A LOWER BOUND FOR THE END-TO-END DELAY IN NETWORKS: APPLICATION TO VOICE OVER IP

Piet Van Mieghem
ARеNA

Alcatel Corporate Research Center
Francis Wellesplein 1, B-2018 Antwerp, Belgium*

Abstract

A closed-form lower bound on the end-to-end delay in a network with N nodes and link density p is presented which can be useful for the dimensioning of connectionless 'homogenous' networks. In particular, several initial considerations on networks specifically designed for carrying voice over IP only, may be estimated.

1 Introduction.

The explosive growth of the Internet has attracted many industries to focus on opportunities to enlarge their markets or to enter into the highly promising telecommunication business. As a recent example, the Internet Protocol (IP) has opened doors for new entrants to the traditional, monopoly market of ordinary voice telephony. Due to the stringent delay bounds imposed on the transport of voice, a natural mode of operation is to employ connection oriented networking principles which have resulted in the well-known circuit switched telephony network.

Originally, the Internet was designed to transport data generated by computers. The timeliness of data communications is generally delay tolerant. Delay constraints were not viewed as important as routing flexibility and connectivity. To provide the latter, a connectionless mode of operations, where each data packet of a same flow is treated independently of the others, is preferable. From this underlying connectionless philosophy, the Internet has grown to a worldwide network excelling in an unreliable, connectionless, best effort delivery of variable length packets.

Hence, using IP to transport voice over a connectionless network seems contradictory to the basic requirement of the voice service: a timely delivery of voice samples. However, if an IP-based network is only used for voice communication (so avoiding detrimental interference with other, delay non-sensitive services), tolerable end-to-end delay bounds (ranging from 200 ms to maximum 300 ms) may be achieved. The purpose of this article is to focus on a method to provide analytic expressions to assess feasibility issues of what is now coined 'voice over IP'. In particular, questions as "how large is the network allowed to be to fulfil tolerable end-to-end delays" or "what is the maximum loading of the network" can be estimated with the present approach.

2 The idea of the method.

Performance analyses of communication networks basically rely on two different approaches. In only a few cases, the problem is simple enough to allow for an analytic solution. Usually, the analytic intractability forces us to simulate the network behaviour. However, for dimensioning purposes or to decide on the break-even point in operating cost, simulations are awkward, consume much time and still remain approximate. Indeed, most often simulation results are computed based on some reference network and al-

*Currently at the Technical University of Delft, Faculty of Information Technology and Systems, Department TVS, Room 19.270, Mekelweg 4, 2628 CD Delft, The Netherlands. *Email:* P.VanMieghem@its.tudelft.nl

though different loading scenario's (e.g. several traffic profiles) are considered, in many cases the influence of the chosen reference topology is not questioned. At last, when the behaviour of several parameters is desired, simulations may become unfeasible as the whole parameter space cannot be skimmed in sufficient detail. In short, closed-form analytic estimates are highly valuable to deal with dimensioning problems.

Since a queueing network approach [11] rarely leads to manageable, closed-form expressions and also assumes a certain network topology, we propose a different stochastic method relying on properties of random graphs, derived elsewhere and briefly summarized here and in the appendix 1.

The two most frequently occurring models for random graphs (r.g.) [1] are $G_p(N)$ and $G(N, E)$. The class of r.g. denoted by $G_p(N)$ consists of all graphs with N nodes in which the edges (or links) are chosen independently and with probability p. A natural refinement of $G_p(N)$ is the model $G_{\{p_{ij}\}}(N)$ where the edges are still chosen independently but where the probability of $i \to j$ being an edge is exactly p_{ij}. The class $G(N, E)$ constitutes the set of graphs with N nodes and E edges. Only connected r.g. have been taken into account. This confinement limits the value of the 'link density' p (or equivalent E) from below by some critical threshold p_c. The computation of this threshold is rather complicated and has been investigated for many years as a challenging topic in the theory of r.g. [7, 5, 6] and in percolation theory in solid state physics [9].

In a r.g., the number of paths from a source node, say A, to a destination node, say B, can be categorised according to the number of hops from A to B over different intermediate nodes. An exact analysis for the probability distribution function (pdf) of the hopcount is possible for r.g. of the class $G_p(N)$, but, unfortunately, not for the class $G(N, E)$. An interesting and simple results is,

Theorem 1 *The pdf of the hopcount in connected r.g. of $G_p(N)$ is bounded from below by a Poisson distribution,*

$$P[\text{path} = \text{i hops}] > \frac{\left(\frac{1}{p}\right)^{N-i-1}}{(N-i-1)!}\, e^{-\frac{1}{p}} \quad (p > p_c) \tag{1}$$

Moreover, (1) is a very good approximation.

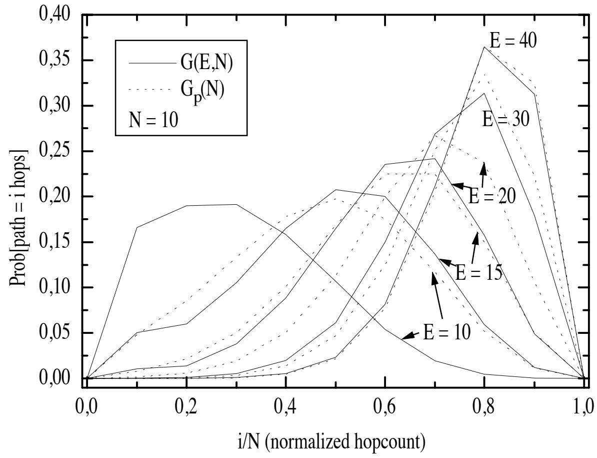

Figure 1: A comparison of the pdf of the hopcount for $G(N, E)$ and $G_p(N)$ computed respectively via the simulations and approximated by (1) for a fixed number of nodes $N = 10$ and various links $E = pE_{max}$.

This theorem is proved in the appendix 1.

In addition, the Poisson probability distribution also seems a reasonable approximation for the class $G_{p_{ij}}(N)$ (with as frequently used member the Waxman random graph [3, 12]) and for the class $G(N, E)$ as shown in Fig. 1. For increasing link density $p \to 1$ all r.g. (and corresponding pdf's) converge rapidly towards a full mesh configuration with $E = E_{max} = \binom{N}{2}$ links as illustrated in Fig. 1.

In section 4 our combination of queueing theory and the theory of r.g. is presented.

3 Assumptions.

As voice over IP has recently evoked considerable attention [10], we will give in this article an estimate of the end-to-end delay experienced by two parties communicating via a voice over IP service. It is assumed that the network only carries voice over IP traffic and no other Internet data. The end-to-end delay computation does not include the encoding time nor the packetization delay[1] at each communicating end system, but only the queueing delay of IP packets in a certain network.

A number of assumptions are adopted. First, we consider each node in the network consisting of a number of separate, single queues, each devoted to traffic for a same output port. This allows us to confine to a single server queueing system. Second, we assume that the input arrival law for the IP packets at each stage is well approximated by a general independent process (such as e.g. a Poisson process) with average number of IP packets per unit time equal to λ. Consequently, the delay in each node may be modeled as in a GI/G/1 queue [8]. Third, the influence of the network topology is reflected via the pdf of the hop count. As shown above, the Poisson distribution is a reasonable approximation for the pdf of the hop count. Only two parameters, the number of nodes N and the link density p characterize a network topology. Fourth, an evenly spread load (average of the arrival process) is assumed. This corresponds more or less to a well designed network exhibiting load balancing as a desired feature. Another interpretation of this assumption is that of a perfect connectionless mode of operation where optimal throughput in the network is achieved. This set of approximations clearly idealizes the network performance, most likely furnishing a lower bound on the (real) average end-to-end delay.

4 Derivation.

Let us denote $W(z)$ and $D(z)$ as the generating functions of respectively the delay per node and the end-to-end delay in a r.g. of the class $G_p(N)$. The variable z refers to a discrete time generating function while we will use below the variable s for continuous time. On the assumption of load balancing each node in the network experiences a same (input) load $\lambda(p)$ which is related to the 'link density p', because the total capacity of a network with identical links is proportional to $p\,E_{max}$. In order words, if C denotes the total capacity of inflows to the network, each node - under load balancing - receives a portion $\frac{C}{E} = \frac{1}{p}\frac{C}{E_{max}}$ of this total capacity. Hence, there holds $\lambda(p) = \frac{\lambda^*}{p}$. The relation between $W(z)$ and $D(z)$ is

$$D(z) = \sum_{h=1}^{N-1} \text{Prob}[\text{hop} = h].W^h(z)$$

Using the Poisson approximation for the pdf of the hop count (1) in a class $G_p(N)$ r.g. yields for $p > p_c$

$$\begin{aligned} D(z) &= e^{-\frac{1}{p}} \sum_{h=1}^{N-1} \frac{\left(\frac{1}{p}\right)^{N-h-1}}{(N-h-1)!} W^h(z) \\ &= e^{-\frac{1}{p}}\, W^{N-1}(z) \sum_{k=0}^{N-2} \frac{\left(\frac{1}{p}\right)^{k}}{k!} W^{-k}(z) \\ &\approx e^{\frac{1}{p}\left[\frac{1}{W(z)}-1\right]}\, W^{N-1}(z) \qquad (2) \end{aligned}$$

where the last approximation is justified if N is sufficiently large. Observe from (2) that $D(1) = 1$ (or in continuous time, $D(0) = 1$) since $W(1) = 1$ (or $W(0) = 1$) guaranteeing the basic normalization condition for the discrete-time (or continuous-time) probability generating function (dpgf) (or cpgf). The derivative evaluated at $z = 1$ (or $s = 0$) equals the average end-to-end delay. Since

$$\begin{aligned} D'(z) &\approx e^{\frac{1}{p}\left[\frac{1}{W(z)}-1\right]}\, W^{N-3}(z)\, W'(z) \\ &\quad \times \left((N-1)\, W(z) - \frac{1}{p} \right) \end{aligned}$$

we obtain taken into account that $W(1) = 1$ (or $W(0) = 1$),

$$D'(1) \approx W'(1) \left((N-1) - \frac{1}{p} \right) \qquad (p > p_c) \qquad (3)$$

This surprisingly simple results means that the average end-to-end delay approximately equals the average delay spent at a node multiplied by the factor $(N-1) - \frac{1}{p}$. The latter mirrors the longest possible number of hops decreased by the inverse of the 'link density p'. In a load balanced topology where all paths are equally probable to be used, a full mesh topology causes the longest average end-to-end delay (because the fraction of longer paths is just larger than in a sparse topology as illustrated in Fig. 1). However, the

[1] An analysis including these delay components is found in [4].

quantity $W'(1)$ involves an average input$\lambda \sim \frac{1}{p}$. This (queueing) dependence on p outweighs the sensitivity of the second factor in (3).

A further analysis requires the knowledge of $W(z)$. The dpgf for the delay in a GI/D/1 system is [2]

$$W(z) = \frac{1 - U'(1)}{U'(1)} \frac{z\,(U(z) - 1)}{z - U(z)}$$

where $U(z)$ is the dpgf of the GI-arrival process. Thus, for a Poisson arrival process with parameter λ holds that $U(z) = e^{\lambda(z-1)}$ or

$$W(z) = \frac{1 - \lambda}{\lambda} \frac{z\,(e^{\lambda(z-1)} - 1)}{z - e^{\lambda(z-1)}}$$

Perhaps a continuous-time equivalent (where z is changed for s) as that of the M/M/1 system may be more appropriate since the IP packets may have a variable length and, hence, may need a variable service time (as opposed to the GI/D/1 system where the service time is deterministic and fixed). From [8, eq. (5.117)], we have

$$W(s) = \frac{(1 - \rho)\mu}{s + \mu(1 - \rho)}$$

where $\rho = \frac{\lambda}{\mu}$ is the traffic intensity and μ the average service time. Introduced into (2), this cpgf leads to a remarkably simple expression for $D(s)$,

$$D(s) \approx ((1 - \rho)\mu)^{N-1} \frac{e^{\overline{\frac{s}{p(1-\rho)\mu}}}}{(s + \mu(1 - \rho))^{N-1}}$$

The inverse Laplace transform, $d(y) \equiv \text{Prob}[D > y] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D(s)\, e^{ys}\, ds$ with $c > 0$, is readily verified with $\rho = \frac{\rho^*}{p}$ and $p > p_c$ as

$$\text{Prob}[D > y] = \frac{\left(1 - \frac{\rho^*}{p}\right)\mu}{(N-2)!} \left(\frac{y(p - \rho^*)\mu + 1}{p}\right)^{N-2} e^{-\frac{y(p-\rho^*)\mu+1}{p}} \quad (4)$$

Relation (4) exhibits that the cumulative pdf $d(y)$ is proportional to a Poisson law with parameter $\frac{y(p-\rho^*)\mu+1}{p}$ evaluated at index $N-2$. In addition, (4) is a simple expression to estimate a lower bound on the end-to-end delay in a network with N nodes and link density p.

An evalution of the goodness of this simple estimate (4) stands on the agenda for further work.

References

[1] B. Bollobas. *Random Graphs.* Academic Press, London, 1985.

[2] H. Bruneel and B. G. Kim. *Discrete-Time Models for Communication System Including ATM.* Kluwer Academic Publishers, Boston, 1993.

[3] K. Calvert, M. Doar, and E. Zegura. Modeling internet topology. *IEEE Comm. Magazine*, pages 160–163, June 1997.

[4] D. De Vleeschauwer, B. Steyaert, and G. H. Petit. Performance analysis of voice over packet-based networks. *submitted to ITC 16, Edingburgh, UK (June 7-10)*, 1999.

[5] P. Erdos and A. Renyi. On random graphs i. *Publ. Math. Debrecen.*, 6:290–297, 1959.

[6] P. Erdos and A. Renyi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*, 5:17–61, 1960.

[7] E. N. Gilbert. Enumeration of labelled graphs. *Canad. J. Math.*, 8:405–411, 1956.

[8] Leonard Kleinrock. *Queueing Systems*, volume 1 : Theory. John Wiley and Sons, N.Y., 1975.

[9] B. I. Shklovskii and A. L. Efros. *Electronic Properties of Doped Semiconductors.* Springer Verlag, Berlin, 1984.

[10] K. Van der Wal, M. Mandjes, and H. Bastiaansen. Delay performance analysis of the new internet services with guaranteed qos. *Proceeding of the IEEE*, 85(12):1947–1957, Dec. 1997.

[11] J. Walrand. *An Introduction to Queueing Networks.* Prentice-Hall, New York, 1988.

[12] B. M. Waxman. Routing of multipoint connections. *IEEE JSAC*, 6(9):1617–1622, dec. 1998.

1 The pdf of the hop count in $G_p(N)$.

A path $\mathbf{p}$ of length i is completely characterised by a list of $i+1$ different nodes, $\mathbf{p_{A\to B}} = [n_1, n_2, \cdots, n_{i+1}]$ with $n_1 = A$, $n_{i+1} = B$ and $n_k \neq n_j$ for all $k, j \in [1, i+1]$. Sometimes a more illustrative representation is given such as $\mathbf{p_{A\to B}} = (n_1 \to n_2)(n_2 \to n_3)\cdots(n_i \to n_{i+1})$. The maximum number of hops is clearly $N-1$. There are two different approaches to compute the probability distribution function (pdf) of the hop count of a path from A to B. Either we fix the topology and vary the source and destination over all possible couples or we choose a particular source and destination and let the topology change over all possible forms. Here, the latter is preferred with $A = 1$ and $B = N$.

The connectivity of network topology with N nodes can be represented by a symmetrical connection matrix T consisting of elements T_{ij} that are either one or zero depending on whether there is a link or edge between node i and j or not.

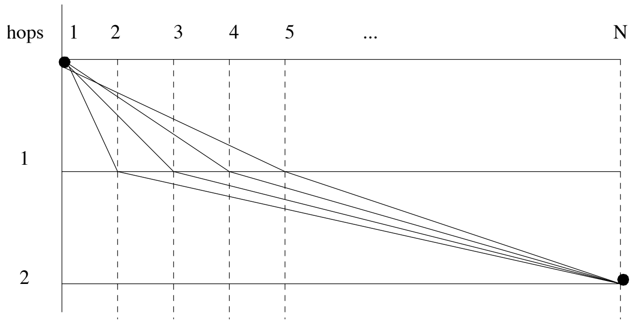

Figure 2: A sketch of the possible constructions of paths with 2 hops.

The average number of paths with 1 hop equals p, because it is related to the probability that $T_{1N} = 1$. The maximum number of different links used is clearly one, which we denote as $L_1 = 1$.

The average number of paths with 2 hops is $E\left[\sum_{i=2}^{N-1}(1 \xrightarrow{p} i).(i \xrightarrow{p} N)\right] = (N-2).p^2$. Indeed as illustrated in Fig. 2, initially, all paths with 2 hops start at 1, visit an intermediate node i different from 1 and N from which they depart to the final destination N. Also, the maximum number of different links used to construct a path with 2 hops equals $L_2 = 2(N-2)$.

Analogous, the average number of paths with 3 hops is

$$E\left[\sum_{i\neq(1,N)}\sum_{j\neq(1,i,N)}(1 \xrightarrow{p} i).(i \xrightarrow{p} j).(j \xrightarrow{p} N)\right] = (N-2)(N-3).p^3$$

The maximum number of links used is now $L_3 = E_{max} - 2$. Indeed, only two link transition are forbidden, T_{1N} and one out of the $N-2$ remaining last column values.

In general, we have that the average (or expected) number of paths with i hops, H_i, equals

$$\begin{aligned}
H_i &= E\Bigg[\sum_{j_1\neq\{1,N\}}\sum_{j_2\neq\{1,j_1,N\}}\cdots\sum_{j_{i-1}\neq\{1,j_1,\cdots,j_{i-2},N\}} \\
&\qquad (1 \xrightarrow{p} j_1).(j_1 \xrightarrow{p} j_2).\cdots.(j_{i-1} \xrightarrow{p} N)\Bigg] \\
&= \sum_{j_1\neq\{1,N\}}\sum_{j_2\neq\{1,j_1,N\}}\cdots\sum_{j_{i-1}\neq\{1,j_1,\cdots,j_{i-2},N\}} \\
&\qquad E[(1 \xrightarrow{p} j_1)].E[(j_1 \xrightarrow{p} j_2)].\cdots.E[(j_{i-1} \xrightarrow{p} N)] \\
&= \sum_{j_1\neq\{1,N\}}\sum_{j_2\neq\{1,j_1,N\}}\cdots\sum_{j_{i-1}\neq\{1,j_1,\cdots,j_{i-2},N\}} p^i \\
&= p^i \sum_{j_1\neq\{1,N\}}\sum_{j_2\neq\{1,j_1,N\}}\cdots\sum_{j_{i-1}\neq\{1,j_1,\cdots,j_{i-2},N\}} 1 \\
&= \frac{(N-2)!}{(N-i-1)!}\,p^i \qquad (5)
\end{aligned}$$

The maximum number of links used equals $L_i = E_{max} - i + 1$ because there are, besides T_{1N}, now $i-2$ last column elements not allowed.

Lemma 1 *The maximum total number of paths in any graph is upper bounded by*

$$M = [e(N-2)!] \qquad (N \geq 3) \qquad (6)$$

where $[x]$ is the integer smaller than or equal to x.

Proof. That maximum M will clearly be attained in case $E = E_{max}$ or $p = 1$

Using (5) and summing over all hops yields

$$\begin{aligned}
M &= \sum_{i=1}^{N-1}\frac{(N-2)!}{(N-i-1)!} = (N-2)!\sum_{j=0}^{N-2}\frac{1}{j!} \\
&\equiv (N-2)!e - R
\end{aligned}$$

where

$$R = (N-2)!\sum_{j=N-1}^{\infty}\frac{1}{j!} = \sum_{j=0}^{\infty}\frac{(N-2)!}{N-1+j!}$$

$$
\begin{aligned}
&= \frac{1}{N-1} + \frac{1}{(N-1)N} + \\
&\quad + \frac{1}{(N-1)N(N+1)} + \cdots \\
&< \sum_{j=1}^{\infty} \left(\frac{1}{N-1}\right)^j = \frac{1}{N-2}
\end{aligned}
$$

meaning that for $N \geq 3$, $R < 1$. Since M must be an integer, this brings us to the surprisingly simple, exact result (6) for the total number of paths in a full mesh. ■

We now turn to the probability distribution function (pdf) of the hop count of a path.

Theorem 2 *The pdf*[2] *of the hop count in connected r.g. of $G_p(N)$ is*

$$
P[\text{path} = \text{i hops}] = \frac{\frac{\left(\frac{1}{p}\right)^{N-i-1}}{(N-i-1)!}}{\sum_{k=0}^{N-2} \frac{\left(\frac{1}{p}\right)^k}{k!}} \qquad (p > p_c) \tag{7}
$$

where $p_c > \frac{1}{N-1}$ is a critical threshold

Proof. We observe

$$
\sum_{i=1}^{N-1} P[\text{path} = \text{i hops}] = 1 \tag{8}
$$

where

$$
P[\text{path} = \text{i hops}] = C\, \frac{(N-2)!}{(N-i-1)!}\, p^i \tag{9}
$$

On the assumption that all graphs are connected, the probability that there is no path from 1 to N, $P[\text{path} = 0\,\text{hops}]$, is exactly zero. Notice, however, that this assumption restricts the value of p from below by a critical threshold, i.e. $p > p_c$, where p_c corresponds with the link density leading to disconnectivity in the r.g.. The proportionality factor C follows from (8) as

$$
C = \frac{1}{(N-2)!\, p^{N-1} \sum_{j=0}^{N-2} \frac{p^{-j}}{j!}} \tag{10}
$$

[2]Expression (7) bears resemblance to the famous Erlang B-formula[8, pp. 106],

$$
p_m = \frac{\frac{\rho^m}{m!}}{\sum_{k=0}^{m} \frac{\rho^k}{k!}}
$$

where p_m is the probability that, in a $M/M/m/m$ system with traffic intensity ρ, all m servers are busy.

Using (10) in (9) leads to (7). ■

The proof of theorem 1 is immediate from theorem 2 and the inequality

$$
\sum_{k=0}^{N-2} \frac{\left(\frac{1}{p}\right)^k}{k!} < e^{\frac{1}{p}}
$$