Properties of a triangular matrix and its linear differential equation

P. Van Mieghem*

28 July 2023

Abstract

The analytic – not the numerical – solution of a set of linear differential equations corresponding to an upper triangular matrix is derived. When all diagonal elements of the triangular matrix are different, the explicit solution can be written as an enumeration over paths.

1 Introduction

We consider a set of m linear differential equations

$$\frac{ds\left(t\right)}{dt}=Qs\left(t\right) \tag{1}$$

where $s\left(t\right)=\left[\begin{array}{cccc} s_1\left(t\right) & s_2\left(t\right) & \cdots & s_m\left(t\right)\end{array}\right]^T$ is an $m\times 1$ vector and Q is an $m\times m$ upper triangular matrix,

$$Q=\left[\begin{array}{ccccc} q_{11} & q_{12} & \cdots & q_{1,m-1} & q_{1m} \\ 0 & q_{22} & \cdots & q_{2,m-1} & q_{2m} \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & \cdots & q_{m-1,m-1} & q_{m-1,m} \\ 0 & 0 & \cdots & 0 & q_{mm}\end{array}\right] \tag{2}$$

The matrix differential equation (1) is explicitly written as

$$\left[\begin{array}{c} \frac{ds_1(t)}{dt} \\ \frac{ds_2(t)}{dt} \\ \vdots \\ \frac{ds_{m-1}(t)}{dt} \\ \frac{ds_m(t)}{dt}\end{array}\right]=\left[\begin{array}{ccccc} q_{11} & q_{12} & \cdots & q_{1,m-1} & q_{1m} \\ 0 & q_{22} & \cdots & q_{2,m-1} & q_{2m} \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & \cdots & q_{m-1,m-1} & q_{m-1,m} \\ 0 & 0 & \cdots & 0 & q_{mm}\end{array}\right]\left[\begin{array}{c} s_1\left(t\right) \\ s_2\left(t\right) \\ \vdots \\ s_{m-1}\left(t\right) \\ s_m\left(t\right)\end{array}\right]$$

or as a linear set,

$$\left\{\begin{array}{l} \frac{ds_1(t)}{dt}=q_{11}s_1+q_{12}s_2+\cdots+q_{1,m-1}s_{m-1}+q_{1m}s_m \\ \frac{ds_2(t)}{dt}=q_{22}s_2+\cdots+q_{2,m-1}s_{m-1}+q_{2m}s_m \\ \vdots \\ \frac{ds_{m-1}(t)}{dt}=q_{m-1,m-1}s_{m-1}+q_{m-1,m}s_m \\ \frac{ds_m(t)}{dt}=q_{mm}s_m\end{array}\right. \tag{3}$$

*Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, P.O Box 5031, 2600 GA Delft, The Netherlands; *email*: P.F.A.VanMieghem@tudelft.nl

We take the Laplace transform from both sides of the set (3). Denoting the Laplace transform as $\mathcal{L}\left(s_k\left(t\right)\right)=\int_0^\infty e^{-pt}s_k\left(t\right)=S_k\left(p\right)$ and invoking $\mathcal{L}\left(\frac{ds_k(t)}{dt}\right)=pS_k\left(p\right)-s_k\left(0\right)$, which follows by partial integration, the Laplace transformed set of the set (3) is

$$\begin{cases} pS_1\left(p\right)-s_1\left(0\right)=q_{11}S_1\left(p\right)+q_{12}S_2\left(p\right)+\cdots+q_{1,m-1}S_{m-1}\left(p\right)+q_{1m}S_m\left(p\right) \\ pS_2\left(p\right)-s_2\left(0\right)=q_{22}S_2\left(p\right)+\cdots+q_{2,m-1}S_{m-1}\left(p\right)+q_{2m}S_m\left(p\right) \\ \vdots \\ pS_{m-1}\left(p\right)-s_{m-1}\left(0\right)=q_{m-1,m-1}S_{m-1}\left(p\right)+q_{m-1,m}S_m\left(p\right) \\ pS_m\left(p\right)-s_m\left(0\right)=q_{mm}S_m\left(p\right) \end{cases}$$

The matrix form,

$$\begin{bmatrix} q_{11}-p & q_{12} & \cdots & q_{1,m-1} & q_{1m} \\ 0 & q_{22}-p & \cdots & q_{2,m-1} & q_{2m} \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & \cdots & q_{m-1,m-1}-p & q_{m-1,m} \\ 0 & 0 & \cdots & 0 & q_{mm}-p \end{bmatrix} \begin{bmatrix} S_1\left(p\right) \\ S_2\left(p\right) \\ \vdots \\ S_{m-1}\left(p\right) \\ S_m\left(p\right) \end{bmatrix} = - \begin{bmatrix} s_1\left(0\right) \\ s_2\left(0\right) \\ \vdots \\ s_{m-1}\left(0\right) \\ s_m\left(0\right) \end{bmatrix} \tag{4}$$

demonstrates that the algebraic set is $\left(Q-pI\right)S\left(p\right)=-s\left(0\right)$, which resembles an eigenvalue equation. We simplify the notation of the matrix in (4) by denoting $\xi_j=q_{jj}-p$,

$$A=\begin{bmatrix} \xi_1 & q_{12} & \cdots & q_{1,m-1} & q_{1m} \\ 0 & \xi_2 & \cdots & q_{2,m-1} & q_{2m} \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & \cdots & \xi_{m-1} & q_{m-1,m} \\ 0 & 0 & \cdots & 0 & \xi_m \end{bmatrix} \tag{5}$$

which is an $m\times m$ triangular matrix where only the diagonal elements $\xi_1,\xi_2,\ldots,\xi_m$ depend upon the Laplace transform parameter p. The linear equation in (4) becomes $A\left(p\right)S\left(p\right)=-s\left(0\right)$ with solution

$$S\left(p\right)=-A^{-1}\left(p\right)s\left(0\right) \tag{6}$$

and, per component,

$$S_j\left(p\right)=-\sum_{i=1}^{m}s_i\left(0\right)\left(A^{-1}\right)_{ji}\left(p\right)$$

If a process starts in one state l, then $s_i\left(0\right)=\delta_{il}$ or $s\left(0\right)=e_l$, where the Kronecker delta $\delta_{il}=1$ if $i=l$, otherwise $\delta_{il}=0$ and where e_l is the l-th basic vector with $\left(e_l\right)_i=1$ if $i=l$, otherwise $\left(e_l\right)_i=0$. The corresponding solution (6) then simplifies to $S\left(p\right)=-\left(A^{-1}\right)_{\mathrm{col}\,l}$ or $S_j\left(p\right)=-\left(A^{-1}\right)_{jl}$, where the inverse matrix $A^{-1}\left(p\right)$ is a function of the Laplace parameter p. After inverse Laplace transformation $\mathcal{L}^{-1}\left(S_j\left(p\right)\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}S_j\left(p\right)e^{pt}dp=s_j\left(t\right)$, we arrive at

$$s\left(t\right)=-A^{-1}\left(t\right)s\left(0\right)$$

For the initial condition $s_i\left(0\right)=\delta_{il}$, the solution of the set of differential equations in (3) is,

$$s_j\left(t\right)=-\left(A^{-1}\right)_{jl}\left(t\right) \tag{7}$$

Our main result in this article is the analytic computation of $\left(A^{-1}\right)_{jl}(t)$ in (7). If all diagonal elements of the matrix Q in (2) are different and non-zero, then $\left(A^{-1}\right)_{jl}(t)$ is explicitly given in (22). The other case, where not all diagonal elements are different, is clearly more complicated (see Theorem 3 on p. 8) and (19) illustrates that also integer powers of t, beside exponential functions, color the dynamics.

The article is structured as follows. Section 2 derives general properties of an $m \times m$ upper triangular matrix. Section 3 computes the inverse A^{-1} of the triangular matrix A, which is the more elegant way instead of focusing on the eigenstructure of A, which is derived in Appendix A. Indeed, the general solution of the inverse A^{-1} allows Laplace transformation that leads to a general solution, derived in Section 4, while the eigenvalue problem is already complicated from the start on when multiple eigenvalues occur (as illustrated in the proof of Theorem 4). The simplest form solution in (22) assumes that all eigenvalues of the triangular matrix A are different.

2 Properties of an upper triangular matrix A

If A is an $m \times m$ upper triangular matrix, then the structure in (5) shows that its elements obey $a_{kj} = 0$ if $j < k$ (fix row k and vary the columns j) and, equivalently, $a_{jn} = 0$ if $j > n$ (fix column n and vary the rows j). The determinant of a triangular matrix (both upper and lower triangular) equals the product of its diagonal elements as proved in [3, art. 207]. Hence, for the upper triangular matrix A in (5), it holds that $\det A = \prod_{j=1}^{m} \xi_j$, from which we conclude that $\det A = 0$ only if at least one diagonal element ξ_j for $1 \leq j \leq m$ is zero.

Theorem 1 *If A and B are $m \times m$ upper triangular matrices, so is the product $C = AB$. Inversely, if C and A are $m \times m$ upper triangular matrices, then also B is an $m \times m$ upper triangular matrix.*

Proof: The elementwise matrix product for any matrix A and B is $c_{kl} = \sum_{j=1}^{m} a_{kj} b_{jl}$. Since A is here an upper triangular matrix with $a_{kj} = 0$ if $j < k$ (fix row k and vary columns j), we obtain $c_{kl} = \sum_{j=k}^{m} a_{kj} b_{jl}$. If also B is an upper triangular matrix, with $b_{jl} = 0$ if $j > l$ (fix column l and vary the rows j), then we arrive at

$$c_{kl} = \sum_{j=k}^{l} a_{kj} b_{jl} \tag{8}$$

If $l < k$, then (8) indicates that $c_{kl} = 0$, because, by convention, $\sum_{j=k}^{l} f(j) = 0$ if $k > l$. Alternatively, for $j \geq k > l$, the triangular structure shows that $b_{jl} = 0$. Consequently, the property "$c_{kl} = 0$ if $l < k$ (fix row k and vary the columns l)" implies that C is an $m \times m$ upper triangular matrix.

The inverse property follows from $c_{kl} = \sum_{j=k}^{m} a_{kj} b_{jl}$, where the triangular property "$c_{kl} = 0$ if $l < k$ (fix row k and vary the columns l)" requires that B is triangular. Indeed, if B is not triangular, then any element $c_{kl} = \sum_{j=k}^{m} a_{kj} b_{jl}$ can be non-zero, in which case the matrix C is not triangular. □

Corollary 1 *If A is an $m \times m$ upper triangular matrix, then A^k is also an $m \times m$ upper triangular matrix for non-negative integers $k \geq 0$. Any integer power $k \in \mathbb{Z}$ of a triangular matrix without zero diagonal elements is also a triangular matrix without zero diagonal elements.*

Proof: Replacing B in Theorem 1 by A shows that A^2 is an $m \times m$ upper triangular matrix. Repeating the argument by replacing B by A^2, then A^3 and so on, proves the first part of Corollary

1. From $A^k = A^{k-1}A$ and assuming that $\det A \neq 0$ (i.e. none of the diagonal elements of A is zero), it follows that $A^{-1}A^k = A^{k-1}$. Since both A^k and A^{k-1} are upper triangular, Theorem 1 states that also the inverse matrix A^{-1} must be an $m \times m$ upper triangular matrix. Hence, any integer power $k \in \mathbb{Z}$ of a triangular matrix without zero diagonal elements is also a triangular matrix without zero diagonal elements. $\square$

If $k = l$ in (8), then $c_{kk} = a_{kk}b_{kk}$, which illustrates that diagonal elements of C are the direct product of the corresponding diagonal elements in A and B; a property related to the Hadamard product of two matrices.

Corollary 2 *If A and B are $m \times m$ upper triangular matrices, the elements in the n-th upper diagonals of $C = AB$ possess the same number of elements, i.e. $n+1$ terms of products. In other words, if $l - k = n$, then c_{kl} consists of a sum of $n+1$ products.*

Proof: Corollary 2 is a consequence of (8). If $l = k + n$, then (8) becomes

$$c_{k,k+n} = \sum_{j=k}^{k+n} a_{kj} b_{j,k+n} = \sum_{j=0}^{n} a_{k,j+k} b_{j+k,k+n}$$

illustrating that the elements in the n-th upper diagonal of $C = AB$ consist of a sum of $n+1$ terms of products. $\square$

3 The inverse A^{-1} of the triangular matrix A

From the definition $A^{-1}A = I$, the matrix product and the Kronecker delta $\delta_{jl} = 1_{\{j=l\}}$, we deduce that

$$\delta_{jl} = \sum_{n=1}^{m} a_{jn} \left(A^{-1}\right)_{nl}$$

Since A is an upper triangular matrix with $a_{jn} = 0$ if $j < n$, we obtain for the matrix in (5)

$$\delta_{jl} = \sum_{n=j}^{m} a_{jn} \left(A^{-1}\right)_{nl} = \xi_j \left(A^{-1}\right)_{jl} + \sum_{n=j+1}^{m} q_{jn} \left(A^{-1}\right)_{nl} \tag{9}$$

Taking the determinant from both sides of $A^{-1}A = I$ leads to $\det\left(A^{-1}\right) = \frac{1}{\det A} = \prod_{k=1}^{m} \frac{1}{\xi_k}$. Theorem 1 with $C = I$ and $B = A^{-1}$ indicates that the inverse matrix A^{-1} of an $m \times m$ upper triangular matrix A is also an $m \times m$ upper triangular matrix.

Before concentrating on the general case, the situation where $j = l$ in (9), $1 = \xi_j \left(A^{-1}\right)_{jj} + \sum_{n=j+1}^{m} q_{jn} \left(A^{-1}\right)_{nj}$, is equivalent by $c_{kk} = a_{kk}b_{kk}$ after Corollary 1, to $\sum_{n=j+1}^{m} q_{jn} \left(A^{-1}\right)_{nj} = 0$, which is correct, because the triangular matrix satisfies $\left(A^{-1}\right)_{nj} = 0$ if $n > j$. If $j < l$, then $\left(A^{-1}\right)_{jl} = 0$ and (9) is again obeyed. If $j > l$, then the governing relation $\delta_{jl} = \sum_{n=j}^{m} q_{jn} \left(A^{-1}\right)_{nl}$ in (9) reduces with $\left(A^{-1}\right)_{nl} = 0$ if $n > l$ to

$$0 = \sum_{n=j}^{m} q_{jn} \left(A^{-1}\right)_{nl} = \sum_{n=j}^{l} q_{jn} \left(A^{-1}\right)_{nl}$$

leading for $j > l$ to the recursion $\left(A^{-1}\right)_{jl} = -\frac{1}{\xi_j}\sum_{n=j+1}^{l} q_{jn}\left(A^{-1}\right)_{nl}$. Invoking $\left(A^{-1}\right)_{ll} = \frac{1}{\xi_l}$, the non-zero elements of the inverse matrix A^{-1} can be iteratively computed from

$$\left(A^{-1}\right)_{jl} = -\frac{q_{jl}}{\xi_j\xi_l} - \frac{1}{\xi_j}\sum_{n=1}^{l-j-1} q_{j,n+j}\left(A^{-1}\right)_{n+j,l} \qquad \text{for } l > j \tag{10}$$

The recursion (10) illustrates that the number of terms increases with the difference $l-j$ as in Corollary 2. If $l = j+1$, then the recursion (10) shows that $\left(A^{-1}\right)_{j,j+1} = -\frac{q_{j,j+1}}{\xi_j\xi_{j+1}}$. The recursion (10) reduces for $l = j+2$ to

$$\left(A^{-1}\right)_{j,j+2} = -\frac{q_{j,j+2}}{\xi_j\xi_{j+2}} - \frac{q_{j,1+j}}{\xi_j}\left(A^{-1}\right)_{1+j,j+2} = -\frac{q_{j,j+2}}{\xi_j\xi_{j+2}} + \frac{q_{j,j+1}q_{j+1,j+2}}{\xi_j\xi_{j+1}\xi_{j+2}}$$

and for $l = j+3$ to

$$\begin{aligned}\left(A^{-1}\right)_{j,j+3} &= -\frac{q_{j,j+3}}{\xi_j\xi_{j+3}} - \frac{1}{\xi_j}q_{j,1+j}\left(A^{-1}\right)_{1+j,j+3} - \frac{1}{\xi_j}q_{j,2+j}\left(A^{-1}\right)_{2+j,j+3} \\ &= -\frac{q_{j,j+3}}{\xi_j\xi_{j+3}} + \frac{q_{j,1+j}}{\xi_j}\left(\frac{q_{j+1,j+3}}{\xi_{j+1}\xi_{j+3}} - \frac{q_{j+1,j+2}q_{j+2,j+3}}{\xi_{j+1}\xi_{j+2}\xi_{j+3}}\right) + \frac{q_{j,2+j}}{\xi_j}\left(\frac{q_{j+2,j+3}}{\xi_{j+2}\xi_{j+3}}\right)\end{aligned}$$

which is simplified as

$$\left(A^{-1}\right)_{j,j+3} = -\frac{q_{j,j+3}}{\xi_j\xi_{j+3}} + \frac{q_{j,1+j}q_{j+1,j+3}}{\xi_j\xi_{j+1}\xi_{j+2}} + \frac{q_{j,2+j}q_{j+2,j+3}}{\xi_j\xi_{j+2}\xi_{j+3}} - \frac{q_{j,1+j}q_{j,1+j}q_{j+1,j+2}}{\xi_j\xi_j\xi_{j+1}\xi_{j+2}}$$

The structure of $\left(A^{-1}\right)_{j,j+3}$ resembles the summations of paths between node j and $j+3$ in a complete graph with four nodes, $j, j+1, j+2, j+3$ and link weight q_{lk} between node k and l. The first term is the direct, one hop path, the second term contains all two-hop paths between node j and $j+3$ and the third term contains the single three-hops path [3, art. 20]. Continuing the evaluation of the recursion (10) leads to Theorem 2:

Theorem 2 *If A is an $m \times m$ upper triangular matrix with the structure in (5), then the elements of the inverse matrix A^{-1} satisfy, for $l > j$,*

$$\left(A^{-1}\right)_{jl} = -\frac{q_{jl}}{\xi_j\xi_l} + \frac{1}{\xi_j\xi_l}\sum_{h=2}^{l-j}(-1)^h \sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1}^{h-1}\xi_{k_r}} \tag{11}$$

and

$$\left(A^{-1}\right)_{jj} = \frac{1}{\xi_j} \tag{12}$$

From a graph perspective, the general formal structure (11) for $l > j$ is written as a sum over all paths $\mathcal{P}_h$ with h hops (and with maximum $h = l-j$ hops) in the complete, directed graph formed by $h+1$ nodes, labelled from $j, j+1, \ldots, l$, as

$$\left(A^{-1}\right)_{jl} = -\frac{q_{jl}}{\xi_j\xi_l} + \frac{1}{\xi_j\xi_l}\sum_{h=2}^{l-j}(-1)^h \sum_{\substack{\mathcal{P}_h=(k_0,k_1,\ldots,k_h)\\ k_0=j,k_h=l}} \frac{\prod_{r=0}^{h-1} q_{k_r,k_{r+1}}}{\prod_{r=1}^{h-1}\xi_{k_r}} \tag{13}$$

where a path of h-hops from node $k_0 = j$ to $k_h = l$ is $\mathcal{P}_h = (j \to k_1)(k_1 \to k_2)\cdots(k_{h-1} \to l)$ and all intermediate nodes $\{k_r\}_{1\le r\le h-1}$ are different, different from $\{j, l\}$ and $j < k_r < l$.

If $l=j$ in (11), then the first term $-\frac{q_{jl}}{\xi_j\xi_l}=-\frac{1}{\xi_j}$, because $q_{jj}=\xi_j$ and second sum is zero, due to the convention that $\sum_{h=a}^{b} f(h)=0$ if $a>b$, which has the opposite sign of (12) and, therefore, the case $l=j$ is excluded from (11). The case $l=j$ corresponds to a zero-hop path or a self-loop, where $h=0$ in (11) and (13). A hop count $h=1$ means the direct link between node j and node l, which is the first term in (11) and (13). If $h=2$, equivalent to one intermediate node k_1 in the path $\mathcal{P}_2$ between starting node j and destination node l, then the second multiple sum in (11) with $k_{h-1}=k_1$ reduces to a single sum

$$\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{\prod_{r=0}^{h-1}q_{k_r,k_{r+1}}}{\prod_{r=0}^{h}\xi_{k_r}}=\frac{1}{\xi_j\xi_l}\sum_{k_1=j+1}^{l-1}\frac{q_{j,k_1}q_{k_1,l}}{\xi_{k_1}}$$

while $h=3$ becomes

$$\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{\prod_{r=0}^{h-1}q_{k_r,k_{r+1}}}{\prod_{r=0}^{h}\xi_{k_r}}=\frac{1}{\xi_j\xi_l}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\frac{q_{j,k_1}q_{k_1,k_2}q_{k_1,l}}{\xi_{k_1}\xi_{k_2}}$$

and so on. Relations (11) and (13) resemble an inclusion-exclusion formula (see e.g. [2, p. 10-12]).

Proof: In order to prove the validity of (11), we introduce the solution (11) into the recursion $\left(A^{-1}\right)_{jl}=-\frac{q_{jl}}{\xi_j\xi_l}-\frac{1}{\xi_j}\sum_{n=j+1}^{l-1}q_{jn}\left(A^{-1}\right)_{nl}$ in (10) for $l>j$,

$$\left(A^{-1}\right)_{jl}=-\frac{q_{jl}}{\xi_j\xi_l}+\frac{1}{\xi_j\xi_l}\sum_{n=j+1}^{l-1}\frac{q_{jn}q_{nl}}{\xi_n}+R_{jl} \tag{14}$$

where

$$R_{jl}=\frac{1}{\xi_j\xi_l}\sum_{n=j+1}^{l-1}\sum_{h=2}^{l-n}(-1)^h\sum_{k_1=n+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{jn}q_{n,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\xi_n\prod_{r=1}^{h-1}\xi_{k_r}}$$

The term $\frac{1}{\xi_j\xi_l}\sum_{n=j+1}^{l-1}\frac{q_{jn}q_{n,l}}{\xi_n}$ in (14) represents the $h=2$ term in (11). Interchanging the n- and h-summation in R_{jl} yields

$$R_{jl}=\frac{1}{\xi_j\xi_l}\sum_{h=2}^{l-1-j}(-1)^h\sum_{n=j+1}^{l-h}\sum_{k_1=n+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{jn}q_{n,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\xi_n\prod_{r=1}^{h-1}\xi_{k_r}}$$

Increasing the index of h by one,

$$R_{jl}=-\frac{1}{\xi_j\xi_l}\sum_{h=3}^{l-j}(-1)^h\sum_{n=j+1}^{l+1-h}\sum_{k_1=n+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-2}=k_{h-3}+1}^{l-1}\frac{q_{jn}q_{n,k_1}\left(\prod_{r=1}^{h-3}q_{k_r,k_{r+1}}\right)q_{k_{h-2},l}}{\xi_n\prod_{r=1}^{h-2}\xi_{k_r}}$$

replacing the index n by k_1 and the index k_r by k_{r+1} for all r, then results in

$$R_{jl} = -\frac{1}{\xi_j \xi_l} \sum_{h=3}^{l-j} (-1)^h \sum_{k_1=j+1}^{l+1-h} \sum_{k_2=k_1+1}^{l-1} \sum_{k_3=k_2+1}^{l-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{jk_1} q_{k_1,k_2} \left(\prod_{r=1}^{h-3} q_{k_{r+1},k_{r+2}} \right) q_{k_{h-1},l}}{\xi_{k_1} \prod_{r=1}^{h-2} \xi_{k_{r+1}}}$$

$$= -\frac{1}{\xi_j \xi_l} \sum_{h=3}^{l-j} (-1)^h \sum_{k_1=j+1}^{l+1-h} \sum_{k_2=k_1+1}^{l-1} \sum_{k_3=k_2+1}^{l-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{jk_1} q_{k_1,k_2} \left(\prod_{r=2}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},l}}{\xi_{k_1} \prod_{r=2}^{h-1} \xi_{k_r}}$$

$$= -\frac{1}{\xi_j \xi_l} \sum_{h=3}^{l-j} (-1)^h \sum_{k_1=j+1}^{l+1-h} \sum_{k_2=k_1+1}^{l-1} \sum_{k_3=k_2+1}^{l-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{jk_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},l}}{\prod_{r=1}^{h-1} \xi_{k_r}}$$

Substituting the above into (14) and comparing with (11) demonstrates Theorem 2. □

4 Inverse Laplace transform

The denominators in (11) only contain ξ-terms that are function of p by $\xi_j = q_{jj} - p$ and each term can be straightforwardly inverse Laplace transformed as demonstrated in this Section 4. Explicitly in terms of the Laplace transform parameter p with $\xi_j = q_{jj} - p$, formula (11) becomes

$$\left(A^{-1}\right)_{jl}(p) = -\frac{q_{jl}}{(q_{jj} - p)(q_{ll} - p)} + \sum_{h=2}^{l-j} (-1)^h \sum_{k_1=j+1}^{l-1} \sum_{k_2=k_1+1}^{l-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{j,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},l}}{(q_{jj} - p) \prod_{r=1}^{h-1} (q_{k_r,k_r} - p)(q_{ll} - p)} \tag{15}$$

The inverse Laplace transform of $\mathcal{L}(f(t)) = \int_0^\infty e^{-pt} f(t)\, dt = F(p)$ is computed via a contour integral $\mathcal{L}^{-1}(F(p)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp = f(t)$, where the real number $c > 0$,

$$\left(A^{-1}\right)_{jl}(t) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{q_{jl} e^{pt} dp}{(q_{jj} - p)(q_{ll} - p)}$$
$$+ \sum_{h=2}^{l-j} (-1)^h \sum_{k_1=j+1}^{l-1} \sum_{k_2=k_1+1}^{l-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{q_{j,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},l} e^{pt}}{\prod_{r=0}^{h} (q_{k_r,k_r} - p)} dp$$

We define the integral

$$I_{\{q_{k_r,k_r}\}_{0 \le r \le h}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{\prod_{r=0}^{h} (q_{k_r,k_r} - p)} dp \tag{16}$$

where the set $\{q_{k_r,k_r}\}_{0 \le r \le h} = \{q_{jj}, q_{k_1,k_1}, \ldots, q_{k_{h-1},k_{h-1}}, q_{ll}\}$ is a selection or subset of diagonal elements $\{q_{nn}\}_{j \le n \le m}$ in the matrix Q in (2) with all different elements (as follows from the path structures in Theorem 2). The product $\prod_{r=0}^{h} (q_{k_r,k_r} - p)$ is a polynomial of order $h+1$ in the complex variable p with zeros at the elements of the set $\{q_{k_r,k_r}\}_{0 \le r \le h}$. In terms of the integral $I_{\{q_{k_r,k_r}\}_{0 \le r \le h}}$ in (16), the

inverse Laplace transform of $\left(A^{-1}\right)_{jl}(p)$ in (15) is written as

$$\left(A^{-1}\right)_{jl}(t) = -q_{jl} I_{\{q_{jj},q_{ll}\}} + \sum_{h=2}^{l-j} (-1)^h \sum_{k_1=j+1}^{l-1} \sum_{k_2=k_1+1}^{l-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{l-1} q_{j,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l} I_{\{q_{k_r,k_r}\}_{0\le r\le h}} \tag{17}$$

In spite of the fact that the path structure indicates that all intermediate nodes $k_1, k_2, \ldots, k_{h-1}$ are different and different from begin node $k_0 = j$ and end node $k_h = l$, the corresponding values of the diagonal elements $\{q_{ll}\}_{j\le l\le m}$ in the matrix Q in (2) can be the same, i.e. $q_{k_r,k_r} = q_{k_n,k_n}$ for some pair of (different) nodes k_r and k_n. Depending on the number of same diagonal elements $\{q_{ll}\}_{j\le l\le m}$ in the matrix Q in (2), the contour integral (16) has a different form (see also [2, p. 45]). We first concentrate on the computation of the contour integral (16) in Section 4.1 and then proceed with the implication for the expression (17) in Section 4.2.

4.1 The contour integral in (16)

Theorem 3 *If all diagonal elements $\{q_{ll}\}_{j\le l\le m}$ in the matrix Q in (2) are different and $\mathrm{Re}\,(q_{ll}) \le 0$, then*

$$I_{\{q_{k_r,k_r}\}_{0\le r\le h}} = \sum_{s=0}^{h} \frac{e^{q_{k_s,k_s}t}}{\prod_{r=0;r\neq s}^{h} (q_{k_r,k_r} - q_{k_s,k_s})} \tag{18}$$

If some diagonal elements are the same, then the polynomial can be written as $\prod_{r=0}^{h} (q_{k_r,k_r} - p) = \prod_{u=0}^{b} (q_{k_u,k_u} - p)^{\nu_u}$, where the sum of the multiplicities of the zeros satisfies $\sum_{u=0}^{b} \nu_u = h+1$, and the integral in (16) becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{\prod_{u=0}^{b} (q_{k_u,k_u} - p)^{\nu_u}} dp = \sum_{u=0}^{b} \sum_{i=0}^{\nu_u-1} \frac{t^{\nu_u-1-i} e^{q_{k_u,k_u}t}}{(\nu_u-1-i)!} \frac{\sum_{\sum_{\substack{n=0\\n\neq u}}^{b} k_n = i; k_n\ge 0} \prod_{\substack{n=0\\n\neq u}}^{b} \frac{\binom{-\nu_n}{k_n}}{(q_{k_n,k_n} - q_{k_u,k_u})^{k_n}}}{\prod_{\substack{n=0\\n\neq u}}^{b} (q_{k_n,k_n} - q_{k_u,k_u})^{\nu_n}} \tag{19}$$

Proof: If all diagonal elements $\{q_{ll}\}_{j\le l\le m}$ are different, then also all $\{q_{k_r,k_r}\}_{0\le r\le h}$ are different by Theorem 2 and the contour can be closed over the negative $\mathrm{Re}\,(p)$-plane. Cauchy's residue theorem then states that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{\prod_{r=0}^{h} (q_{k_r,k_r} - p)} dp = \sum_{s=0}^{h} \lim_{p\to q_{k_s,k_s}} \frac{e^{pt}}{\prod_{r=0;r\neq s}^{h} (q_{k_r,k_r} - p)} = \sum_{s=0}^{h} \frac{e^{q_{k_s,k_s}t}}{\prod_{r=0;r\neq s}^{h} (q_{k_r,k_r} - q_{k_s,k_s})}$$

provided $\mathrm{Re}\,(q_{k_s,k_s}) \le 0$, because positive poles are not encircled by the contour.

If some diagonal elements are the same and $\prod_{r=0}^{h} (q_{k_r,k_r} - p) = \prod_{u=0}^{b} (q_{k_u,k_u} - p)^{\nu_u}$ with $\sum_{u=0}^{b} \nu_u = h+1$, then the integral (16) is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{\prod_{u=0}^{b} (q_{k_u,k_u} - p)^{\nu_u}} dp = \sum_{u=0}^{b} \lim_{p\to q_{k_u,k_u}} \frac{1}{(\nu_u-1)!} \frac{d^{\nu_u-1}}{dp^{\nu_u-1}} \left(\frac{e^{pt}}{\prod_{n=0;n\neq u}^{b} (q_{k_n,k_n} - p)^{\nu_n}}\right) \tag{20}$$

where the Cauchy integral $\frac{1}{k!}\left.\frac{d^k f(z)}{dz^k}\right|_{z=z_0} = \frac{1}{2\pi i}\int_{C(z_0)} \frac{f(\omega)\,d\omega}{(\omega-z_0)^{k+1}}$ has been used. Leibniz' rule gives us

$$\begin{aligned}\frac{d^{\nu_u-1}}{dp^{\nu_u-1}}\left(\frac{e^{pt}}{\prod_{n=a;n\neq u}^{b}(q_{k_n,k_n}-p)^{\nu_n}}\right) &= \sum_{i=0}^{\nu_u-1}\binom{\nu_u-1}{i}\frac{d^{\nu_u-1-i}}{dp^{\nu_u-1-1}}e^{pt}\frac{d^i}{dp^i}\left(\prod_{n=0;n\neq u}^{b}(q_{k_n,k_n}-p)^{-\nu_n}\right)\\ &= e^{pt}\sum_{i=0}^{\nu_u-1}\binom{\nu_u-1}{i}t^{\nu_u-1-i}\frac{d^i}{dp^i}\left(\prod_{n=0;n\neq u}^{b}(q_{k_n,k_n}-p)^{-\nu_n}\right)\end{aligned}$$

Substitution into (20) yields

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{pt}}{\prod_{u=0}^{b}(q_{k_u,k_u}-p)^{\nu_u}}dp = \sum_{u=0}^{b}e^{q_{k_u,k_u}t}\sum_{i=0}^{\nu_u-1}\frac{t^{\nu_u-1-i}}{i!\,(\nu_u-1-i)!}\lim_{p\to q_{k_u,k_u}}\frac{d^i}{dp^i}\left(\prod_{n=0;n\neq u}^{b}(q_{k_n,k_n}-p)^{-\nu_n}\right) \tag{21}$$

The remaining limit in (21) is most elegantly computed after Taylor expansion around $p = q_{k_u,k_u}$. Thus,

$$\begin{aligned}\prod_{n=0;n\neq u}^{b}(q_{k_n,k_n}-p)^{-\nu_n} &= \prod_{n=0;n\neq u}^{b}((q_{k_n,k_n}-q_{k_u,k_u})-(p-q_{k_u,k_u}))^{-\nu_n}\\ &= \prod_{n=0;n\neq u}^{b}(q_{k_n,k_n}-q_{k_u,k_u})^{-\nu_n}\prod_{n=0;n\neq u}^{b}\left(1-\frac{(p-q_{k_u,k_u})}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{-\nu_n}\end{aligned}$$

Newton's binomial series $(1+z)^\alpha = \sum_{j=0}^{\infty}\binom{\alpha}{j}z^j$, convergent for all α and $|z|<1$, yields

$$H = \prod_{n=0;n\neq u}^{b}\left(1-\frac{(p-q_{k_u,k_u})}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{-\nu_n} = \prod_{n=0;n\neq u}^{b}\sum_{j=0}^{\infty}\binom{-\nu_n}{j}\left(\frac{1}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^j x^j$$

where $x = p - q_{k_u,k_u}$ and $\frac{d^i}{dp^i} = \frac{d^i}{dx^i}$ for all $i \geq 0$. Further,

$$\begin{aligned}H &= \sum_{k_0=0}^{\infty}\binom{-\nu_0}{k_0}\left(\frac{1}{q_{k_0,k_0}-q_{k_u,k_u}}\right)^{k_0}x^{k_0}\cdots\sum_{k_b=0}^{\infty}\binom{-\nu_b}{k_b}\left(\frac{1}{q_{k_b,k_b}-q_{k_u,k_u}}\right)^{k_b}x^{k_b}\\ &= \sum_{k_0=0}^{\infty}\cdots\sum_{k_b=0}^{\infty}\binom{-\nu_0}{k_0}\left(\frac{1}{q_{k_0,k_0}-q_{k_u,k_u}}\right)^{k_0}\cdots\binom{-\nu_b}{k_b}\left(\frac{1}{q_{k_b,k_b}-q_{k_u,k_u}}\right)^{k_b}x^{\sum_{n=a;n\neq u}^{b}k_n}\\ &= \sum_{k_0=0}^{\infty}\cdots\sum_{k_b=0}^{\infty}\prod_{n=0;n\neq u}^{b}\binom{-\nu_n}{k_n}\left(\frac{1}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{k_n}x^{\sum_{n=0;n\neq u}^{b}k_n}\end{aligned}$$

After letting $m = \sum_{n=0;n\neq u}^{b}k_n$ with $k_n \geq 0$ for each $0 \leq n \neq u \leq b$, we obtain

$$\prod_{n=0;n\neq u}^{b}\left(1-\frac{(p-q_{k_u,k_u})}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{-\nu_n} = \sum_{m=0}^{\infty}\left(\sum_{\sum_{n=0;n\neq u}^{b}k_n=m;k_n\geq 0}\prod_{n=0;n\neq u}^{b}\binom{-\nu_n}{k_n}\left(\frac{1}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{k_n}\right)x^m$$

and

$$
\begin{aligned}
M &= \frac{d^i}{dp^i}\left(\prod_{n=0;n\neq u}^{b}\left(q_{k_n,k_n}-p\right)^{-\nu_n}\right)\\
&= \prod_{n=0;n\neq u}^{b}\left(q_{k_n,k_n}-q_{k_u,k_u}\right)^{-\nu_n}\frac{d^i}{dp^i}\left(\prod_{n=0;n\neq u}^{b}\left(1-\frac{\left(p-q_{k_u,k_u}\right)}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{-\nu_n}\right)\\
&= \prod_{n=0;n\neq u}^{b}\left(q_{k_n,k_n}-q_{k_u,k_u}\right)^{-\nu_n}\times\\
&\quad \frac{d^i}{dx^i}\left(\sum_{m=0}^{\infty}\left(\sum_{\sum_{n=0;n\neq u}^{b}k_n=m;k_n\geq 0}\prod_{n=0;n\neq u}^{b}\binom{-\nu_n}{k_n}\left(\frac{1}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{k_n}\right)x^m\right)\\
&= \prod_{n=a;n\neq u}^{b}\left(q_{k_n,k_n}-q_{k_u,k_u}\right)^{-\nu_n}\times\\
&\quad \sum_{m=i}^{\infty}\left(\sum_{\sum_{n=0;n\neq u}^{b}k_n=m;k_n\geq 0}\prod_{n=0;n\neq u}^{b}\binom{-\nu_n}{k_n}\left(\frac{1}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{k_n}\right)\frac{m!}{(m-i)!}\,x^{m-i}
\end{aligned}
$$

After taking the limit $p\to q_{k_u,k_u}$, which is equivalent to $x\to 0$, we arrive at

$$
\lim_{p\to q_{k_u,k_u}}\frac{d^i}{dp^i}\left(\prod_{n=a;n\neq u}^{b}\left(q_{k_n,k_n}-p\right)^{-\nu_n}\right)=i!\frac{\sum_{\sum_{n=0;n\neq u}^{b}k_n=i;k_n\geq 0}\prod_{n=0;n\neq u}^{b}\binom{-\nu_n}{k_n}\left(\frac{1}{q_{k_n,k_n}-q_{k_u,k_u}}\right)^{k_n}}{\prod_{n=0;n\neq u}^{b}\left(q_{k_n,k_n}-q_{k_u,k_u}\right)^{\nu_n}}
$$

Introducing into (21) results into (19). □

The simplest case of (19), when two diagonal elements q_{jj} and q_{ll} of the matrix Q are the same, is differently computed in Appendix C.

4.2 The time-dependent solution $\left(A^{-1}\right)_{jl}(t)$

We confine ourselves to the situation where all diagonal elements of the matrix Q are different. Introducing the explicit form of the integral $I_{\left\{q_{k_r,k_r}\right\}_{0\leq r\leq h}}$ in (18) into (17) yields

$$
\begin{aligned}
\left(A^{-1}\right)_{jl}(t) &= -q_{jl}\left(\frac{e^{q_{jj}t}-e^{q_{ll}t}}{q_{ll}-q_{jj}}\right)+\sum_{h=2}^{l-j}(-1)^h\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}\\
&\quad\times\sum_{n=0}^{h}\frac{e^{q_{k_n,k_n}t}}{\prod_{r=0;r\neq n}^{h}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)}
\end{aligned}
$$

We take into account that the begin node $k_0=j$ and $k_h=l$ are fixed,

$$\left(A^{-1}\right)_{jl}(t) = -q_{jl}\left(\frac{e^{q_{jj}t}-e^{q_{ll}t}}{q_{ll}-q_{jj}}\right) + \sum_{h=2}^{l-j}(-1)^h \sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}$$
$$\times\left\{\frac{\frac{e^{q_{jj}t}}{(q_{ll}-q_{jj})}}{\prod_{r=1}^{h-1}(q_{k_r,k_r}-q_{jj})} + \sum_{n=1}^{h-1}\frac{\frac{e^{q_{k_n,k_n}t}}{(q_{jj}-q_{k_n,k_n})(q_{ll}-q_{k_n,k_n})}}{\prod_{r=1;r\neq n}^{h-1}(q_{k_r,k_r}-q_{k_n,k_n})} + \frac{\frac{e^{q_{ll}t}}{(q_{jj}-q_{ll})}}{\prod_{r=1}^{h-1}(q_{k_r,k_r}-q_{ll})}\right\} \tag{22}$$

illustrating that the intermediate nodes $k_1,\ldots,k_{h-1}$ in a path $\mathcal{P}_h$ from j to l changes for different paths and that each intermediate node k_r has a node label in the set $\{j+1,j+2,\ldots,l-1\}$. Explicit in the direct and first hop, (22) is

$$\left(A^{-1}\right)_{jl}(t) = -q_{jl}\left(\frac{e^{q_{jj}t}-e^{q_{ll}t}}{q_{ll}-q_{jj}}\right) + \sum_{k_1=j+1}^{l-1} q_{j,k_1}q_{k_1,l}\left(\frac{\frac{e^{q_{jj}t}}{(q_{ll}-q_{jj})}}{(q_{k_1,k_1}-q_{jj})} + \frac{e^{q_{k_1,k_1}t}}{(q_{jj}-q_{k_1,k_1})(q_{ll}-q_{k_1,k_1})} + \frac{\frac{e^{q_{ll}t}}{(q_{jj}-q_{ll})}}{(q_{k_1,k_1}-q_{ll})}\right)$$
$$-\sum_{h=3}^{l-j}(-1)^h \sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}$$
$$\times\left\{\frac{\frac{e^{q_{jj}t}}{(q_{ll}-q_{jj})}}{\prod_{r=1}^{h-1}(q_{k_r,k_r}-q_{jj})} + \sum_{n=1}^{h-1}\frac{\frac{e^{q_{k_n,k_n}t}}{(q_{jj}-q_{k_n,k_n})(q_{ll}-q_{k_n,k_n})}}{\prod_{r=1;r\neq n}^{h-1}(q_{k_r,k_r}-q_{k_n,k_n})} + \frac{\frac{e^{q_{ll}t}}{(q_{jj}-q_{ll})}}{\prod_{r=1}^{h-1}(q_{k_r,k_r}-q_{ll})}\right\}$$

The path structure $\mathcal{P}_h = (j,k_1,\ldots,k_{h-1},l)$ or $\mathcal{P}_h = j \to k_1 \to \ldots \to k_{h-1} \to l$ in a directed complete graph is encoded in (22) via the indices in the product $q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}$. As shown in [2, Sec. 15.2] and [3, art. 20], the total number of different paths with h hops, which arises when all elements of the matrix Q are non-zero (as in the complete graph with $N = l-j+1$ nodes), equals $\frac{(N-2)!}{(N-h-1)!}$. For large N, the asymptotic law is $\frac{(N-2)!}{(N-h-1)!} = \frac{\Gamma(N-1)}{\Gamma(N-h)} = N^{h-1}\left(1+O\left(\frac{1}{N}\right)\right)$ and summed over all $h = N-1$ hop paths in (22) leads to a maximum total number of different paths equal to $[e\,(N-2)!]$. In particular when the matrix Q is sparse, the total number of different paths can be substantially smaller. Another observation from (22) is that only pure exponentials $e^{q_{k_n,k_n}t}$ for $0 \le n \le h$ specify the time-dependence of $\left(A^{-1}\right)_{jl}(t)$.

It is convenient to rewrite (22) in terms of a sum of exponentials in the time t,

$$\left(A^{-1}\right)_{jl}(t) = \left(q_{jl} + \sum_{h=2}^{l-j}(-1)^h \sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}}{\prod_{r=1}^{h-1}(q_{k_r,k_r}-q_{jj})}\right)\frac{e^{q_{jj}t}}{(q_{jj}-q_{ll})}$$
$$-\left(q_{jl} + \sum_{h=2}^{l-j}(-1)^h \sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}}{\prod_{r=1}^{h-1}(q_{k_r,k_r}-q_{ll})}\right)\frac{e^{q_{ll}t}}{(q_{jj}-q_{ll})}$$
$$-\sum_{h=2}^{l-j}\sum_{n=1}^{h-1}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{(-1)^h q_{j,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}}{(q_{jj}-q_{k_n,k_n})(q_{ll}-q_{k_n,k_n})\prod_{r=1;r\neq n}^{h-1}(q_{k_r,k_r}-q_{k_n,k_n})} e^{q_{k_n,k_n}t}$$

We reverse the h- and n-summation, $\sum_{h=2}^{l-j}\sum_{n=1}^{h-1}\cdots=\sum_{h=1}^{l-j-1}\sum_{n=1}^{h}\cdots=\sum_{n=1}^{l-j-1}\sum_{h=n}^{l-j-1}\cdots$, to arrive at

$$\left(A^{-1}\right)_{jl}(t)=\left(q_{jl}+\sum_{h=2}^{l-j}(-1)^{h}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)}\right)\frac{e^{q_{jj}t}}{\left(q_{jj}-q_{ll}\right)}$$
$$-\left(q_{jl}+\sum_{h=2}^{l-j}(-1)^{h}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{ll}\right)}\right)\frac{e^{q_{ll}t}}{\left(q_{jj}-q_{ll}\right)}$$
$$-\sum_{n=1}^{l-j-1}\left(\sum_{h=n}^{l-j-1}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)}\right)\frac{(-1)^{h}e^{q_{k_n,k_n}t}}{\left(q_{jj}-q_{k_n,k_n}\right)\left(q_{ll}-q_{k_n,k_n}\right)}$$

which can be summarized, with $k_0=j$ and $k_{h_{\max}}=k_{l-j}=l$ as

$$\left(A^{-1}\right)_{jl}(t)=\sum_{n=0}^{l-j}\Psi_{jl;n}e^{q_{k_n,k_n}t} \tag{23}$$

where, as told before, any node k_n has a label between $k_0=j$ and $k_{l-j}=l$. The first coefficients

$$\Psi_{jl;0}=\frac{q_{jl}+\sum_{h=2}^{l-j}(-1)^{h}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)}}{q_{jj}-q_{ll}}$$

$$\Psi_{jl;l-j}=-\frac{q_{jl}+\sum_{h=2}^{l-j}(-1)^{h}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{ll}\right)}}{q_{jj}-q_{ll}}$$

possess the same structure, whereas for the intermediate nodal indices $1\leq n\leq l-j$,

$$\Psi_{jl;n}=\frac{\sum_{h=n}^{l-j-1}(-1)^{h}\sum_{k_1=j+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1}\frac{q_{j,k_1}\left(\prod_{r=1}^{h-2}q_{k_r,k_{r+1}}\right)q_{k_{h-1},l}}{\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)}}{\left(q_{jj}-q_{k_n,k_n}\right)\left(q_{ll}-q_{k_n,k_n}\right)}$$

Since the diagonal elements of a triangular matrix equal the eigenvalues, thus $\lambda_{k_n}=q_{k_n,k_n}$, the exponentials $e^{q_{k_n,k_n}t}$ with rate q_{k_n,k_n} are weighted by the coefficients $\Psi_{jl;n}$, that reflect the total contribution of all paths between two nodes j and l, in a complete graph configuration on the nodes $k_0=j,k_1,\ldots,k_{l-j-1},k_{l-j}=l$. If the matrix Q can be regarded as minus a weighted Laplacian or an infinitesimal generator of a continuous-time Markov chain, then the directed graph G without loops is defined by the weighted off-diagonal elements of the triangular matrix Q.

5 Summary

We have derived the entire, analytic solution of the set of differential equations in (3). The solution $s_j(t)=-\left(A^{-1}\right)_{jl}(t)$ in (7) with either (22) or (23) is the basic building block for the dynamics of

Markovian SIR epidemics on any network [1].

Acknowledgement We are grateful to Massimo Achterberg for checking our computations.

References

[1] M. A. Achterberg and P. Van Mieghem. Analytic solution of Markovian epidemics without re-infections on heterogeneous networks. *in preparation*, 2023.

[2] P. Van Mieghem. *Performance Analysis of Complex Networks and Systems.* Cambridge University Press, Cambridge, U.K., 2014.

[3] P. Van Mieghem. *Graph Spectra for Complex Networks.* Cambridge University Press, Cambridge, U.K., second edition, 2023.

A Eigenvectors of an upper triangular matrix A

The eigenvalue problem for triangular matrices is considerably easier than that for general matrices, because a diagonal element is an eigenvalue which follows from $\det(A-\lambda I)=\prod_{k=1}^{m}(\xi_k-\lambda)=0$ for the matrix A in (5). The eigenvalue equation $Ax_n=\lambda_n x_n$ of the n-th eigenvalue λ_n of an $m\times m$ upper triangular matrix becomes, with $(Ax_n)_k=\sum_{j=1}^{m}a_{kj}(x_n)_j=\sum_{j=k}^{m}a_{kj}(x_n)_j$, for the right-eigenvector component $(x_n)_k$,

$$\lambda_n(x_n)_k=\sum_{j=k}^{m}a_{kj}(x_n)_j \tag{24}$$

which leads to

$$(x_n)_k=\frac{1}{\lambda_n-a_{kk}}\sum_{j=k+1}^{m}a_{kj}(x_n)_j \qquad \text{for } k<m \tag{25}$$

Similarly, the left-eigenvalue equation $y_s^T A=\lambda_s y_s^T$, equivalent to $\left(A^T y_s\right)_l=\lambda_s(y_s)_l$, leads, with $\left(A^T y_s\right)_l=\sum_{j=1}^{m}a_{jl}(y_s)_j=\sum_{j=1}^{l}a_{jl}(y_s)_j$, to

$$(y_s)_l=\frac{1}{\lambda_s-a_{ll}}\sum_{j=1}^{l-1}a_{jl}(y_s)_j \qquad \text{for } l>1 \tag{26}$$

A.1 Right-eigenvectors of an upper triangular matrix A

Theorem 4 *If A is an $m\times m$ upper triangular matrix with different, non-zero diagonal elements, then the right-eigenvector matrix X with the right-eigenvectors $x_1,x_2,\ldots,x_m$ in the columns is an upper triangular matrix with arbitrary non-zero diagonal elements $g_1,g_2,\ldots,g_m$.*

Proof: The right-eigenvalue equation (24) for $k=m$ reduces to $\lambda_n(x_n)_m=a_{mm}(x_n)_m$ or $(\lambda_n-a_{mm})(x_n)_m=0$, implying that either $(x_n)_m=0$ or $\lambda_n=a_{mm}$. Any diagonal element is an eigenvalue and we propose the convention that $\lambda_n=a_{nn}$. Thus, eigenvalues are not ordered, but indexed according to the position of the diagonal element. With the convention that $\lambda_n=a_{nn}$, we enter two cases: (a) if all diagonal elements are different, then $\lambda_n\neq a_{mm}$ for $n\neq m$ and consequently, $(x_n)_m=g_m\,\delta_{nm}$, where g_m is any, non-zero number. In the other case (b) where diagonal elements are the same, the analysis is more involved and the diagonal vector $(a_{11},a_{22},\ldots,a_{mm})$ must be specified in terms of a reduced set $(\alpha_1,\alpha_2,\ldots,\alpha_\rho)$, where $\rho<m$.

We have shown that $(x_n)_m = 0$ if $n < m$ and that $(x_m)_m = g_m$. Using $(x_n)_m = 0$, the right-eigenvalue equation (24) for eigenvectors x_n where $n < m$ becomes $\lambda_n (x_n)_k = \sum_{j=k}^{m-1} a_{kj} (x_n)_j$ which indicates that $(\lambda_n - a_{m-1,m-1}) (x_n)_{m-1} = 0$, leading to $(x_{m-1})_{m-1} = g_{m-1}$ and $(x_n)_{m-1} = 0$ for $n < m-1$. Repeating the argument for $n < m-1$ shows that $(x_{m-2})_{m-2} = g_{m-2}$ and $(x_n)_{m-2} = 0$ for $n < m-2$. Continuing the iteration demonstrates Theorem 4. □

The proof shows that the eigenstructure for a triangular matrix with at least two same diagonal elements is different and more complicated.

Corollary 3 *If A is an $m \times m$ upper triangular matrix with different diagonal elements, then the left-eigenvector matrix Y with the left-eigenvectors $y_1, y_2, \ldots, y_m$ in the columns is a lower triangular matrix with non-zero diagonal elements $g_1^{-1}, g_2^{-1}, \ldots, g_m^{-1}$.*

Proof: The eigenvalue equation in matrix form [3, p. 3] is $AX = X\Lambda$, where Λ is the diagonal matrix with eigenvalues. Theorem 4 implies that $\det(X) = \prod_{k=1}^{m} g_k \neq 0$ so that X^{-1} exists. Hence, the eigenvalue equation is written as $A = X\Lambda X^{-1}$. Since $B = X\Lambda$ is an upper triangular matrix, Theorem 1 states that X^{-1} is also an upper triangular matrix. Generally, the left- and right-eigenvector matrix Y and X satisfy [3, art.238] that $Y^T X = I$ or $Y^T = X^{-1}$, which proves Corollary 3. □

Corollary 3 suggests to take $g_j = 1$ to simplify computations.

A.1.1 The right-eigenvector x_m

If $k = m-1$ in (25), then $(x_n)_{m-1} = \frac{a_{m-1,m}}{\lambda_n - a_{m-1,m-1}} (x_n)_m$ and where $(x_n)_m = \delta_{nm}$. We proceed with $n = m$ and compute the eigenvector x_m. With our convention $\lambda_n = a_{nn}$, we obtain

$$(x_m)_{m-1} = \frac{a_{m-1,m}}{\lambda_m - \lambda_{m-1}}$$

which is reformulated for the matrix in (5) as

$$(x_m)_{m-1} = \frac{q_{m-1,m}}{\xi_m - \xi_{m-1}} \tag{27}$$

If $k = m-2$ and $n = m$, then (25 becomes $(x_m)_{m-2} = \frac{1}{\lambda_m - a_{m-2,m-2}} \left(a_{m-2,m-1} (x_m)_{m-1} + a_{m-2,m} (x_m)_m \right)$. Introducing $(x_m)_m = 1$ and (27) yields

$$(x_m)_{m-2} = \frac{q_{m-2,m}}{\xi_m - \xi_{m-2}} + \frac{q_{m-2,m-1} q_{m-1,m}}{(\xi_m - \xi_{m-1})(\xi_m - \xi_{m-2})}$$

A next iteration with $k = m-3$ and $n = m$ in (25) reveals that

$$\begin{aligned}(x_m)_{m-3} = & \frac{q_{m-3,m}}{\xi_m - \xi_{m-3}} + \left(\frac{q_{m-3,m-2} q_{m-2,m}}{(\xi_m - \xi_{m-3})(\xi_m - \xi_{m-2})} + \frac{q_{m-3,m-1} q_{m-1,m}}{(\xi_m - \xi_{m-3})(\xi_m - \xi_{m-1})} \right) \\ & + \frac{q_{m-3,m-2} q_{m-2,m-1} q_{m-1,m}}{(\xi_m - \xi_{m-3})(\xi_m - \xi_{m-2})(\xi_m - \xi_{m-1})}\end{aligned}$$

Analogously to (11) in Theorem 2, we deduce, for $l < m$, that

$$(x_m)_l = \frac{q_{l,m}}{\xi_m - \xi_l} + \sum_{h=2}^{m-l} \sum_{k_1=l+1}^{m-1} \sum_{k_2=k_1+1}^{m-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{m-1} \frac{q_{l,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},m}}{(\xi_m - \xi_l) \prod_{r=1}^{h-1} (\xi_m - \xi_{k_r})} \tag{28}$$

with the convention $\prod_{r=a}^{b} f(r) = 1$ if $a > b$. Similarly as in the proof of Theorem 2, introducing (28) in the right-eigenvalue equation $(x_m)_l = \frac{1}{\xi_m - \xi_l} \sum_{j=l+1}^{m} q_{lj} (x_m)_j$ in (24) for the eigenvectors x_m for the matrix A in (5) for $l < m$ demonstrates the correctness of (28).

A.1.2 The right-eigenvector x_{m-1}

We repeat the computation in Section A.1.1 for $n = m-1$ in (25), for which we know that $(x_{m-1})_m = 0$ and $(x_{m-1})_{m-1} = 1$. If $k = m-2$ and $n = m-1$ in (25), then $(x_{m-1})_{m-2} = \frac{q_{m-2,m-1}}{\xi_{m-1} - \xi_{m-2}}$, which equals (27) with m decreased by one. The case $k = m-3$ and $n = m-1$ in (25) yields $(x_{m-1})_{m-3} = \frac{q_{m-3,m-2} q_{m-2,m-1}}{(\xi_{m-1} - \xi_{m-2})(\xi_{m-1} - \xi_{m-3})} + \frac{q_{m-3,m-1}}{\xi_{m-1} - \xi_{m-3}}$, which again provides the same form as for $n = m$ in Section A.1.1 after replacing m by $m-1$ everywhere. Hence, (28) transforms for $l < m-1$ into

$$(x_{m-1})_l = \frac{q_{l,m-1}}{\xi_{m-1} - \xi_l} + \sum_{h=2}^{m-1-l} \sum_{k_1=l+1}^{m-2} \sum_{k_2=k_1+1}^{m-2} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{m-2} \frac{q_{l,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},m-1}}{(\xi_{m-1} - \xi_l) \prod_{r=1}^{h-1} (\xi_{m-1} - \xi_{k_r})}$$

Repeating the procedure and replacing m in (28) by n shows that

Theorem 5 *If A is an $m \times m$ upper triangular matrix with different diagonal elements, then the l-th component of the right-eigenvector x_n is, for $l < n$,*

$$(x_n)_l = \frac{q_{l,n}}{\xi_n - \xi_l} + \sum_{h=2}^{n-l} \sum_{k_1=l+1}^{n-1} \sum_{k_2=k_1+1}^{n-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{n-1} \frac{q_{l,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},n}}{(\xi_n - \xi_l) \prod_{r=1}^{h-1} (\xi_n - \xi_{k_r})} \tag{29}$$

and for $l = n$, it holds that $(x_n)_n = 1$. If $l > n$, then $(x_n)_l = X_{ln} = 0$, because X is an upper triangular matrix.

Taken into account that $(x_n)_l = X_{ln}$, (29) is similar in structure than $\left(A^{-1}\right)_{ln}$ in (11),

$$\left(A^{-1}\right)_{ln} = -\frac{q_{l,n}}{\xi_l \xi_n} - \frac{1}{\xi_l \xi_n} \sum_{h=2}^{n-l} (-1)^h \sum_{k_1=j+1}^{n-1} \sum_{k_2=k_1+1}^{n-1} \cdots \sum_{k_{h-1}=k_{h-2}+1}^{n-1} \frac{q_{l,k_1} \left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}} \right) q_{k_{h-1},n}}{\prod_{r=1}^{h-1} \xi_{k_r}}$$

A.2 Left-eigenvectors of an upper triangular matrix A

The left-eigenvalue equation $y_s^T A = \lambda_s y_s^T$, equivalent to $\left(A^T y_s\right)_l = \lambda_s (y_s)_l$, leads for the l-th component of the s-th left-eigenvector to $\sum_{j=1}^{l} a_{jl} (y_s)_j = \lambda_s (y_s)_l$. Thus, for $l = 1$, we find that $(\lambda_s - a_{11}) (y_s)_1 = 0$ from which $(y_1)_1 = 0$, while $(y_s)_1 = 0$ for $s > 1$. The eigenvalue equation (26) for $s = 1$ and for the $m \times m$ upper triangular matrix in (5) becomes

$$(y_1)_l = \frac{1}{\xi_1 - \xi_l} \sum_{j=1}^{l-1} q_{jl} (y_1)_j \qquad \text{for } l > 1$$

After iteration, we obtain for the few first values of l:

$$
\begin{aligned}
(y_1)_1 &= 1 \\
(y_1)_2 &= \frac{q_{12}}{\xi_1-\xi_2} \\
(y_1)_3 &= \frac{q_{13}}{\xi_1-\xi_3} + \frac{q_{12}q_{23}}{(\xi_1-\xi_2)(\xi_1-\xi_3)} \\
(y_1)_4 &= \frac{q_{14}}{\xi_1-\xi_4} + \frac{q_{12}q_{24}}{(\xi_1-\xi_2)(\xi_1-\xi_4)} + \frac{q_{13}q_{34}}{(\xi_1-\xi_3)(\xi_1-\xi_4)} + \frac{q_{12}q_{23}q_{34}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)}
\end{aligned}
$$

from which we deduce, similarly to (28), for $l > 1$,

$$
(y_1)_l = \frac{q_{1l}}{\xi_1-\xi_l} + \sum_{h=2}^{l-1}\sum_{k_1=2}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{1,k_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}}{(\xi_1-\xi_l)\prod_{r=1}^{h-1}(\xi_m-\xi_{k_r})} \tag{30}
$$

The companion of Theorem 5 is

Theorem 6 *If A is an $m\times m$ upper triangular matrix with different diagonal elements, then the l-th component of the left-eigenvector y_n is, for $l > n$,*

$$
(y_n)_l = \frac{q_{nl}}{\xi_n-\xi_l} + \sum_{h=2}^{l-n}\sum_{k_1=n+1}^{l-1}\sum_{k_2=k_1+1}^{l-1}\cdots\sum_{k_{h-1}=k_{h-2}+1}^{l-1} \frac{q_{nk_1}\left(\prod_{r=1}^{h-2} q_{k_r,k_{r+1}}\right) q_{k_{h-1},l}}{(\xi_n-\xi_l)\prod_{r=1}^{h-1}(\xi_n-\xi_{k_r})} \tag{31}
$$

and for $l = n$, it holds that $(y_n)_n = 1$. If $l < n$, then $(y_n)_l = Y_{ln} = 0$, because Y is a lower triangular matrix.

One may verify $XY^T = I$, i.e. $\sum_{i=1}^m (x_i)_l (y_i)_n = \delta_{nl}$, by combining (29) and (31). Next, the eigenvalue equation $A = X\Lambda Y^T$ implies, if all diagonal elements ξ_k for $1 \le k \le m$, are non-zero, that $A^{-1} = X\Lambda^{-1}Y^T$. This would then be a second verification of $\left(A^{-1}\right)_{jl}$ in (11).

B Second method to compute A^{-1}

A direct computation of the inverse for any matrix A [3, p. 324] is

$$
\left(A^{-1}\right)_{ij} = (-1)^{i+j}\,\frac{\det A_{\setminus \text{row } j \setminus \text{col } i}}{\det A} \tag{32}
$$

For an $m\times m$ upper triangular matrix A in (5), where $\det A = \prod_{k=1}^m \xi_k$, it remains to compute $\det A_{\setminus \text{row } j \setminus \text{col } i}$, where $A_{\setminus \text{row } j \setminus \text{col } i}$ is an upper triangular matrix, possibly complemented with one subdiagonal.

Indeed, if $i = j$, then $\det A_{\setminus \text{row } i \setminus \text{col } i} = \prod_{k=1;k\neq i}^m \xi_k$ and $\left(A^{-1}\right)_{jj} = \frac{1}{\xi_{jj}}$, which agrees with (12).

If $i < j$, then $A_{\setminus \text{row } j \setminus \text{col } i}$ is a triangular matrix with at least one zero diagonal element. Hence, we retrieve that $\left(A^{-1}\right)_{ij} = 0$ if $i < j$ and, consequently, that A^{-1} is an upper triangular matrix.

If $i = j+1$, then $A_{\setminus \text{row } j \setminus \text{col } j+1}$ is an $(m-1)\times(m-1)$ triangular matrix, where all diagonal elements equal ξ_k for $1 \le k \le m-1$, except that ξ_j is replaced by $q_{j,j+1}$. Hence, (32) becomes $\left(A^{-1}\right)_{j,j+1} = -\frac{q_{j,j+1}}{\xi_j\xi_{j+1}}$.

If $i > j+1$, then $A_{\setminus \text{row}\, j \setminus \text{col}\, i}$ is an $(m-1)\times(m-1)$ triangular matrix complemented with one subdiagonal. However, the diagonal elements contain not only ξ_k, but also $q_{k,k+1}$. The subdiagonal contains ξ_k and, only if $i > 1$, it contains zeros.

We illustrate the structure and consider first $i > j$:

$$A_{m\times m} = \begin{bmatrix} \xi_1 & \cdots & q_{1j-1} & q_{1j} & q_{1,j+1} & \cdots & q_{1i-1} & q_{1i} & q_{1,i+1} & \cdots & q_{1m} \\ 0 & \ddots & & & & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \xi_{j-1} & q_{j-1,j} & q_{j-1,j+1} & & \vdots & & \vdots & & q_{j-1,m} \\ \vdots & \cdots & 0 & \xi_j & q_{j,j+1} & \cdots & \vdots & & \vdots & & q_{j,m} \\ \vdots & \cdots & 0 & 0 & \xi_{j+1} & \cdots & \vdots & & \vdots & & q_{j+1,m} \\ \vdots & & & & & \ddots & & & & \cdots & \\ \vdots & & & & & & \xi_{i-1} & q_{i-1,i} & q_{i-1,i+1} & \cdots & q_{i-1,m} \\ \vdots & & & & & & 0 & \xi_i & q_{i,i+1} & & q_{i,m} \\ \vdots & \cdots & & & & \cdots & 0 & 0 & \xi_{i+1} & \cdots & q_{i+1,m} \\ \vdots & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & & & & & 0 & 0 & 0 & & \xi_m \end{bmatrix}$$

Removing row j and column i results in

$$A_{\setminus \text{row}\, j \setminus \text{col}\, i} = \begin{bmatrix} \xi_1 & \cdots & q_{1j-1} & q_{1j} & q_{1,j+1} & \cdots & q_{1i-1} & q_{1,i+1} & \cdots & q_{1m} \\ 0 & \ddots & & & & & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \xi_{j-1} & q_{j-1,j} & q_{j-1,j+1} & & \vdots & \vdots & & q_{j-1,m} \\ \vdots & \cdots & 0 & 0 & \xi_{j+1} & \cdots & \vdots & \vdots & & q_{j+1,m} \\ \vdots & & & & 0 & \ddots & & & \cdots & \\ \vdots & & & & & \ddots & \xi_{i-1} & q_{i-1,i+1} & \cdots & q_{i-1,m} \\ \vdots & & & & & & 0 & q_{i,i+1} & & q_{i,m} \\ \vdots & \cdots & & & & \cdots & 0 & \xi_{i+1} & \cdots & q_{i+1,m} \\ \vdots & & & & & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & & & & & 0 & 0 & & \xi_m \end{bmatrix}$$

and contains $i-j$ zero elements on the diagonal. Hence, if $i > j$, then $\det\left(A_{\setminus \text{row}\, j \setminus \text{col}\, i}\right) = 0$ and (32) shows that $\left(A^{-1}\right)_{ij} = 0$ for $i > j$.

If $i < j$, then

$$A_{m\times m} = \begin{bmatrix} \xi_1 & \cdots & q_{1i-1} & q_{1i} & q_{1,i+1} & \cdots & q_{1j-1} & q_{1j} & q_{1,j+1} & \cdots & q_{1m} \\ 0 & \ddots & & & & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \xi_{i-1} & q_{i-1,i} & q_{i-1,i+1} & & \vdots & & \vdots & & q_{i-1,m} \\ \vdots & \cdots & 0 & \xi_i & q_{i,i+1} & \cdots & \vdots & & \vdots & & q_{i,m} \\ \vdots & \cdots & 0 & 0 & \xi_{i+1} & \cdots & \vdots & & \vdots & & q_{i+1,m} \\ \vdots & & & & & \ddots & & & & \cdots & \\ \vdots & & & & & & \xi_{j-1} & q_{j-1,j} & q_{j-1,j+1} & \cdots & q_{j-1,m} \\ \vdots & & & & & & 0 & \xi_j & q_{j,j+1} & & q_{j,m} \\ \vdots & \cdots & & & & \cdots & 0 & 0 & \xi_{j+1} & \cdots & q_{j+1,m} \\ \vdots & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & & & & & 0 & 0 & 0 & & \xi_m \end{bmatrix}$$

from which

$$A_{\setminus \text{row}\, j \setminus \text{col}\, i} = \begin{bmatrix} \xi_1 & \cdots & q_{1i-1} & q_{1,i+1} & \cdots & q_{1j-1} & q_{1j} & q_{1,j+1} & \cdots & q_{1m} \\ 0 & \ddots & & & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \xi_{i-1} & q_{i-1,i+1} & & \vdots & & \vdots & & q_{i-1,m} \\ \vdots & \cdots & 0 & q_{i,i+1} & \cdots & \vdots & & \vdots & & q_{i,m} \\ \vdots & \cdots & 0 & \xi_{i+1} & \cdots & \vdots & & \vdots & & q_{i+1,m} \\ \vdots & & & & \ddots & q_{j-2,j-1} & & & \cdots & \\ \vdots & & & & & \xi_{j-1} & q_{j-1,j} & q_{j-1,j+1} & \cdots & q_{j-1,m} \\ \vdots & \cdots & & & \cdots & 0 & 0 & \xi_{j+1} & \cdots & q_{j+1,m} \\ \vdots & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & & & & 0 & 0 & 0 & & \xi_m \end{bmatrix}$$

Unfortunately, the direct evaluation of $\det A_{\setminus \text{row}\, j \setminus \text{col}\, i}$ is difficult. Theorem 2, on the other hand, offers the complete evaluation of $\det A_{\setminus \text{row}\, j \setminus \text{col}\, i}$, via (32).

C The simplest instance of the general (19) differently computed

Merely as an illustration, we add here a more basic computation when two diagonal elements q_{jj} and q_{ll} of the matrix Q are the same. The computation in Lemma 7 is the simplest instance of the general (19) and shows the appearance of the time dependent factor $te^{q_{jj}t}$ next to pure exponentials $e^{q_{k_n,k_n}t}$ for $1 \le n \le h-1$.

Lemma 7 *If only two diagonal elements q_{jj} and q_{ll} of the matrix Q are the same, i.e. $q_{jj} = q_{ll}$, then*

the integral in (16) with $k_0 = j$ and $k_h = l$ equals

$$\begin{aligned}\lim_{q_{ll}\to q_{jj}} I_{\left\{q_{k_r,k_r}\right\}_{0\le r\le h}} &= \lim_{q_{ll}\to q_{jj}} I_{\left\{q_{jj},q_{k_1,k_1},\ldots,q_{k_{h-1},k_{h-1}},q_{ll}\right\}} \\ &= -\frac{te^{q_{jj}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)} + \sum_{n=1}^{h-1}\frac{e^{q_{k_n,k_n}t}}{\left(q_{jj}-q_{k_n,k_n}\right)^2\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)} \\ &\quad -\sum_{n=1}^{h-1}\frac{e^{q_{jj}t}}{\left(q_{k_n,k_n}-q_{jj}\right)^2\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)}\end{aligned} \tag{33}$$

Proof: We invoke (18), where all q_{k_r,k_r} for $0 \le r \le h$ are different,

$$I_{\left\{q_{k_r,k_r}\right\}_{0\le r\le h}} = \sum_{n=0}^{h}\frac{e^{q_{k_n,k_n}t}}{\prod_{r=0;r\neq n}^{h}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)}$$

and take the definition $k_0 = j$ and $k_h = l$ into account,

$$\begin{aligned}I_{\left\{q_{k_r,k_r}\right\}_{0\le r\le h}} &= \frac{e^{q_{jj}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)\left(q_{ll}-q_{jj}\right)} + \sum_{n=1}^{h-1}\frac{e^{q_{k_n,k_n}t}}{\left(q_{jj}-q_{k_n,k_n}\right)\left(q_{ll}-q_{k_n,k_n}\right)\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)} \\ &\quad + \frac{e^{q_{ll}t}}{\left(q_{jj}-q_{ll}\right)\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{ll}\right)}\end{aligned}$$

Rewritten as

$$\begin{aligned}I_{\left\{q_{k_r,k_r}\right\}_{0\le r\le h}} &= \frac{1}{\left(q_{ll}-q_{jj}\right)}\left\{\frac{e^{q_{jj}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)} - \frac{e^{q_{ll}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{ll}\right)}\right\} \\ &\quad + \sum_{n=1}^{h-1}\frac{e^{q_{k_n,k_n}t}}{\left(q_{jj}-q_{k_n,k_n}\right)\left(q_{ll}-q_{k_n,k_n}\right)\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)}\end{aligned}$$

illustrates when $q_{ll} \to q_{jj}$ that

$$\begin{aligned}\lim_{q_{ll}\to q_{jj}} I_{\left\{q_{jj},q_{k_r,k_r},q_{ll}\right\}_{1\le r\le h-1}} &= \lim_{x\to q_{jj}}\frac{1}{\left(x-q_{jj}\right)}\left\{\frac{e^{q_{jj}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)} - \frac{e^{xt}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}\right\} \\ &\quad + \sum_{n=1}^{h-1}\frac{e^{q_{k_n,k_n}t}}{\left(q_{jj}-q_{k_n,k_n}\right)^2\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{k_n,k_n}\right)}\end{aligned} \tag{34}$$

The remaining limit follows from de l'Hospital's rule as

$$L = \lim_{x\to q_{jj}}\frac{\left(\frac{e^{q_{jj}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)} - \frac{e^{xt}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}\right)}{\left(x-q_{jj}\right)} = -\lim_{x\to q_{jj}}\frac{d}{dx}\frac{e^{xt}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}$$

The derivative is

$$\frac{d}{dx}\frac{e^{xt}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}=\frac{te^{xt}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}+e^{xt}\frac{d}{dx}\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)^{-1}$$

where the last derivative is computed with the logarithmic derivative $f'(x)=f(x)\frac{d\log f(x)}{dx}$ as

$$\begin{aligned}\frac{d}{dx}\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)^{-1}&=-\frac{1}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}\frac{d}{dx}\left(\sum_{n=1}^{h-1}\log\left(q_{k_n,k_n}-x\right)\right)\\&=\frac{1}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}\left(\sum_{n=1}^{h-1}\frac{1}{\left(q_{k_n,k_n}-x\right)}\right)\\&=\sum_{n=1}^{h-1}\frac{1}{\left(q_{k_n,k_n}-x\right)^2\prod_{r=1;r\neq n}^{h-1}\left(q_{k_n,k_n}-x\right)}\end{aligned}$$

Combining all ingredients results in the limit

$$\begin{aligned}L&=-\lim_{x\to q_{jj}}\left(\frac{te^{xt}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-x\right)}+e^{xt}\sum_{n=1}^{h-1}\frac{1}{\left(q_{k_n,k_n}-x\right)^2\prod_{r=1;r\neq n}^{h-1}\left(q_{k_n,k_n}-x\right)}\right)\\&=-\frac{te^{q_{jj}t}}{\prod_{r=1}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)}-\sum_{n=1}^{h-1}\frac{e^{q_{jj}t}}{\left(q_{k_n,k_n}-q_{jj}\right)^2\prod_{r=1;r\neq n}^{h-1}\left(q_{k_r,k_r}-q_{jj}\right)}\end{aligned}$$

Substitution into (34) leads to (33). □