (95). Also, $F(\varphi,k)=a\lim_{n\to\infty}\frac{\varphi_n}{2^na_n}$ becomes $K(k)=\frac{\pi}{2}\frac{a}{M(a,b)}$.

D Proof of Legendre's formula (98) of products of complete elliptic integrals

Replace k^2 in (132) by c and differentiate with respect to c,

$$\begin{aligned}
\frac{d}{dc}\left(E\left(\varphi,\sqrt{c}\right)-F\left(\varphi,\sqrt{c}\right)\right)&=-\frac{d}{dc}\left(c\int_0^{\varphi}\frac{\sin^2\theta d\theta}{\sqrt{1-c\sin^2\theta}}\right)\\
&=-\int_0^{\varphi}\frac{\sin^2\theta}{\sqrt{1-c\sin^2\theta}}d\theta-\frac{1}{2}\int_0^{\varphi}\frac{c\sin^4\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&=\frac{1}{c}\int_0^{\varphi}\frac{1-c\sin^2\theta-1}{\sqrt{1-c\sin^2\theta}}d\theta-\frac{1}{2}\int_0^{\varphi}\frac{c\sin^4\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&=\frac{1}{c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta-\frac{1}{2c}\int_0^{\varphi}\frac{2\left(1-c\sin^2\theta\right)}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&\quad-\frac{1}{2c}\int_0^{\varphi}\frac{c^2\sin^4\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&=\frac{1}{c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta-\frac{1}{2c}\int_0^{\varphi}\frac{1+1-2c\sin^2\theta+c^2\sin^4\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&=\frac{1}{c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta-\frac{1}{2c}\int_0^{\varphi}\frac{1+\left(1-c\sin^2\theta\right)^2}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&=\frac{1}{c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta-\frac{1}{2c}\int_0^{\varphi}\frac{\left(1-c\sin^2\theta\right)^2}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}d\theta\\
&\quad-\frac{1}{2c}\int_0^{\varphi}\frac{d\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}\\
&=\frac{1}{2c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta-\frac{1}{2c}\int_0^{\varphi}\frac{d\theta}{\left(1-c\sin^2\theta\right)^{\frac{3}{2}}}
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\theta}\left(\frac{\sin 2\theta}{\sqrt{1-c\sin^2\theta}}\right) &= \frac{1}{4}\frac{c\sin^2 2\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} + \frac{\cos 2\theta}{\sqrt{1-c\sin^2\theta}} \\
&= \frac{c\sin^2\theta\cos^2\theta + \left(\cos^2\theta-\sin^2\theta\right)\left(1-c\sin^2\theta\right)}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} \\
&= \frac{c\sin^2\theta\cos^2\theta + \cos^2\theta - \sin^2\theta - c\sin^2\theta\cos^2\theta + c\sin^4\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} \\
&= \frac{\cos^2\theta - \sin^2\theta\left(1-c\sin^2\theta\right)}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} = \frac{\cos^2\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} - \frac{\sin^2\theta}{\sqrt{1-c\sin^2\theta}} \\
&= \frac{\cos^2\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} + \frac{1}{c}\frac{1-c\sin^2\theta-1}{\sqrt{1-c\sin^2\theta}} \\
&= \frac{1}{c}\sqrt{1-c\sin^2\theta} - \frac{1}{c}\frac{1}{\sqrt{1-c\sin^2\theta}} + \frac{\cos^2\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}} \\
&= \frac{1}{c}\sqrt{1-c\sin^2\theta} - \frac{1}{c}\frac{1-c\sin^2\theta-c\cos^2\theta}{\left(1-c\sin^2\theta\right)\sqrt{1-c\sin^2\theta}}
\end{aligned}$$

from which

$$\frac{1}{\left(1-c\sin^2\theta\right)^{\frac{3}{2}}} = \frac{1}{1-c}\sqrt{1-c\sin^2\theta} - \frac{c}{1-c}\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\sqrt{1-c\sin^2\theta}}\right)$$

we deduce, denoting $c' = 1-c$, that

$$\begin{aligned}
\frac{d}{dc}\left(E\left(\varphi,\sqrt{c}\right) - F\left(\varphi,\sqrt{c}\right)\right) &= \frac{1}{2c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta - \frac{1}{2c}\int_0^{\varphi}\frac{d\theta}{\left(1-c\sin^2\theta\right)^{\frac{3}{2}}} \\
&= \frac{1}{2c}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta - \frac{1}{2cc'}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta \\
&\quad - \frac{1}{2c'}\int_0^{\varphi}\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\sqrt{1-c\sin^2\theta}}\right)d\theta \\
&= -\frac{1}{2c'}E\left(\varphi,\sqrt{c}\right) - \frac{1}{2c'}\frac{\sin\varphi\cos\varphi}{\sqrt{1-c\sin^2\varphi}}
\end{aligned}$$

If we choose $\varphi = \frac{\pi}{2}$, then the above simplifies to

$$\frac{d}{dc}\left(E\left(\sqrt{c}\right) - K\left(\sqrt{c}\right)\right) = -\frac{1}{2c'}E\left(\sqrt{c}\right) \tag{174}$$

We can verify with $c' = 1-c$ that

$$\frac{d}{dc}\left(E\left(\sqrt{1-c}\right) - K\left(\sqrt{1-c}\right)\right) = \frac{1}{2c}E\left(\sqrt{1-c}\right)$$

Further

$$\frac{d}{dc}E\left(\varphi,\sqrt{c}\right) = \frac{d}{dc}\int_0^{\varphi}\sqrt{1-c\sin^2\theta}d\theta = \frac{1}{2}\int_0^{\varphi}\frac{-\sin^2\theta}{\sqrt{1-c\sin^2\theta}}d\theta = \frac{1}{2c}\int_0^{\varphi}\frac{1-c\sin^2\theta-1}{\sqrt{1-c\sin^2\theta}}d\theta$$

so that

$$\frac{d}{dc}E\left(\varphi,\sqrt{c}\right) = \frac{1}{2c}\left(E\left(\varphi,\sqrt{c}\right) - F\left(\varphi,\sqrt{c}\right)\right)$$

and, similarly with $c' = 1 - c$,

$$\frac{d}{dc} E\left(\varphi, \sqrt{1-c}\right) = -\frac{1}{2c'}\left(E\left(\varphi, \sqrt{c'}\right) - F\left(\varphi, \sqrt{c'}\right)\right)$$

For $\varphi = \frac{\pi}{2}$, the above reduces to

$$\begin{aligned} \frac{d}{dc} E\left(\sqrt{c}\right) &= \frac{1}{2c}\left(E\left(\sqrt{c}\right) - K\left(\sqrt{c}\right)\right) \\ \frac{d}{dc} E\left(\sqrt{1-c}\right) &= -\frac{1}{2c'}\left(E\left(\sqrt{c'}\right) - K\left(\sqrt{c'}\right)\right) \end{aligned} \tag{175}$$

If we denote

$$\begin{aligned} L(\varphi) &= F\left(\varphi, \sqrt{c}\right) E\left(\varphi, \sqrt{1-c}\right) + F\left(\varphi, \sqrt{1-c}\right) E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right) F\left(\varphi, \sqrt{1-c}\right) \\ &= E\left(\varphi, \sqrt{c}\right) E\left(\varphi, \sqrt{1-c}\right) - \left(E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right)\right)\left(E\left(\varphi, \sqrt{1-c}\right) - F\left(\varphi, \sqrt{1-c}\right)\right) \end{aligned}$$

then

$$\begin{aligned} \frac{dL(\varphi)}{dc} &= E\left(\varphi, \sqrt{1-c}\right) \frac{d}{dc} E\left(\varphi, \sqrt{c}\right) + E\left(\varphi, \sqrt{c}\right) \frac{d}{dc} E\left(\varphi, \sqrt{1-c}\right) \\ &\quad - \left(E\left(\varphi, \sqrt{1-c}\right) - F\left(\varphi, \sqrt{1-c}\right)\right) \frac{d}{dc}\left(E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right)\right) \\ &\quad - \left(E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right)\right) \frac{d}{dc}\left(E\left(\varphi, \sqrt{1-c}\right) - F\left(\varphi, \sqrt{1-c}\right)\right) \\ &= \frac{E\left(\varphi, \sqrt{c'}\right)}{2c}\left(E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right)\right) - \frac{E\left(\varphi, \sqrt{c}\right)}{2c'}\left(E\left(\varphi, \sqrt{c'}\right) - F\left(\varphi, \sqrt{c'}\right)\right) \\ &\quad - \left(E\left(\varphi, \sqrt{c'}\right) - F\left(\varphi, \sqrt{c'}\right)\right)\left(-\frac{1}{2c'} E\left(\varphi, \sqrt{c}\right) - \frac{1}{2c'} \frac{\sin\varphi\cos\varphi}{\sqrt{1 - c\sin^2\varphi}}\right) \\ &\quad - \left(E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right)\right)\left(\frac{1}{2c} E\left(\varphi, \sqrt{c'}\right) + \frac{1}{2c} \frac{\sin\varphi\cos\varphi}{\sqrt{1 - c'\sin^2\varphi}}\right) \\ &= \frac{\left(E\left(\varphi, \sqrt{c'}\right) - F\left(\varphi, \sqrt{c'}\right)\right)}{2c'} \frac{\sin\varphi\cos\varphi}{\sqrt{1 - c\sin^2\varphi}} - \frac{\left(E\left(\varphi, \sqrt{c}\right) - F\left(\varphi, \sqrt{c}\right)\right)}{2c} \frac{\sin\varphi\cos\varphi}{\sqrt{1 - c'\sin^2\varphi}} \end{aligned}$$

which is zero if $\varphi = \frac{\pi}{2}$, implying that

$$L\left(\frac{\pi}{2}\right) = E\left(\sqrt{c}\right) E\left(\sqrt{1-c}\right) - \left(E\left(\sqrt{c}\right) - K\left(\sqrt{c}\right)\right)\left(E\left(\sqrt{1-c}\right) - K\left(\sqrt{1-c}\right)\right)$$

is a constant for any c. When choosing $c = 0$, then **art**. 47 shows that $\lim_{c\to 0} E\left(\sqrt{c}\right) = \frac{\pi}{2}$ and $\lim_{c\to 0} E\left(\sqrt{1-c}\right) = 1$. Further, **art**. 47 indicates for $c \to 0$ that $\left(E\left(\sqrt{c}\right) - K\left(\sqrt{c}\right)\right) < \frac{\pi}{2}\frac{c}{\sqrt{1-c}} = O(c)$ and $\left(E\left(\sqrt{1-c}\right) - K\left(\sqrt{1-c}\right)\right) = O\left(\frac{1}{\sqrt{\sqrt{c}}}\right)$. Hence, their product is $O\left(c^{1-1/4}\right)$, which tends to zero for $c \to 0$ resulting in $L\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ for any c, which proves Legendre's formula $K(k)E(k') + K(k')E(k) - K(k)K(k') = \frac{\pi}{2}$ in (98).

When subtracting (175) from (174), we find that

$$\begin{aligned} \frac{d}{dc}\left(K\left(\sqrt{c}\right)\right) &= \frac{1}{2c'} E\left(\sqrt{c}\right) + \frac{1}{2c} E\left(\sqrt{c}\right) - \frac{1}{2c} K\left(\sqrt{c}\right) \\ &= \frac{1}{2c}\left\{\frac{1}{(1-c)} E\left(\sqrt{c}\right) - K\left(\sqrt{c}\right)\right\} \end{aligned}$$

E Expansion of the function $r(y)$ based on its functional equation (78)

The functional equation of $r(y)$ is $\frac{1}{8}r^4(y) = r(y^4)\, S\left(S^2 + r^2(y^4)\right)$ in (78).

First, we rewrite the series expansion $f^N(z) = \left(\sum_{k=0}^{\infty} f_k\, z^k\right)^N = \sum_{j_1=0}^{\infty} \cdots \sum_{j_N=0}^{\infty} \prod_{i=1}^{N} f_{j_i}\; z^{\sum_{i=1}^{N} j_i}$ for an integer N. After letting $m = \sum_{i=1}^{N} j_i$ with $j_i \geq 0$ for each $1 \leq i \leq N$, we obtain

$$f^N(z) = \sum_{m=0}^{\infty} \left(\sum_{\sum_{i=1}^{N} j_i = m; j_i \geq 0} \prod_{i=1}^{N} f_{j_i} \right) z^m \tag{176}$$

The series expansion of the left-hand side in (78) becomes with $N = 4$,

$$\begin{aligned} r^4(y) &= 2^4 y \left(\sum_{k=0}^{\infty} r_k y^k \right)^4 = 16y \sum_{m=0}^{\infty} \left(\sum_{\sum_{i=1}^{4} j_i = m; j_i \geq 0} \prod_{i=1}^{4} r_{j_i} \right) y^m \\ &= 16y \left(1 + \sum_{m=1}^{\infty} \left(\sum_{\sum_{i=1}^{4} j_i = m; j_i \geq 0} \prod_{i=1}^{4} r_{j_i} \right) y^m \right) \end{aligned}$$

The Taylor coefficients can be computed via the recursion relation of our characteristic coefficients (see e.g. [33]). The Taylor coefficient $\sum_{\sum_{i=1}^{4} j_i = m; j_i \geq 0} \prod_{i=1}^{4} r_{j_i}$ for $m > 0$ contains terms in which product $\prod_{i=1}^{4} r_{j_i} = r_m$ in all ways $\sum_{i=1}^{4} j_i = m$, where 3 out of the 4 j_i are equal to zero, because $r_0 = 1$. Thus, in precisely $\binom{4}{3} = 4$ ways and the highest index term is thus $4r_m$, while all other products $\prod_{i=1}^{4} r_{j_i}$ consists of lower indices v in r_v. Another way to compute the Taylor coefficient is by twice evaluating a Cauchy product. Indeed, $\left(\sum_{k=0}^{\infty} r_k y^k\right)^2 = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} r_{k-l} r_l\right) y^k$ and $\left(\sum_{k=0}^{\infty} r_k y^k\right)^4 = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \left(\sum_{l=0}^{i} r_{i-l} r_l\right) \left(\sum_{j=0}^{k-i} r_{k-i-j} r_j\right)\right) y^k$, so that

$$\sum_{\sum_{i=1}^{4} j_i = m; j_i \geq 0} \prod_{i=1}^{4} r_{j_i} = \sum_{i=0}^{m} \sum_{l=0}^{i} r_{i-l} r_l \sum_{j=0}^{m-i} r_{m-i-j} r_j$$

For $m > 0$, we can write

$$\sum_{\sum_{i=1}^{4} j_i = m; j_i \geq 0} \prod_{i=1}^{4} r_{j_i} = 4r_m + 2 \sum_{j=1}^{m-1} r_{m-j} r_j + \sum_{i=1}^{m-1} \sum_{l=0}^{i} r_{i-l} r_l \sum_{j=0}^{m-i} r_{m-i-j} r_j \tag{177}$$

The first factor on the right-hand side in (78) of the series $r(u) = 2u^{\frac{1}{4}} \left(1 + \sum_{k=1}^{\infty} r_k u^k\right)$ in (77) has the Taylor series

$$\begin{aligned} r\left(y^4\right) &= 2y \sum_{k=0}^{\infty} r_k y^{4k} = 2y \sum_{m=0}^{\infty} r_{\frac{m}{4}} 1_{\{4|m\}} y^m \\ &= 2y \left(1 + \sum_{m=1}^{\infty} r_{\frac{m}{4}} 1_{\{4|m\}} y^m \right) \end{aligned}$$

where the indicator $1_{\{4|m\}}$ only equals one if m is a multiple of 4, which is equivalent to requiring that 4 must divide the integer m. If $\frac{m}{4}$ is not an integer, then $1_{\{4|m\}} = 0$.

The second factor on the right-hand side in (78) requires the Taylor expansion of $S = 1 + \sum_{l=2}^{\infty} r\left(y^{2^{2l}}\right)$. We introduce the series (77),

$$\sum_{l=2}^{\infty} r\left(y^{2^{2l}}\right) = 2\sum_{l=2}^{\infty}\left(y^{2^{2l}}\right)^{\frac{1}{4}}\sum_{k=0}^{\infty} r_k y^{2^{2l}k} = 2\sum_{l=2}^{\infty}\sum_{k=0}^{\infty} r_k y^{2^{2(l-1)}+2^{2l}k} = 2\sum_{l=1}^{\infty}\sum_{k=0}^{\infty} r_k y^{2^{2l}(4k+1)}$$

Let $m = 2^{2l}(4k+1)$, then $\frac{m}{2^{2l}}$ must run over the integers $1, 5, 9, 14, \ldots, 1+4k, \ldots$. Hence, the condition for m is that $\frac{1}{4}\left(\frac{m}{2^{2l}} - 1\right) \in \mathbb{N}$, implying that $m \geq 2^{2l}$, so that

$$\sum_{l=2}^{\infty} r\left(y^{2^{2l}}\right) = 2\sum_{l=1}^{\infty}\sum_{m=2^{2l}}^{\infty} r_{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)} 1_{\left\{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)\in\mathbb{N}\right\}} y^m$$

Reversing the l- and m-summation yields, where $[x]$ is the integer smaller than or equal to x,

$$\sum_{l=2}^{\infty} r\left(y^{2^{2l}}\right) = 2\sum_{m=1}^{\infty}\left(\sum_{l=1}^{[\log_4 m]} r_{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)} 1_{\left\{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)\in\mathbb{N}\right\}}\right) y^m$$

The Taylor series of S is

$$S = \sum_{m=0}^{\infty} s_m y^m$$

where $s_0 = 1$ and, for $m > 0$,

$$s_m = 2\sum_{l=1}^{[\log_4 m]} r_{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)} 1_{\left\{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)\in\mathbb{N}\right\}}$$

In fact, $s_1 = s_2 = s_3 = 0$ and $s_4 = 2$. The highest index of r_v in s_m occurs for $l = 1$,

$$s_m = 2\left(r_{\frac{1}{4}\left(\frac{m}{4}-1\right)} 1_{\left\{\frac{1}{4}\left(\frac{m}{4}-1\right)\in\mathbb{N}\right\}} + \sum_{l=2}^{[\log_4 m]} r_{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)} 1_{\left\{\frac{1}{4}\left(\frac{m}{2^{2l}}-1\right)\in\mathbb{N}\right\}}\right)$$

The third factor $S^2 + r^2\left(y^4\right)$ on the right-hand side in (78) consists of the sum of two Cauchy products of Taylor series, namely

$$\begin{aligned} r^2\left(y^4\right) &= 4y^2\sum_{m=0}^{\infty} r_{\frac{m}{4}} 1_{\{4|m\}} y^m \sum_{l=0}^{\infty} r_{\frac{l}{4}} 1_{\{4|l\}} y^l = 4y^2\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m} r_{\frac{m-l}{4}} 1_{\{4|m-l\}} r_{\frac{l}{4}} 1_{\{4|l\}}\right) y^m \\ &= \sum_{m=2}^{\infty}\left(4\sum_{l=0}^{m-2} r_{\frac{m-2-l}{4}} 1_{\{4|m-2-l\}} r_{\frac{l}{4}} 1_{\{4|l\}}\right) y^m \end{aligned}$$

and

$$S^2 = \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m} s_{m-l} s_l\right) y^m = 1 + \sum_{m=2}^{\infty}\left(\sum_{l=0}^{m} s_{m-l} s_l\right) y^m$$

because $\sum_{l=0}^{1} s_{1-l} s_l = 2 s_0 s_1 = 0$. Hence, the Taylor series is

$$S^2 + r^2\left(y^4\right) = 1 + \sum_{m=2}^{\infty}\left(\sum_{l=0}^{m} s_{m-l} s_l + 4\sum_{l=0}^{m-2} r_{\frac{m-2-l}{4}} 1_{\{4|m-2-l\}} r_{\frac{l}{4}} 1_{\{4|l\}}\right) y^m$$

which we write as

$$S^2 + r^2\left(y^4\right) = \sum_{m=0}^{\infty} U_m y^m$$

where $U_0 = 1$ and for $m > 0$,

$$U_m = \sum_{l=0}^{m} s_{m-l} s_l + 4 \sum_{l=0}^{m-2} r_{\frac{m-2-l}{4}} 1_{\{4|m-2-l\}} r_{\frac{l}{4}} 1_{\{4|l\}}$$

The Taylor series of $r\left(y^4\right) S$ again consists of the Cauchy product of two Taylor series $r\left(y^4\right) S = 2y \sum_{m=0}^{\infty} r_{\frac{m}{4}} 1_{\{4|m\}} y^m \sum_{m=0}^{\infty} s_m y^m$ and equals

$$r\left(y^4\right) S = 2y \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} r_{\frac{m-l}{4}} 1_{\{4|m-l\}} s_l \right) y^m$$

The Taylor series of the right-hand side $\frac{1}{8} r^4(y) = r\left(y^4\right) S\left(S^2 + r^2\left(y^4\right)\right)$ in (78) is

$$r\left(y^4\right) S\left(S^2 + r^2\left(y^4\right)\right) = 2y \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l \right) y^m$$

Let us investigate the Taylor coefficient $W_m = \sum_{k=0}^{m} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l$ for $m > 0$, because $W_0 = 1$. We rewrite

$$\begin{aligned} W_m &= \sum_{l=0}^{m} r_{\frac{m-l}{4}} 1_{\{4|m-l\}} s_l + \sum_{k=0}^{m-1} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l \\ &= r_{\frac{m}{4}} 1_{\{4|m\}} + \sum_{l=1}^{m} r_{\frac{m-l}{4}} 1_{\{4|m-l\}} s_l + \sum_{k=0}^{m-1} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l \end{aligned}$$

The second and third sum only contain terms in r_v with a lower index $v < \frac{m}{4}$.

Finally, both left-hand an right-hand side Taylor series in $\frac{1}{8} r^4(y) = r\left(y^4\right) S\left(S^2 + r^2\left(y^4\right)\right)$ in (78)

$$2y \left(\sum_{m=0}^{\infty} \left(\sum_{\sum_{i=1}^{4} j_i = m; j_i \geq 0} \prod_{i=1}^{4} r_{j_i} \right) y^m \right) = 2y \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l \right) y^m$$

leads, after equating corresponding powers in y for[49] $m > 0$ with (177), to

$$4r_m + 2 \sum_{j=1}^{m-1} r_{m-j} r_j + \sum_{i=1}^{m-1} \sum_{l=0}^{i} r_{i-l} r_l \sum_{j=0}^{m-i} r_{m-i-j} r_j = \sum_{k=0}^{m} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l$$

Since the right-hand side does not contain r_v with index equal to m (but at most $r_{\frac{m}{4}}$), all Taylor coefficients for $m > 0$ are found by the recursion

$$r_m = \frac{1}{4} \left(\sum_{k=0}^{m} U_{m-k} \sum_{l=0}^{k} r_{\frac{k-l}{4}} 1_{\{4|k-l\}} s_l - 2 \sum_{j=1}^{m-1} r_{m-j} r_j - \sum_{i=1}^{m-1} \sum_{l=0}^{i} r_{i-l} r_l \sum_{j=0}^{m-i} r_{m-i-j} r_j \right)$$

Executing the recursion returns $r_m = 1_{\left\{m = \left(k+\frac{1}{2}\right)^2 - \frac{1}{4}\right\}}$ for $k \geq 0$. Hence,

$$r(y) = 2y^{\frac{1}{4}} \left(1 + y^2 + y^6 + y^{12} + y^{20} + y^{30} + \cdots + y^{\left(k+\frac{1}{2}\right)^2 - \frac{1}{4}} + \cdots \right)$$

which agrees with Gauss's last series in [15, art. 16, p. 383].

[49] If $m = 0$, then both sides are equal to 1.