A Scaling Law for the Hopcount in Internet

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Abstract

The discrete probability density function for the hopcount of the shortest path between two arbitrary points in the Internet is studied. Similar to complex physical systems, a stochastic approach to model the Internet is presented. Various topology models are analysed and compared with Internet data obtained via the `trace-route` utility. It was found that topologies with constant link weights hardly fit the data, which suggests that the link weights in Internet vary significantly. One model, random graphs of the class $G_p(N)$ with N nodes, link density p and with exponentially or, equivalently, uniformly distributed link weights is demonstrated to explain the Internet hopcount properties surprisingly well. The resulting generating function of the hopcount h_N

$$E[x^{h_N}] = \sum_{k=1}^{N-1} \Pr[h_N = k]\, x^k \sim \frac{N^{x-1}}{\Gamma(x+1)}(1 + o(1)), \qquad N \to \infty,$$

is invariant in the link density p and seems robust under other minor changes of the model.

1 Introduction.

This article focuses on the behavior of the probability density function (pdf) of the hopcount of the shortest path between two arbitrary points in the Internet. In Internet, routers forward IP packets to the next hop router which is found by routing protocols (such as OSPF) based on a *shortest path* algorithm (e.g. the Dijkstra algorithm). The motivation to consider the shortest path between two *arbitrary* nodes stems from the facts that (a) the IP address does not reflect a precise geographical location and (b) that uniformly distributed world wide communication (especially on the web) seems natural because the information stored in servers can be located in places unexpected and unknown to browsing users. The Internet type of communication is different from classical telephony since (a) telephone numbers have a direct binding with physical location and (b) the intensity of average human interaction rapidly decreases with distance. Finally, the emphasis lies on the *hopcount* because it is simple to measure via the `trace-route` utility, it is an integer, dimensionless and not blurred by

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illusive factors, and the quality of service (QoS) measures (such as packet delay, jitter and packet loss) depend on the number of traversed routers. The recent introduction of QoS and service differentiation in Internet has extended the previously 'connectivity only' forwarding of best effort to end-to-end QoS constraint routing [14, 22]. Especially real-time services such as telephony on Internet become unacceptable if the end-to-end delay exceeds 200 ms. Knowledge of the pdf of the shortest path hopcount enables estimates of end-to-end delay in the Internet [19] and, hence, of the feasibility of offering service differentiation [3] in the current Internet topology.

In addition to QoS aspects, the results also contribute to our understanding of the Internet topological structure. Recently, modeling the Internet topology has received increased attention (see e.g. [27]). Many questions regarding the details of the Internet topology are being posed. Undoubtedly, so far there are more questions than results which is a general theme in the broad field of networks as described in the recent book of Watts [25]. When considering the Internet as a growing organism, similarities with other complex networks appearing in nature evoke more and more interest (see e.g. [2]). Faloutsos *et al.* [7] have suggested that many Internet topological properties obey a power law. Power laws happen to receive much interest due to the seminal work of Mandelbrot [13] and, in communication networks, of Willinger *et al.* who showed that the nature of Internet traffic is self-similar (even multi-fractal) and long range dependent (see for references [26]). In spite of the esoteric fascination for power laws, alternative explanations [24] of the long range dependence as caused entirely deterministically by TCP, caution to draw conclusions too fast. In [17], we demonstrate that the observed power law for the multicast efficiency in Internet, coined the Chuang-Sirbu law, holds only approximately in a limited regime.

We consider random graphs (r.g.) as a model for the Internet. Of course, the Internet topology is *not* random. However, the Internet *is* chaotic and unstructured, and one way to model this chaotic structure is by introducing randomness. This principle first appeared in physics. When considering particles in a gas their behavior is completely deterministic. Unfortunately, it is too complex to be described explicitly. An efficient way to deal with this complexity is to use a stochastic description. We will use the same approach to model the Internet. However, the above still does not explain *how* to model the Internet as a random graph. We will investigate several topology models with different link weights, both constant as well as variable link weights[1]. When computing the hopcount distribution in these models, we test which one fits the available Internet data best. It turns out that the extreme case where all weights are equal does not describe the hopcount distribution well, and we conclude that in the Internet the link weights have considerable variability. We also conclude that the structured or regular graphs do not fit well, and we are lead to random graphs. We finally find one model, random graphs of the class $G_p(N)$ with exponentially or, equivalently, with uniformly distributed link weights, that describes the measurements remarkably well. A further achievement of this article is the precise determination of the limiting generating function $H_N(x)$ of the hopcount h_N of the shortest path between two arbitrary nodes when the number of nodes $N \to \infty$ in that model,

$$H_N(x) = E[x^{h_N}] = \sum_{k=1}^{N-1} \Pr[h_N = k]\, x^k \sim \frac{N^{x-1}}{\Gamma(x+1)}(1 + o(1)), \tag{1}$$

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[1] In Cisco's OSPF implementation, it is suggested to use $w(i \to j) = \frac{10^8}{B(i\to j)}$where $B(i \to j)$ denotes the bandwidth (in bit/s) of the link between nodes i and j.

from which the average hopcount $E[h_N]$, the variance $\mathrm{var}[h_N]$ and the probability that the hopcount of the shortest path consists precisely of k hops follow as

$$E[h_N] = \ln N + \gamma - 1 + o(1) \tag{2}$$

$$var[h_N] = \ln N + \gamma - \frac{\pi^2}{6} + o(1) \tag{3}$$

$$\Pr[h_N = k] = \frac{(1+o(1))}{N} \sum_{m=0}^{k} c_{m+1} \frac{\ln^{k-m} N}{(k-m)!}, \tag{4}$$

where γ is Euler's constant ($\gamma = 0.5772156\ldots$) and c_k are the Taylor coefficients of $\frac{1}{\Gamma(x)}$ listed in [1, 6.1.34]. We argue that this key result is quite universal in that it holds for many classes of random graphs, it is independent of the precise details of the topology (e.g. of the number of links) but it is sensitive to the behavior of the distribution of the link metrics w, $F_w(x) = \Pr[w \leq x]$, for $x \downarrow 0$. Moreover, the limit law (4) agrees remarkably well with Internet measurements. We can conclude that either the limiting distribution (4) is robust under minor changes to the model detailed above, and that the Internet is a minor change to that model, or that the model we have chosen describes the hopcount of the shortest path in Internet well.

First, we present Internet measurements of the hopcount, followed by general modelling assumptions. Based on these Internet measurements, some topology models with constant link weights for which the pdf of the hopcount can be computed analytically are discussed in section 4. Section 5 presents observations and properties of the hopcount of the shortest path in r.g.'s with variable link weights deduced from simulations. These observations are explained in section 6 via a mathematical analysis. The details of the mathematical derivations and proofs of the theorems are found in a companion paper [18]. The limit law (4) is statistically verified in section 7. Finally, we conclude with a brief discussion.

2 Internet Measurements.

Recently, Vanhastel *et al.* [23] have reported measurements of the hopcount (of the shortest path) in the Internet obtained via the well-known `trace-rout` utility [15] and performed medio 1998. They found that the average number of hops was around 18. Also, this number hardly varied over the different regions in the world, which seems to indicate that the Internet topology is quite homogeneous or isotropic and, at the same time, chaotic and unstructured.

Paxson [15] has reported both the average and the standard deviation (sdev) of Internet measurements of the hopcount of the shortest path. In his measurement performed in November-December 1994, $E[h_N] = 15.6$ and $sdev = 4.5$ ($var[h_N] = 20.25$), while in the November-December 1995 measurements, he found $E[h_N] = 16.2$ and also $sdev = 4.5$ ($var[h_N] = 20.25$). However, in these measurements the sample size S was rather small (around 30). From these measurements, the ratio $\frac{E[h_N]}{var[h_N]}$ in the first period is 0.77 and 0.8 in the latter. Data from Demeester *et al.* [23] show that $\frac{E[h_N]}{var[h_N]} = 1.4$ and are plotted in Figure 1 together with our measurements at Delft University of Technology with $\frac{E[h_N]}{var[h_N]}$ around 1. The latter two measurements have chosen routers uniformly spread over the continents which were being reached from Gent (Belgium) and Delft (The Netherlands), respectively. Although measured from different locations and at different times, Figure 1 demonstrates a reasonable agreement.

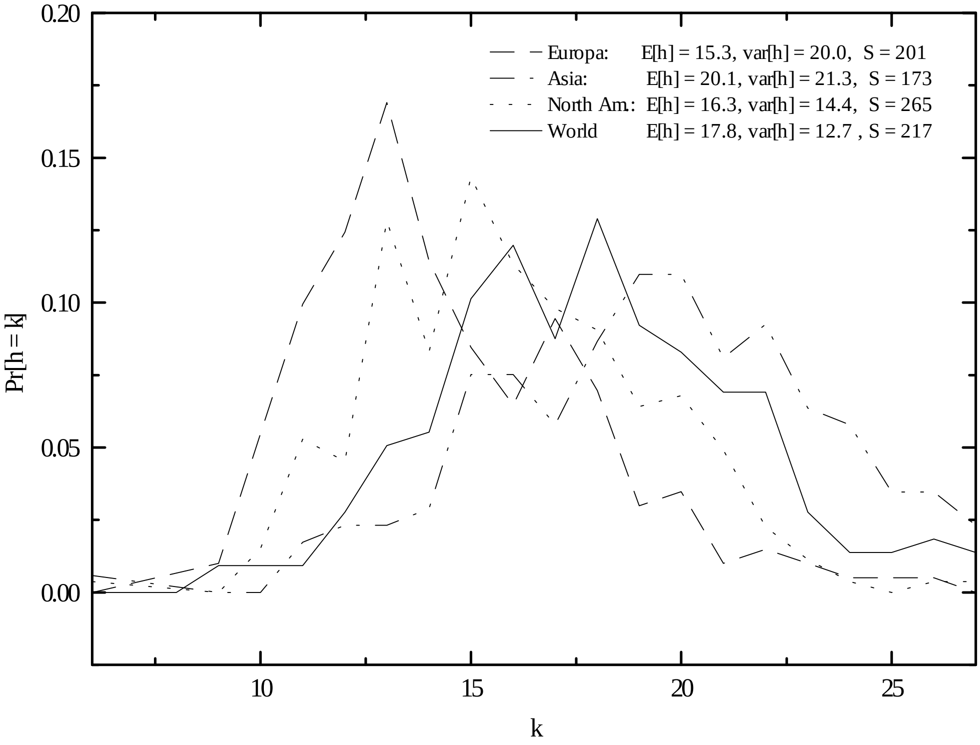

Figure 1: The pdf of the hopcount in Europe, Asia and Northern America as measured at Delft University of Technology (August, 2000), whereas that of the world at the University of Gent (medio 1998). The last column gives the sample size S of each data set.

3 Topology Models.

The interconnection pattern of a network with N nodes (routers in the Internet) can be represented[2] by a $N \times N$ *connectivity matrix* T consisting of elements T_{ij} that are either one or zero depending on whether there is a link or edge between node i and j or not, but for any j, we have that $T_{jj} = 0$.

Apart from the topological structure specified via the connectivity matrix T, the link between node i and j is further characterized by a vector $w(i \to j)$, with m positive components, each reflecting a QoS measure (such as delay, jitter, loss, cost, administrative weight, etc.). Here we confine to one ($m = 1$) additive link metric. Hence, the shortest path from $A \to B$ is the path $P_{A\to B}$ that minimizes $\sum_{i\to j\in P_{A\to B}} w(i \to j)$.

We assume symmetry in both directions, $w(i \to j) = w(j \to i)$ and, hence, also end-to-end symmetry such that the shortest path from $A \to B$ equals that from $B \to A$. Although this assumption seems rather trivial, we point out that in telecommunications, transport of information in up-link and down-link is, in general, not symmetrical. Via Internet measurements, Paxson [15] found that, in

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[2] Other representations are the adjacency list and the link state table (as in OSPF).

1995, about 50% of the paths from $A \rightarrow B$ were different from those from $B \rightarrow A$. Fortunately, as explained in section 5, the limiting behavior of the pdf for $N \rightarrow \infty$ of the hopcount of the shortest path is insensitive to this asymmetry.

Initially in Internet with RIP, all link weights $w(i \rightarrow j) = 1$ and both routing and TTL were using hopcount as metrics. In OSPF today, operators have the freedom[3] to specify the link metric $w(i \rightarrow j) > 0$ on the interfaces of their routers. We argue that it is realistic to consider different weights for different links. Conform to our approach, we assume that the link weights are characterized via a certain distribution $F_w(x) = \Pr[w \leq x]$. The difficulty then lies in determining the precise details of $F_w(x)$ for the Internet.

Apart from the link weight distribution $F_w(x)$, also the topological structure of the Internet is unknown. There are three reasons why hierarchical graphs are not considered, although routing in Internet consists of two-levels, interdomain and intradomain. First, the measurements of Vanhastel *et al.* [23] and ours in Figure 1 show a remarkable degree of homogeneity over the different continents which is difficult to explain with a hierarchical model since the number of service providers (intradomains) is not so uniformly distributed over the world. Second, regarding the hopcount, there is no difference between traversing a border gate way router or a interior domain router: both contribute 1 hop. Therefore, when only counting the hops of the end-to-end shortest path, we cannot distinguish between these two levels. The influence of BGP policy routing can be modeled via the probability that there is a link between two nodes for the packet under investigation. Policy routing somehow enhances the randomness and decouples the physical topology from the actual followed path. The third reason obviously involves the modeling of the hierarchy: Which parameters are enough to determine an Internet hierarchical structure? Even if everybody agrees on the particular hierarchical structure (as e.g. in PNNI), the next question is how to choose the realistic values of the parameters that describe that hierarchy. Even for the well-defined PNNI hierarchy, node and link aggregation quantifiers (see e.g. [20]) remain very difficult to extract from measurements.

Therefore, we will consider *homogeneous* topology models for which the hopcount can be computed. We will start analyzing different topology models all with constant link weights equal to 1. Comparing the resulting hopcount behavior with Internet measurements suggests to investigate models with variable link weights.

4 Constant Link Weights.

In this section, we consider several topological structures in which each link has unit weight. In these cases, every shortest path between two arbitrary chosen nodes A and B also minimizes the hopcount. We emphasize that the considered topologies are in general not good models for the Internet topology. Rather, the purpose of the analysis is to illustrate the highly unlikeliness of constant link weights.

4.1 A d-lattice.

An extremely regular graph is a d-lattice where each nodal position corresponds to a point with integer coordinates within a d dimensional hyper-cube with size Z. Apart from border nodes, each node has

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[3] An approach to optimize the OSPF weights to reflect actual traffic loads is presented by Fortz and Thorup [9].

a constant degree (number of neighbors), precisely equal to $2d$. Assuming that all link metrics are equal to one, the probability distribution of the hopcount of the shortest path between two uniformly chosen points is readily computed. Indeed, in one dimension, the hopcount of the shortest path between two uniformly chosen points x_A and x_B equals the distance between x_A and x_B. Thus, $\Pr[|x_A - x_B| = k] = \frac{1}{Z}1_{k=0} + \frac{2(Z-k)}{Z^2}1_{1\leq k\leq Z-1}$ with corresponding generating function

$$\varphi(x) = \sum_{k=0}^{Z-1} \Pr[|x_A - x_B| = k]x^k = \frac{Z - Zx^2 + 2x(x^Z - 1)}{Z^2(x-1)^2}.$$

Since the nodes are uniformly chosen, all coordinate dimensions are independent and the generating function of the hopcount of the shortest path in a d-lattice is $\varphi^d(x)$. Since $E[h_N] = (\varphi^d)'(1)$ and $var[h_N] = (\varphi^d)''(1) + (\varphi^d)'(1) - [(\varphi^d)'(1)]^2$, the average number of hops is immediate as $E[h_N] = \frac{d}{3Z}(Z^2-1)$ and the variance as $var[h_N] = \frac{d(Z^2-1)(Z^2+2)}{18Z^2}$, while $\frac{E[h_N]}{var[h_N]} = \frac{6Z}{Z^2+2}$. The total number of nodes in the d-lattice is $N = Z^d$ such that, for large N, we obtain

$$E[h_N] \simeq \frac{d}{3}N^{1/d}, \tag{5}$$

and

$$var[h_N] \simeq \frac{d}{18}N^{2/d}, \tag{6}$$

both increasing in $d > 1$ (for constant N) as in N (for constant d). However $\frac{E[h_N]}{var[h_N]} \simeq 6N^{-1/d}$ is decreasing and tends for large N to zero (for finite d), which contradicts Internet measurements.

4.2 The k-ary Tree.

Let us consider the k-ary tree of depth[4] D with the node A at the root (level 0) of the tree and the node $B \neq A$ randomly in the k-ary tree. The k-ary tree may be a model for the shortest path tree of node A. All link weights are equal to one. Since every node at level $j < D$ has precisely k-children nodes at level $j+1$, the total number of nodes satisfies

$$N = 1 + k + k^2 + \ldots + k^D = \frac{k^{D+1}-1}{k-1}, \tag{7}$$

The probability of the hopcount of the shortest path between A and B equals $\Pr[h_N = j] = \frac{k^j}{N-1}$, for $1 \leq j \leq D$, and the corresponding generating function is $\varphi_k(x) = E\left[x^{h_N}\right] = \frac{kx}{N-1}\frac{(kx)^D-1}{kx-1}$. After appropriate differentiation of $\varphi_k(x)$ we obtain

$$\begin{aligned} E\left[h_N\right] &= \frac{ND}{N-1} - \frac{1}{k-1} + \frac{D}{(N-1)(k-1)}, \\ var\left[h_N\right] &= \frac{k}{(k-1)^2} - \frac{D^2N}{(k-1)(N-1)^2}\frac{D^2}{(k-1)^2(N-1)^2}. \end{aligned}$$

For large N, we find with $D = \left\lfloor \frac{\log[1+N(k-1)]}{\log k} - 1 \right\rfloor \sim \log_k N + \log_k(1-1/k) + O(1/N)$, that

$$\begin{aligned} E[h_N] &= \log_k N + \log_k(1-1/k) - \frac{1}{k-1} + O\left(\frac{\log_k N}{N}\right) \\ var\left[h_N\right] &= \frac{k}{(k-1)^2} + O\left(\frac{\log_k^2 N}{N}\right) \end{aligned}$$

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[4] The depth D is equal to the number of hops from the root to a node at the leaves.

Hence, for hopcount of the shortest path in the k-ary tree we find that, for large N, $var\left[h_N\right]$ is almost constant and $\frac{E[h_N]}{var[h_N]} \simeq \frac{(k-1)^2}{k}\log_k N$, which grows unbounded, contrary to Internet measurements.

4.3 Random Graphs

There exists an astonishingly large amount of literature on properties of random graphs (r.g.'s). We refer to the book of Bollobas [4] for an excellent discussion, the work of Janson *et al.* [11] on evolutionary processes in r.g.'s and the survey paper on recursive trees by Smythe and Mahmoud [16]. The two most frequently occurring models for r.g.'s are $G_p(N)$ and $G(N,E)$. The class of r.g.'s denoted by $G_p(N)$ consists of all graphs with N nodes in which the edges (or links) are chosen independently and with probability p. A natural refinement of $G_p(N)$ is the model $G_{\{p_{ij}\}}(N)$ where the edges are still chosen independently but where the probability of $i \rightarrow j$ being an edge is exactly p_{ij}. The Waxman graph, investigated in section 5.3, is an example of $G_{\{p_{ij}\}}(N)$. The class $G(N,E)$ constitutes the set of graphs with N nodes and E edges. Bollobas [4, pp. 37] remarks that *'the nearer the r.g. are to being independent, the easier it is to handle them. This is the reason why $G_p(N)$ is more pleasant to work with than $G(N,E)$. In $G_p(N)$ the edges are chosen independently, while in the model $G(N,E)$ the choice of an edge does have a (fortunately small) effect on the choice of another edge'.* In the class $G_p(N)$ the number of links is not deterministic, but known on average as $p\,E_{max}$ where the maximum number of links E_{max} in a (bi-directional) topology with N nodes is $E_{max} = \frac{N(N-1)}{2} \equiv \binom{N}{2}$. This situation is coined a *full mesh* and $G_1(N) = G(N, E_{max})$ is called the complete graph K_N. In the class $G(N,E)$, the number of different network topologies with N nodes and E links we can construct equals $\binom{E_{max}}{E}$ which corresponds to the number of ways we can distribute a set of E ones in the E_{max} possible places in the upper triangular part above the diagonal of the matrix T. In constructing a r.g. in $G(N,E)$, each of the possible E_{max} links has equal probability. Furthermore, there are precisely E links equivalent to E non-zero elements in the upper triangle of T. The probability that $T_{ij} = 1$ equals $p = \frac{E}{E_{max}}$ and conversely, the probability that $T_{ij} = 0$ equals $q = 1 - p$. In the sequel, we merely consider the class $G_p(N)$ because, for large N, the total number of links E is close to $\frac{pN^2}{2}$ and $G_p(N)$ is almost in $G(N, \frac{pN^2}{2})$.

From the point of view of telecommunication networks, by far the most interesting graphs are those with connected topology. This limitation restricts the value of p from below by a critical threshold, i.e. $p > p_c$, where, for large N, $p_c \sim \frac{\ln N}{N}$ corresponds to the link density leading to disconnectivity in the r.g.'s. Connectedness of r.g.'s has received considerable attention in the past [4, chapt. 7]. The properties of $G_p(N)$ change dramatically below the connectivity threshold $p_c \sim \frac{\ln N}{N}$. The drawback of the connectivity requirement in $G_p(N)$ is that the minimal nodal degree is $p_c N \sim \ln N$, which means that the average number of neighboring nodes becomes unbounded as $N \rightarrow \infty$. Fortunately, the tendency to this limit is slow: if $N \simeq 10^6$, the average number of neighbors is $p_c N \simeq \ln N = 13.8$.

4.3.1 The random graph with constant weights

The average degree per node in the Internet, here denoted by λ, is a finite number (around 3), which implies, for large N, that $p = \frac{\lambda}{N}$ with $\lambda > 1$. In that case, there is a positive probability that a given node is not connected to the largest cluster. This largest cluster is of the order N, and we therefore confine to the situation that both the nodes A and B are in this largest cluster. We claim that the number of nodes that can be reached in n steps from either of these nodes A or B grows like $W_A\lambda^n$, respectively, $W_B\lambda^n$, as long as λ^n is much smaller than N. From the theory of branching processes [10, p. 150], it follows, for N large, that W_A, W_B are independent identically distributed random variables with mean almost equal to 1. Indeed, we can view the number of edges connected to each node as the offspring in a branching process $\{Z_k\}_{k\geq 0}$. As long as the clusters remain of order smaller than N, that number of edges has a binomial distribution with the parameters, in leading order, N and p. For Np^2 small, the binomial distribution can be approximated by the Poisson distribution with mean $\lambda = N \cdot p$. From the martingale convergence theorem [10, p. 309], more specifically from the example on page 311 of this reference, it follows that $W_n = \frac{Z_n}{E[Z_n]}$, where Z_n is the number of individuals in the n-th generation, converges almost surely to a limit W. In our example the number of individuals in the n-th generation equals the number of nodes that can be reached in n steps and $E\left[Z_n\right] \approx \lambda^n$. This motivates the claim. The generating function of the offspring distribution (or rather of the approximating Poisson(λ) distribution) is $\varphi(s) = \sum_{k=0}^{\infty} s^k e^{-\lambda}\frac{\lambda^k}{k!} = \exp\{-\lambda(1-s)\}$. We condition the nodes to be in the largest cluster, which effectively conditions on $W_A, W_B > 0$. The probability that $W_A = 0$, which is called the *extinction probability* p_∞, is, for large N, close to the smallest non-negative root value of the equation $s = \varphi(s)$ [10, p. 153] and, hence, p_∞ is close to $e^{-\lambda}$ for λ large, so that the probability that $W_A > 0$ is approximately $1 - e^{-\lambda}$.

Denote by $C_A(l)$, respectively $C_B(l)$, the set of points that can be reached from A, respectively B, in l or less steps. We know that the hopcount is larger than $2l$ precisely when $C_A(l) \cap C_B(l)$ is empty. Conditionally on $|C_A(l)| = C_A$, respectively $|C_B(l)| = C_B$, this happens with probability

$$\begin{aligned}
\frac{\binom{N-1}{C_A}\binom{N-C_A-1}{C_B}}{\binom{N-1}{C_A}\binom{N-1}{C_B}} &= \frac{(N-C_A-1)(N-C_A-2)\ldots(N-C_A-C_B)}{(N-1)(N-2)\ldots(N-C_B)} \\
&= \frac{(1-\frac{C_A+1}{N})(1-\frac{C_A+2}{N})\ldots(1-\frac{C_A+C_B}{N})}{(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{C_B}{N})}.
\end{aligned}$$

For N large, approximating $1 - \frac{i}{N}$ by $e^{-\frac{i}{N}}$ leads to

$$\Pr\left[h_N > 2l \middle| |C_A(l)| = C_A, |C_B(l)| = C_B\right] \approx \frac{e^{-\sum_{i=1}^{C_B}(C_A+i)/N}}{e^{-\sum_{i=1}^{C_B} i/N}} = e^{-C_A C_B/N}.$$

Hence, we obtain that

$$\Pr[h_N > 2l] \approx E\left[e^{-\frac{|C_A(l)||C_B(l)|}{N}} \middle| A, B \text{ are both in the largest cluster}\right].$$

Noting that $|C_A(l)| \sim W_A \sum_{k=0}^{l} \lambda^k \sim W_A\lambda^{l+1}/(\lambda-1)$, and similar for $C_B(l)$, we have

$$\Pr[h_N > 2l] \approx E\left[e^{-\frac{W_A W_B \lambda^{2l+2}}{N(\lambda-1)^2}} \middle| W_A, W_B > 0\right].$$

Since the same asymptotics also holds for odd values of the hopcount, we finally arrive, for $k \geq 1$, at

$$\Pr[h_N > k] \sim E\left[e^{-\frac{W_A W_B \lambda^{k+2}}{N(\lambda-1)^2}} \middle| W_A, W_B > 0\right].$$

Now consider a discrete random variable Y_N with support $\{1, 2, \ldots, N-1\}$ and survival probabilities

$$\Pr[Y_N > k] = e^{-c\frac{\lambda^k}{N}}, \qquad k = 1, 2, \ldots, N-2$$

The mean value of this random variable equals

$$E[Y_N] = \sum_{k=1}^{N-1} k \Pr[Y_N = k] = \sum_{k=0}^{N-2} \Pr[Y_N > k] = \sum_{k=0}^{N-2} e^{-c\frac{\lambda^k}{N}}.$$

The main contribution of the sum stems from values of k, where $\lambda^k \leq (N/c)$. Indeed, when k runs through the values 0 up to the largest integer with $\lambda^k \leq (N/c)$, for large N, the tail probability $e^{-c\frac{\lambda^k}{N}}$ decreases from 1 to the value e^{-1}; when λ^k is significantly larger than N/c (and since $\lambda > 1$, we only need a finite number of terms beyond $k = [\log_\lambda(N/c)]$ to reach this goal), the tail probability $e^{-c\frac{\lambda^k}{N}}$ is almost 0. This explains that the typical value of $E[Y_N]$ is such that $\lambda^k = N/c$, i.e., $k \approx \log_\lambda(N/c)$. For this value of k, because of the connection $k = 2l$, both $|C_A(l)|$ and $|C_B(l)|$ are of the order $W_A\lambda^{l+1}/(\lambda - 1) = O(\sqrt{N}) = o(N)$ and the initial approximation $1 - \frac{C_A+C_B}{N} \approx \exp\{-\frac{C_A+C_B}{N}\}$ is accurate. Hence, we have shown heuristically that

$$E[h_N] \approx \log_\lambda(N) - E\left[\log_\lambda\left(\frac{W_A W_B \lambda^2}{(\lambda-1)^2}\right)\right] \geq \log_\lambda(N) - 2\log_\lambda\left(\frac{\lambda}{(\lambda-1)}\right),$$

where we have used Jensen's inequality and $E[W_A] \approx E[W_B] \approx 1$. For the variance of Y_N we take $c = 1$, because we can rescale N by the transformation $N \mapsto N/c$. Similar to the expression for $E[Y_N]$ we find

$$var[Y_N] = \sum_{k=0}^{N-2} ((2k+1) - E[Y_N]) \Pr[Y_N > k] \approx \sum_{k=0}^{N-2} (2k + 1 - \log_\lambda(N))\, e^{-\frac{\lambda^k}{N}}.$$

The terms with $k \geq \log_\lambda(N)$ only contribute a number independent of N. The terms with $k < (\log_\lambda N)/2$ give a negative contribution which is in absolute value larger than the contribution from the terms with $(\log_\lambda N)/2 < k \leq \log_\lambda N$, because $e^{-\frac{\lambda^k}{N}}$ is decreasing. Hence $var[Y_N]$ remains bounded when N increases and this indicates that the same is true for $var[h_N]$. Hence, the ratio $\frac{E[h_N]}{var[h_N]}$ increases unboundedly when N increases.

The above considerations hold for fixed λ and $p = \frac{\lambda}{N}$. At the other end of the scale consider the case where $p = \frac{c}{\sqrt{N}}$ which corresponds to $\lambda = c \cdot \sqrt{N}$, and the above analysis no longer applies for such large values of λ. Fortunately, in that case, an exact asymptotic analysis is possible. It is not hard to show that $E[h_N] \simeq 2 - p$ and $var[h_N] \simeq p(1-p)$. In fact h_N is either 1 or 2 with probability p or $(1-p)\left(1 - (1-p^2)^{N-2}\right)$ respectively, and higher values of h_N are extremely unlikely for $p \sim \frac{1}{\sqrt{N}}$ since $\Pr[h_N > 2] = (1-p)\left[1-p^2\right]^{N-2}$ tends to zero. These results agree with those for fixed λ since, for $\lambda = c \cdot \sqrt{N}$, $E[h_N] \approx \frac{\ln N}{\ln \lambda} = \frac{\ln N}{\ln c + \frac{1}{2}\ln N} \to 2$, as $N \to \infty$ and the variance is bounded. Intuitively, for larger λ, the number of edges in first approximation is equal to Np and we can compare the model with the k-ary tree with the parameter k equal to the binomial mean $Np = \lambda$. If we extrapolate the results of Section 4.2 for the k-ary tree, then we obtain

$$E[h_N] \approx \log_k N = \log_\lambda N; \qquad var[h_N] \approx \frac{k}{(k-1)^2} \approx \frac{1}{\lambda}.$$

Computer simulations with $N = 10,000$ and various values of λ confirm the above sketched picture: in almost all simulations the hopcount equals $[\log_\lambda(N)] \pm 1$, indicating that the hopcount is almost deterministic. The above considerations lead inevitably to the conclusion that r.g.'s of the class $G_p(N)$ with constant weights do not fit Internet measurements for the shortest path.

4.4 Conclusion.

Although the analysis above is not exclusive nor decisive, it nevertheless suggests that graphs with constant link weights are likely to possess different properties than those deduced from Internet measurements. In the sequel, therefore, we will investigate the influence of variations in the link weights.

5 The Shortest Path between A and B: Observations.

In this section, we present - not always intuitively expected - observations from simulations, which will be explained in section 6 based on a rigorous mathematical analysis. The precise modeling assumptions are:

1. We confine ourselves to r.g.'s of the class $G_p(N)$.
2. Each node in the r.g. in $G_p(N)$ has on average $E[L] = p(N-1)$ links with other nodes and the variance of the number of links L per node $var[L] = (N-1)p(1-p)$ is small compared to the square of the average, i.e. $var[L] << (E[L])^2$. For the r.g., this condition means that $pN >> 1$ and it is needed for the coupling argument in the proof (see sec. 6.3).
3. The weight $w(i \to j)$ of the link between node i and node j is (a) independent of other links and (b) described by the same distribution function $F_w(x) = \Pr[w \leq x]$.

For bi-directional links where $w(i \to j)$ and $w(j \to i)$ are identical and independent, the shortest path tree from $A \to B$ and that from $B \to A$ are identical in distribution. This argument explains why we further confine to the symmetrical case.

5.1 Simulations in the class $G_p(N)$.

We will start presenting results obtained for the class $G_p(N)$ with uniformly $[0,1]$ distributed link metrics w, i.e. $F_w(x) = \Pr[w \leq x] = x$, for $0 \leq x \leq 1$. Clearly, the hopcount h_N of the shortest path is independent from a link weight scaling factor and the confinement to the interval $[0,1]$ does not affect the generality. Using Prim's minimum spanning tree algorithm [5], every r.g. is first tested to assure that the r.g. is connected. Disconnected r.g.'s are removed and a total of one million connected r.g.'s have been constructed for each simulation. In each r.g. of the class $G_p(N)$, the shortest path between node 1 and N has been computed with Dijkstra's algorithm. The number of hops of this shortest path has been stored in a histogram, from which the pdf of the hopcount h_N is deduced. The special interest in this article lies in the behavior of the pdf of the hopcount h_N of the shortest path as N increases. For that purpose, the mean $E[h_N]$ and variance $var[h_N]$ are plotted in Figure 2, for a same value of $p = 0.8$, as a function of N.

5.2 Insensitivity of the link density p for large N.

The dependency on the link density p becomes vanishingly small as N grows as illustrated in Figure 3. We observe that the curves for $p = 0.2$ and $p = 0.8$ computed for a same number of nodes N tend to each other when N increases. This means that, for sufficiently large graphs (in practice for $N > 50$

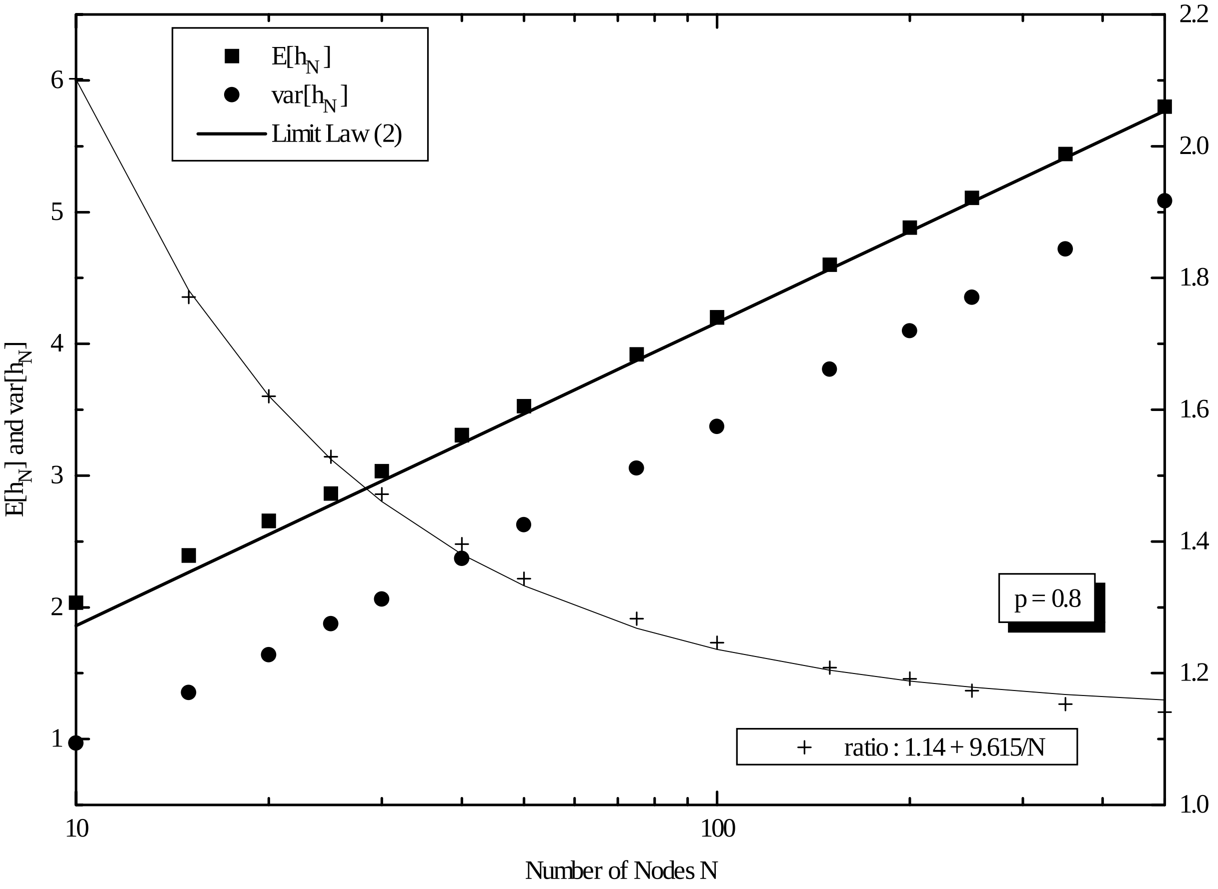


Figure 2: The mean $E[h_N]$ and the variance $var[h_N]$ as a function of N deduced from $G_{0.8}(N)$ with uniformly distributed link metrics w. The bold line is the theoretical limit law.

in $G_p(N)$), the behavior of the hopcount of the shortest path only depends on the number of nodes N in the topology of the class $G_p(N)$. In other words, as long as $p > 0$ is constant, the precise value of p is not important in the limit $N \to \infty$. This phenomenon is explained in section 6.1.

5.3 The precise details of the topology of the graph are not important.

Waxman graphs are believed to be better representatives of telecommunication networks than r.g.'s of the class $G_p(N)$. The Waxman graph belongs to the family $G_{p_{ij}}(N)$ with $p_{ij} = f(\vec{r_i} - \vec{r_j})$, where the vector $\vec{r_i}$ represents the position of a node i and all nodes are uniformly distributed in a hyper-cube of size Z in the m-dimensional space. The dependence on distance $f(\vec{r})$ is a positive, real function of the m coordinates of the vector $\vec{r}$. For example, the Waxman graphs are specified by $f(\vec{r}) = e^{-\alpha|\vec{r}|}$, where $|\vec{r}|$ is a norm, denoting a distance from the origin. The idea of relating the probability of a link between node i and j to some function of the distance between those nodes stems from the correspondence with realistic telecommunications networks. The farther two nodes lie separated, the smaller the need for a direct link between them.

From the definition of the link density $p = \frac{E}{E_{\max}}$, where E denotes the average number of links in

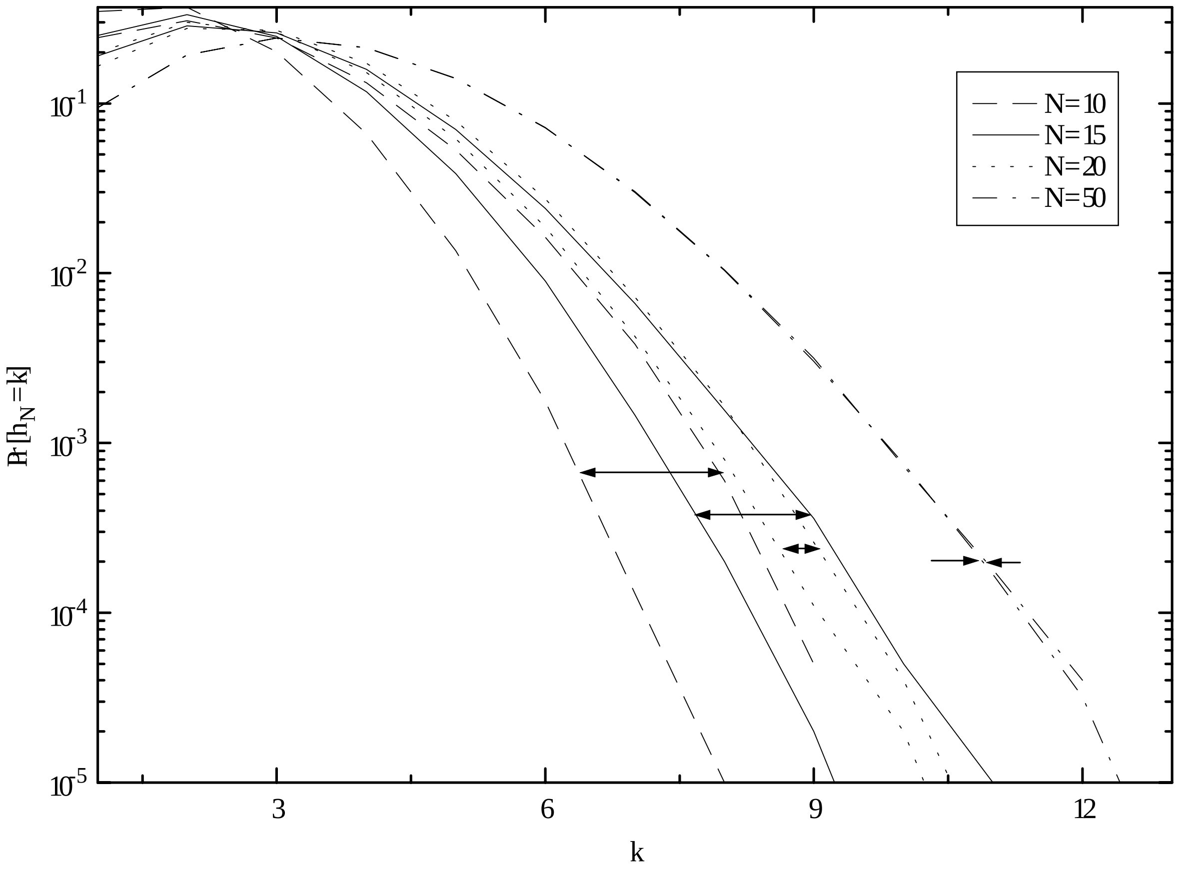


Figure 3: For various values of N, the pdf of the hop count of the shortest path is drawn for $p = 0.2$ (upper curve) and $p = 0.8$ (lower curve). The link metric w is uniformly distributed in $[0, 1]$.

the Waxman graph, the decay rate $a = \alpha Z$ of the existence of the link is expressed [21] uniquely in terms of p as, for $a > 0$,

$$p(a) = \frac{2}{a^4}\left[6\left(1 - 2\,e^{-a} + e^{-\sqrt{2}a}\right) + 2a\left(-4 - 2\,e^{-a} + 3\sqrt{2}\,e^{-\sqrt{2}a}\right) + a^2\left(4e^{-\sqrt{2}a} + \pi\right)\right] + \frac{8\,g_1(a)}{a} + \frac{8\,g_2(a)}{a^2}, \tag{8}$$

with, of course, $p(0) = 1$ and where

$$g_1(y) = \frac{dg_2(y)}{dy} = \int_1^{\sqrt{2}} e^{-yx}\sqrt{x^2 - 1}\,dx \tag{9}$$

$$g_2(y) = \int_1^{\sqrt{2}} e^{-yx}\sqrt{1 - 1/x^2}\,dx \tag{10}$$

Relation (8) shows that the link density $p(a)$ is only a function of the decay rate a (and not of other parameters as Z). Zegura *et al.* [27] have considered $p_{ij} = \eta\exp(-|\vec{r_i} - \vec{r_j}|/\beta L)$ and were led to the same conclusion concerning $p(a)$ via extensive simulations.

In Figure 4, the pdf of the hopcount of the shortest path is drawn together with the corresponding pdfs in $G_p(10)$. Surprisingly perhaps, for identical link densities as computed via (8), there does not seem to be any significant difference between the shortest path behavior in $G_p(N)$ and the Waxman

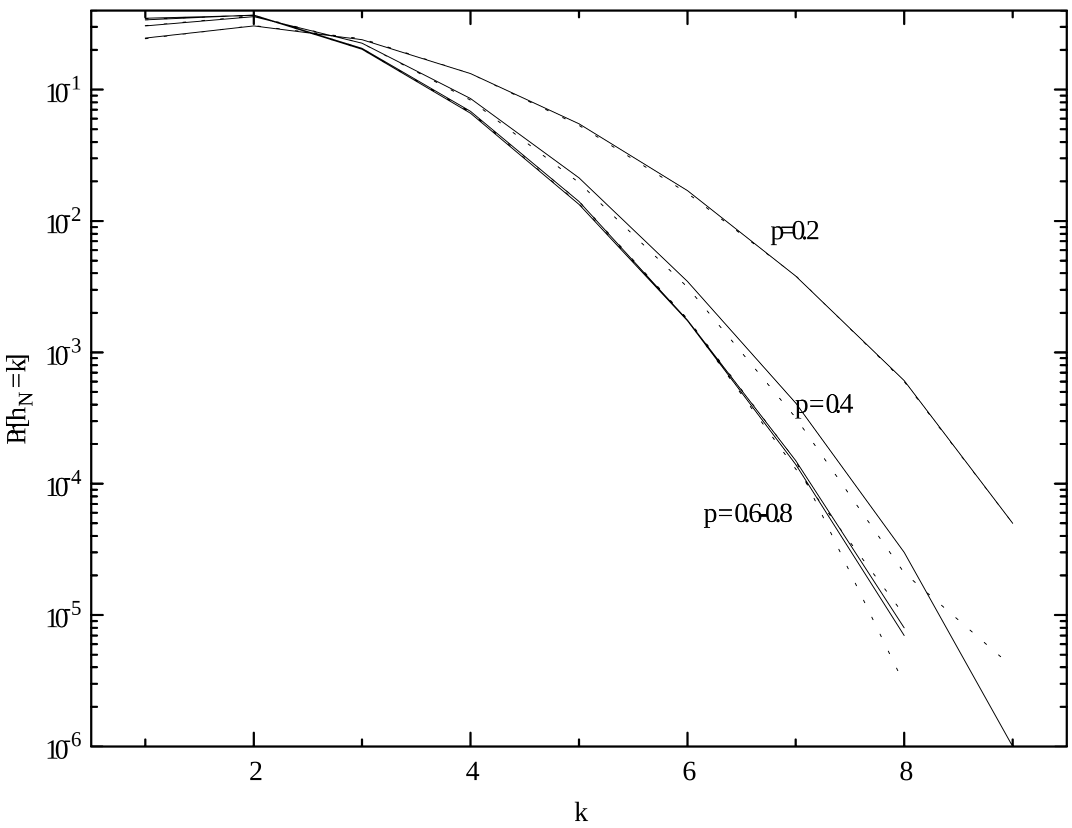
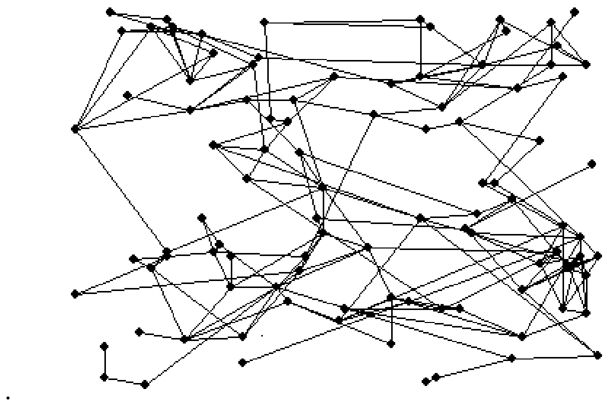
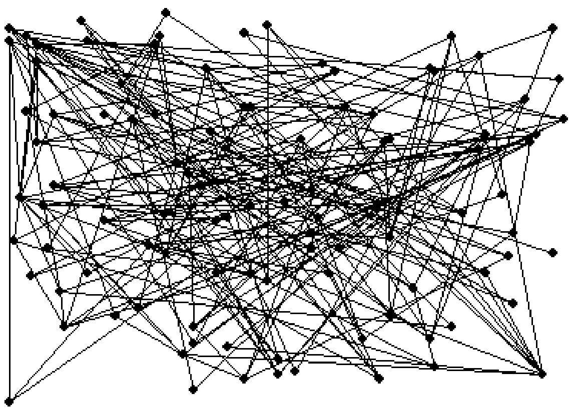


Figure 4: The pdf of the shortest path in a Waxman graphs with $N = 10$ nodes and various link densities $p = \{0.2, 0.4, 0.6, 0.8\}$. In dotted line, we have also added the corresponding pdf's of $G_p(10)$.

graph, even not for small network sizes as $N = 10$, in spite of their definitely different topologies as illustrated below. This result indicates that the details of the topological structure of the random graph become obsolete when using random link measures.



A Waxman graph ($N = 100$, $a = 11$, $p = 0.04$).



The graph $G_{0.04}(100)$.

5.4 The precise details of $F_w(x)$ are important.

Apart from uniformly distributed link metrics w, other types such as the exponential $F_w(x) = 1 - e^{-x}$, with $x \geq 0$ and polynomial distribution functions $F_w(x) = x^\alpha$ with $x \in [0, 1]$ and $0 < \alpha < \infty$, have been considered. The uniform distribution is a special case of a polynomial distribution with $\alpha = 1$, whereas the exponential distribution has the same limit for $x \downarrow 0$ as the uniform distribution. Note that for $\alpha \neq 1$, $\lim_{x \downarrow 0} F'_w(x) = 0$ for $\alpha > 1$, while $\lim_{x \downarrow 0} F'_w(x) = \infty$ for $\alpha < 1$. The parameter α is known as the extreme value index of the probability distribution of w. Figure 5 shows that the ratio $\frac{E[h_N]}{var[h_N]} \to \alpha$ for $N \to \infty$, which will be explained in section 6.1. Figure 5 further illustrates that a uniform ($\alpha = 1$) and exponential distribution for w lead to nearly identical results, especially for large N. All other polynomial distributions with exponents $\alpha \neq 1$, lead to distinctly different behavior for the hopcount h_N.

The dependence of the hopcount on α is explained intuitively as follows. The shortest path is most sensitive to small values of w. The more mass the pdf of the distribution of w has around zero (i.e. the smaller α), the larger we may expect the hopcount of the shortest path because the shortest length is hardly changed by including an additional hop. On the other hand, for large α, most of the link weights have a value around 1 and the shortest path changes roughly by one if an additional hop is included. Hence, the affinity to include many hops is low.

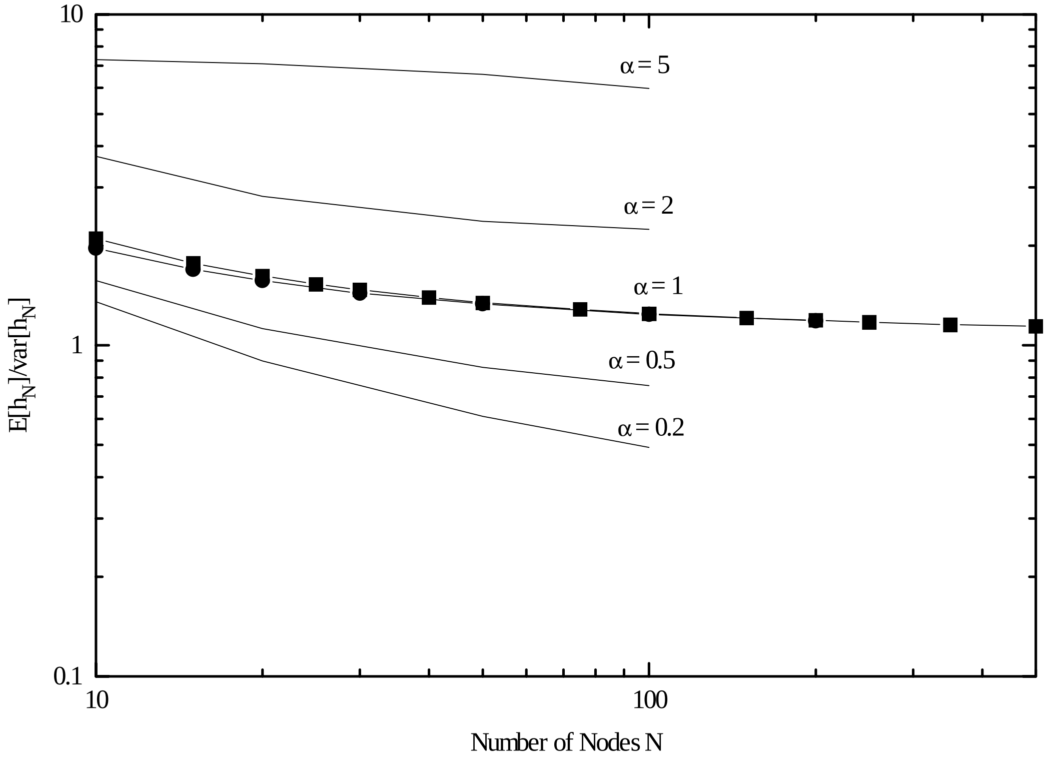


Figure 5: The ratio $\frac{E[h_N]}{var[h_N]}$ versus number of nodes N (logarithmic scale) for various α

More rigorously, the dependence on α is reflected through the distribution of the minimum of the link weights. Since the link weights w_j are independent with distribution $F_w(x) = x^\alpha$ with $x \in [0, 1]$, at each hop, the minimum link weight has distribution

$$\Pr[\min(w_1, \cdots, w_n) > x] = \Pr[w_1 > x, \cdots, w_n > x] = (\Pr[w > x])^n = (1 - x^\alpha)^n .$$

Hence, for large n,

$$\Pr\left[\min(w_1, \cdots, w_n) > \frac{x}{n^{1/\alpha}}\right] = \left(1 - \frac{x^\alpha}{n}\right)^n \to e^{-x^\alpha}$$

is a limit distribution with extreme value index α. Thus, a typical minimum link weight is of order $n^{-1/\alpha}$ which tends to 1 if $\alpha \to \infty$, and to 0 if $\alpha \to 0$. In case Cisco's suggestion to use OSPF link weights inverse equal to $\frac{10^8}{BW}$, where BW denotes the bandwidth of the link in bit/s, is followed by a majority of network administrators, the relation

$$\Pr\left[BW > x\right] = \Pr\left[\frac{1}{BW} \leq \frac{1}{x}\right] = \Pr\left[w \leq \frac{10^8}{x}\right] = \left(\frac{10^8}{x}\right)^\alpha$$

indicates that the bandwidth distribution in Internet follows a power law with exponent α close to 1.

6 The shortest path between A and B: theory.

The analysis that leads to the key results derived from (1) is presented. First, the observed independence on p is discussed via an order calculus where polynomially distributed link weights are treated. We then concentrate on the particular and most easy case of the complete graph K_N, where $p = 1$ and where all weights or link metrics are exponentially distributed with parameter 1. Finally, the general case for the link density $p = p_N = f(N)$ is outlined. As mentioned before, the rigorous mathematical proofs can be found in [18].

6.1 Order calculus

In this section we consider the random graph $G_p(N)$, with weights chosen independently according to the distribution function

$$F_w(x) = x^\alpha 1_{[0,1]}(x) + 1_{[1,\infty)}(x), \quad \alpha > 0, \tag{11}$$

with corresponding density $f_w(x) = \alpha x^{\alpha-1}$, $0 < x < 1$. We will give a heuristic argument that the hopcount h_N satisfies the central limit theorem:

$$\lim_{N\to\infty} \Pr\left[\frac{h_N - (\ln N)/\alpha}{\sqrt{(\ln N)/\alpha^2}} \leq x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \tag{12}$$

which indicates in particular that for N large

$$E[h_N] \sim \frac{\ln N}{\alpha}, \tag{13}$$

$$var[h_N] \sim \frac{\ln N}{\alpha^2}, \tag{14}$$

and that the hopcount is (at least asymptotically) independent of the link density p.

For this argument we introduce some notation: we fix the nodes A and B, and denote by Γ_l the set of all paths in $G_p(N)$ from A to B, with l links or hops. A particular path of the set Γ_l is denoted by γ_l. The probability that an arbitrary path $\gamma_l \in \Gamma_l$ is present in $G_p(N)$ is p^l, by independence. We denote the weight of γ by $w(\gamma)$. Let w_N be the (random) weight of the shortest path between A and B in $G_p(N)$ with i.i.d. link weights chosen according to (11). We will focus on the event

$$\{h_N \le k, w_N \le z\}.$$

This event implies that there is a path $\gamma_l \in G_p(N)$, $l \le k$, with weight $w(\gamma_l) \le z$ and, therefore,

$$\Pr[h_N \le k, w_N \le z] \le \Pr[\cup_{l=1}^{k} \cup_{\gamma\in\Gamma_l} \{w(\gamma) \le z\}] \le \sum_{l=1}^{k} \Pr[\cup_{\gamma\in\Gamma_l} w(\gamma) \le z], \tag{15}$$

where the second inequality follows from Boole's inequality ($\Pr[\cup A_j] \le \sum \Pr[A_j]$). Again using Boole's inequality and the independence of the link and the link weights,

$$\begin{aligned}\sum_{l=1}^{k} \Pr[\cup_{\gamma\in\Gamma_l} w(\gamma) \le z] &\le \sum_{l=1}^{k} \sum_{all\ \gamma} \Pr[\gamma \in \Gamma_l, w(\gamma) \le z] \\ &= \sum_{l=1}^{k} \sum_{all\ \gamma} \Pr[\gamma \in \Gamma_l] \Pr[w(\gamma) \le z] = \sum_{l=1}^{k} E[|\Gamma_l|] \Pr[w(\gamma_l) \le z]\end{aligned}$$

The average number of paths from A to B with l hops [19] equals $E[|\Gamma_l|] = \binom{N-2}{N-l-1} p^l$. With the inequality $\binom{N-2}{N-l-1} \le N^{l-1}$, we obtain

$$\sum_{l=1}^{k} \Pr[\cup_{\gamma\in\Gamma_l} w(\gamma) \le z] \le \sum_{l=1}^{k} N^{l-1} p^l \Pr[w(\gamma_l) \le z]\,. \tag{16}$$

Combining (15) and (16) leads to

$$\Pr[h_N \le k, w_N \le z] \le \sum_{l=1}^{k} N^{l-1} p^l \Pr[w(\gamma_l) \le z].$$

From this rigorous inequality we infer the heuristic statement

$$\Pr[h_N = k, w_N \le z] \simeq N^{k-1} p^k F_w^{k*}(z),$$

where the distribution function $F_w^{k*}(z)$ is the probability that a sum of k independent random variables each with d.f. F_w is at most z and is given by the k-fold convolution:

$$F_w^{k*}(z) = \int_0^z F_w^{(k-1)*}(z-y) f_w(y)\, dy, \quad k \ge 2,$$

and where $F_w^{1*} = F_w$. By induction it readily follows from (11), that for $z \downarrow 0$,

$$F_w^{k*}(z) \sim \frac{z^{\alpha k} (\alpha\Gamma(\alpha))^k}{\Gamma(\alpha k + 1)}.$$

Hence

$$\Pr\left[h_N = k, w_N \le z\right] \simeq N^{k-1} p^k \frac{z^{\alpha k} (\alpha\Gamma(\alpha))^k}{\Gamma(\alpha k + 1)}. \tag{17}$$

For a typical value of z, the probabilities in (17) should sum to 1, yielding:

$$\begin{aligned}
1 &= \frac{1}{N}\sum_{k=1}^{N-1}\frac{(\alpha Np\,\Gamma(\alpha)z^{\alpha})^{k}}{\Gamma(\alpha k+1)} \simeq \frac{1}{N}\int_0^\infty \frac{(\alpha Np\,\Gamma(\alpha)z^{\alpha})^{x}}{\Gamma(\alpha x+1)}\,dx \\
&= \frac{1}{\alpha N}\int_0^\infty \frac{(\alpha Np\,\Gamma(\alpha)z^{\alpha})^{u/\alpha}}{\Gamma(u+1)}\,du \simeq \frac{1}{\alpha N}\sum_{k=0}^{\infty}\frac{(\alpha Np\,\Gamma(\alpha)z^{\alpha})^{k/\alpha}}{\Gamma(k+1)} \\
&= \frac{1}{\alpha N}\exp\{(\alpha Np\,\Gamma(\alpha))^{1/\alpha}z\}.
\end{aligned}$$

Hence a *typical* value for the weight of the shortest path is the solution of $\alpha N = \exp\{(\alpha Np\,\Gamma(\alpha))^{1/\alpha}z\}$, yielding

$$z \sim \frac{\ln(\alpha N)}{(\alpha Np\,\Gamma(\alpha))^{1/\alpha}}.$$

Substitution of this result in (17) shows that the hopcount probability should satisfy

$$\Pr[h_N = k] \simeq \frac{1}{N}\frac{(\ln \alpha N)^{\alpha k}}{\Gamma(\alpha k+1)}, \tag{18}$$

where the event $\{w_N \le z\}$ is deleted because we substitute the *typical* value of z, so that $P\{w_N \le z\}$ is close to 1.

For $\alpha = 1$, the right-hand side of (18) equals the probability that a Poisson variable with parameter $(\ln N)$ equals k,

$$\frac{1}{N}\frac{(\ln \alpha N)^{\alpha k}}{\Gamma(\alpha k+1)} = e^{-\ln N}(\ln N)^{k}/k!,$$

so that for $\alpha = 1$ and N large $E[h_N] \simeq var[h_N] \simeq \ln N$. This result has been proven rigorously in [18] with the *uniformly* distributed weights replaced by exponential weights. This replacement is irrelevant, because, as was pointed out in the introduction to this section, the exponential and the uniform distribution are in the same minimal domain of attraction. The proof for exponential weights will be sketched in section 6.3.

For $\alpha \neq 1$, a rough first order approximation for the Gamma-function is $\Gamma(\alpha k+1) \simeq (\alpha k)^{\alpha k}$, and hence (18) suggests that $\alpha k \simeq \ln \alpha N = \ln N + \ln \alpha$. To confirm this and to calculate the asymptotic variance of the hopcount we substitute

$$k = \frac{1}{\alpha}\ln N + v, \tag{19}$$

in (18). Using Stirling's formula ($\Gamma(u+1) \sim \left(\frac{u}{e}\right)^{u}\sqrt{2\pi u}$),

$$\begin{aligned}
\frac{1}{N}\frac{(\ln(\alpha N))^{\alpha k}}{\Gamma(\alpha k+1)} &\simeq \left(\frac{\ln N + \ln \alpha}{\ln N + \alpha v}\right)^{\ln N + \alpha v}\frac{e^{\alpha v}}{\sqrt{2\pi(\ln N + \alpha v)}} \\
&\simeq \frac{e^{\alpha v}}{\sqrt{2\pi \ln N}}\exp\left\{-(\ln N + \alpha v)\cdot \ln\left(\frac{\ln N + \alpha v}{\ln N + \ln \alpha}\right)\right\}.
\end{aligned}$$

Now use $\ln(1+x) = x - x^2/2$, up to second order to obtain

$$\frac{e^{\alpha v}}{\sqrt{2\pi \ln N}}\exp\left\{-(\ln N + \alpha v)\cdot \ln\left(\frac{1+\alpha v/(\ln N)}{1+\ln \alpha/(\ln N)}\right)\right\} \simeq \frac{\alpha e^{-\alpha^2 v^2/(2\ln N) + \frac{\ln \alpha}{\ln N}\{\alpha v - (\ln \alpha)/2\}}}{\sqrt{2\pi \ln N}}.$$

To interpret this result properly note that we compared (asymptotically) a discrete probability by a continuous function in v, whereas (compare (12)), we should approximate the cumulative probabilities of the standardized hopcount with the standard normal distribution function. If a random variable X_N is asymptotically normally distributed, then

$$\Pr\left[\frac{X_N - \mu_N}{\sigma_N} \leq x\right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

where $\mu_N = E[X_N]$ and $\sigma_N^2 = var[X_N]$. This statement almost always (but not necessarily) implies local convergence of the densities:

$$\sigma_N \Pr\left[\frac{X_N - \mu_N}{\sigma_N} \in [x, x + dx)\right] / dx \rightarrow \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \Re.$$

Now substitution of $v = x\sqrt{(\ln N)/\alpha^2}$ in (18) and (19) gives

$$\Pr\left[\frac{h_N - (\ln N)/\alpha}{\sqrt{(\ln N)/\alpha^2}} \in [x, x + dx)\right] / dx \simeq \frac{\alpha e^{-x^2/2 + O((\ln N)^{-1/2})}}{\sqrt{2\pi \ln N}}.$$

This indicates that (12) indeed holds with $\mu_N \simeq (\ln N)/\alpha$ and $\sigma_N^2 \sim (\ln N)/\alpha^2$. Hence, we expect also (13) and (14) to hold. It then follows that the ratio $\frac{E[h_N]}{var[h_N]} \rightarrow \alpha$ for $N \rightarrow \infty$, as observed in Figure 5. The statements (12)-(14), for $\alpha \neq 1$, although very plausible, seem hard to prove rigorously, because of lack of order between the nodes of the random graph. Classical central limit theory with some kind of mixing and or the central limit theorem for martingale difference sequences seems therefore hard to apply. The proof for $\alpha = 1$ relies on the equivalence of uniform and exponential weights in the shortest path problem and the theory of continuous time Markov chains, as explained in the next section.

6.2 The complete graph K_N

The previous results point to the independence of the hopcount on p and suggest to consider the simplest case of the complete graph K_N where $p = 1$. The problem of finding the shortest path between two nodes A and B in K_N with exponentially distributed link metrics can be rephrased in terms of a Markov discovery process. The discovery process evolves as a function of time and stops at a random time T when node B is found. The process is shown in Figure 6.

The evolution of the discovery process can be described by a continuous time Markov chain $X(t)$, where $X(t)$ denotes the number of discovered nodes at time t, because the characteristics of a Markov chain [8] are based on the exponential distribution and the memoryless property. Of particular interest here is the property that the minimum of n independent exponential variables each with parameter α_i is again an exponential variable with parameter $\sum_{i=1}^{n} \alpha_i$.

The discovery process starts at time $t = T_0$ with the source node A and for the initial distribution of the Markov chain, we have $\Pr[X(T_0) = 1] = 1$. The state space of the continuous Markov chain is the set S_N consisting of all positive integers n with $n \leq N$. For the complete graph K_N, the transition rates are given by

$$\lambda_{n,n+1} = n(N - n), \qquad n \in S_N. \tag{20}$$

Indeed, initially there is only the source node A, hence $n = 1$. From this first node A precisely $N - 1$ new nodes can be reached in the complete graph K_N. Alternatively one can say that $N - 1$ nodes are competing with each other each with exponentially distributed strength to be discovered and the winner amongst them, say C, is the one reached in shortest time which corresponds to an exponential

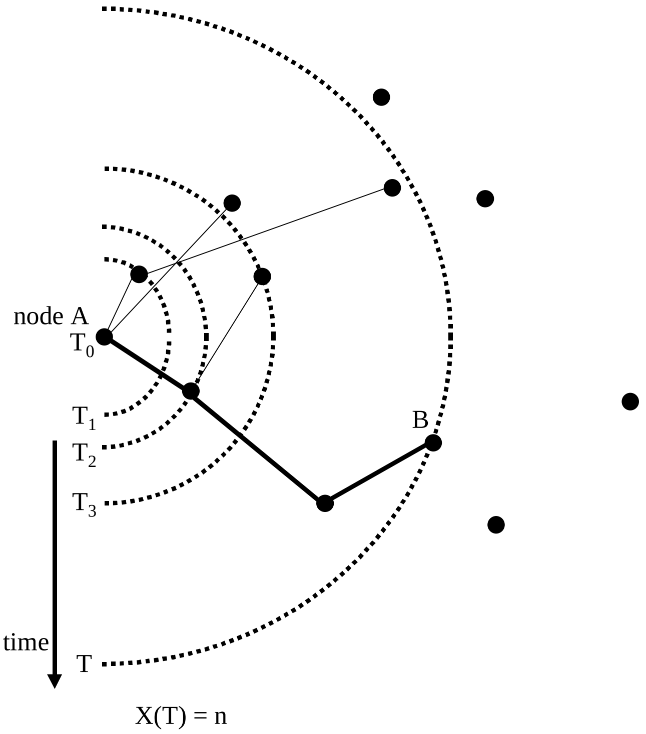


Figure 6: The discovery process as function of time. The time horizon is drawn as a dotted semi-circle centered at the starting node A. At time T the process stops when B is found.

variable with rate $N-1$. After having reached C from A at hitting time T_2, $n=2$, and the discovery process restarts from both A and C. Although at time T_2 we were already progressed a certain distance towards each of the $N-2$ other, not yet discovered nodes, the memoryless property of the exponential distribution tells us that the remaining distance to these $N-2$ nodes is again exponentially distributed with the same parameter 1. Hence, this allows us to restart the process from A and C by erasing the previously partial distance as if we ignore that it were ever travelled. From T_2 on, the discovery process has double strength to reach precisely $N-2$ new nodes. Hence, the next winner, say D, is reached in the minimum time out of $2(N-2)$ links. This node D has equal probability to be attached to A or C because of symmetry. When D is attached to C (the argument below holds similarly for attachment to A), symmetry appears to be broken, because D and A have only one link used whereas C has already two links used. However, since we are interested in the shortest path problem, the direct link from A to D is longer than the path $A \to C \to D$ and, thus, we exclude this link in the discovery process hereby establishing again the full symmetry in the Markov chain. This exclusion also means that the Markov chain maintains single paths from A to each newly discovered node and this path is also the shortest path. Hence, there are no cycles possible. Furthermore, similar to Dijkstra's shortest path algorithm, each newly reached node is withdrawn from the next competition round guaranteeing that the Markov chain eventually terminates. Besides terminating by extinction of all available nodes, after each transition (or newly discovered node), the Markov chain stops with probability equal to $\frac{1}{N-n}$ since each of the n already discovered nodes has precisely 1 possibility out of the remaining $N-n$ to reach B and only one of them is the discoverer. The stopping time T is defined as the infimum for

$t \geq 0$ at which destination node B is discovered. In summary, the described Markov discovery process models exactly the shortest path for all values of N.

Next, since we are merely interested in the hopcount of the shortest path, we need to determine for each node the smallest number of hops to A. Therefore, we consider an associated tree to the Markov process with equal number of nodes $X(t)$ at time t constructed as follows. Just as the discovery process, the associated tree starts at the root A. After each transition in the continuous time Markov chain, $X(t) \to X(t)+1$, an edge of *unit* length is attached *randomly* to one of the n already discovered nodes in the associated tree because a new edge is equally likely to be attached to any of the n discovering nodes. Hence, by construction of the associated tree, all configurations are equally likely and the probability $p_N^{(k)}$ that a uniformly chosen node in the tree has hopcount k [16], is $p_N^{(k)} = \frac{(-1)^{N-k} S_N^{(k)}}{N!}$, where $S_N^{(k)}$ is the Stirling number of the first kind [1, 24.1.3]. On the exact reformulation of the shortest path problem into the Markov discovery process and as shown in the appendix A, the generating function for the hopcount of the shortest path in the complete graph K_N is

$$H_N(x) = \frac{N}{N-1}\left(\frac{\Gamma(N+x)}{\Gamma(N+1)\Gamma(x+1)} - \frac{1}{N}\right). \tag{21}$$

Although exact for all N, the final step involves the asymptotic behavior for large N which is best obtained from the Taylor series expansion around $x = 0$ of the generating function of the hopcount of the shortest path $H_N(x)$ for large N given by (1).

6.3 Beyond the Complete Graph.

In this section we demonstrate that the result derived for the complete graph, thus assuming $p = 1$, still holds for any constant $p \in (0,1)$, and moreover for sequences $p = p_N$ such that $p_N \to 0$ and $Np_N \to \infty$ at a certain rate.

The basic ingredient is again the associated uniform tree as described for the complete graph. The uniformity of the tree for the complete graph is based on a same *deterministic* number of $(N-n)$ outgoing links from each node in the tree of size n. In the random graph $G_p(N)$, the corresponding number of outgoing links per node is a binomial distributed random variable with parameters $N-n$ and p. Thus, in $G_p(N)$, the number of outgoing links is not necessarily the same for each node in the cluster of size n, but has an average value $(N-n)p$ and variance $(N-n)p(1-p)$. In [18], it is proven that for each sequence $p_N \in (0,1)$, with $Np_N/(\ln N)^3 \to \infty$, the asymptotic behavior for the hopcount h_N still holds. We will give a sketch of that proof here. In the quoted paper it is first shown that the number of outgoing links from each node in the cluster of size n can, with overwhelming probability, be sandwiched between the *deterministic* values

$$(N-n)p_N \pm \sqrt{A(N-n)p_N \ln N},$$

if A is large. Consecutively, a Markov discovery process $X_N^-(t)$, with transition rate

$$\lambda_{n,n+1}^- = n \cdot \left((N-n)p_N - \sqrt{A(N-n)p_N \ln N}\right)$$

is constructed. This Markov chain generates a (uniform) recursive tree of size N^-, where $n = N^-$ is the largest number for which $\lambda^-_{n,n+1}$ is positive. This gives

$$N^- \sim N - \frac{A \ln N}{p_N}, \quad N \to \infty. \tag{22}$$

The value N^- is the final size of the uniform tree associated with X^-_N. The hopcount h_N of $G_{p_N}(N)$ is then proven to have asymptotically the same distribution as the height of node B in the uniform tree associated with X^-_N; for this result the technical condition $Np_N/(\ln N)^3 \to \infty$ is needed. This condition is slightly more demanding than the (minimal) condition $N^-/N \to 1$, which as can be seen from (22) holds iff $Np_N/\ln N \to \infty$. The famous asymptotic result of Erdös and Rényi [6] states that the probability that $G(N, [N \ln N/2 + xN])$ (of the class $G(N, E)$) is connected, is asymptotically equal to $\exp(-e^{-2x})$. This result suggests that when $p_N < p_c = \frac{\ln N}{N}$, the graph $G_p(N)$ is with positive probability disconnected. In other words our result (4) holds, apart from a small technical gap ($\frac{\ln N}{N}$ versus $\frac{\ln^3 N}{N}$), for all connected graphs of the class $G_p(N)$.

In order to close the gap between the connectivity threshold $p_c = \frac{\ln N}{N}$ and p_N, satisfying the condition $\frac{Np_N}{\ln^3 N} \to \infty$, we have simulated the hopcount in $G_p(N)$ with $p_N = \frac{\lambda}{N}$ for various values of λ from $\lambda_c = \ln N$ to λ_c^3. For $N = 1000$ where $\lambda_c = 6.90$, the values $\lambda = 6, \lambda = 7$, $\lambda = 49$ and $\lambda = 7^3 = 343$ are simulated.

The simulation reflects the Markov discovery process. We start with an initial node (the source) at height 0 and simulate a binomially distributed number X_1 of links with parameters $N - 1$ and $p = \frac{\lambda}{N}$. In each step the number of free links per node, i.e. the number of links to which a new node can be attached and the height of the node is maintained. The simulation then proceeds as follows. A randomly chosen free link out of the X_1 serves as the endpoint for a second node in the cluster at height 1. The number X_1 of free links of the root is decreased by one and the number X_2 of free links of the second node is determined as a binomially distributed number with parameters $N - 2$ and p. The total number of free links in the cluster discovered so far is equal to $X_1 + X_2 - 1$ and one of these links as attachment for the third node is chosen at random. Proceeding in this way, the k-th node is uniformly chosen from the $X_1 + \ldots + X_{k-1} - (k-2)$ available free links. This node is supplied with a binomially distributed number of links (with parameters $N - k$ and p). The height of the new node is equal to one plus the depth of its ancestor node l to which it was attached. At the same time, one of the free links of node l is removed, i.e. $X_l \to X_l - 1$. The simulation ends when the total number of free links is zero or when all nodes are discovered, i.e. when the cluster has N nodes. Observe that no exponentially distributed random variables are simulated. We use the property that each of the competing free links is equally likely to have the minimum weight and therefore the link is chosen at *random*. This observation simplifies the simulation drastically: only one loop of length N must be simulated.

We performed 1000 *runs* with $N = 1000$ and the three values of λ mentioned above. After each run, one node different from the initial node, was selected at random from the cluster of the initial node and its hopcount was determined as its height. The mean and the variance of these 1000 runs were denoted by $\bar{x}$ and σ^2, respectively. The theoretical values according to (2) and (3) are denoted by $E[h_N]$ and $var[h_N]$, respectively; these values remain constant for varying values of λ, as they do not depend on p. The results are in the table below.

	$E[h_N]$	$\bar{x}$	$var[h_N]$	σ^2
$\lambda = 6$	6.485	6.42	5,480	5.53
$\lambda = 7$	6.485	6.54	5,480	5.89
$\lambda = 49$	6.485	6.62	5,480	6.20
$\lambda = 343$	6.485	6.52	5,480	6.19

The simulations with merly 1000 runs show convincingly that the hopcount is indeed independent of p_N. Longer simulation (with more runs) approach the theoretical values closer. As long as the graphs are connected (even below p_c), simulations indicate the correctness of the theoretical results.

Finally, we present an alternative argument that $E[h_N]$ is, to first order, independent of p. The transition rates can be written as $\lambda_{n,n+1} = np(N-n) + v(n)$, where $v(n)$ is a stochastic variation around the mean number of links at each node due to the randomness of the graph. For large N, $np(N-n) >> v(n)$, such that we can neglect this stochastic effect in a first order computation. The discovery time T_k then equals[5] $\sum_{n=1}^{k} Exp(np(N-n)) \overset{\mathcal{D}}{=} \frac{1}{p}\sum_{n=1}^{k} Exp(n(N-n))$ or, the node k is discovered in the Markov chain corresponding to $G_p(N)$ precisely a factor $\frac{1}{p}$ later than in the complete graph. Since a time scaling does not alter the structure of the discovered cluster nor the associated recursive tree, the argument demonstrates that $E[h_N]$ does indeed not depend on p.

7 Statistical Verification of (4).

In order to (statistically) verify that the Internet hopcount distribution fits (4), we have used the measurements (consisting of 217 datapoints) on the hopcount distribution provided by Demeester and co-workers. The classical χ^2-goodness of fit test is based on

$$T_m = \sum_{k=1}^{m} \frac{(o_k - e_k)^2}{e_k}, \tag{23}$$

where o_k is the observed frequency of a hopcount k; m is the length of the vector o (which is the number of different values of the observed hopcount) and e_k the expected frequency obtained from (4) as

$$e_k = \frac{217}{N} \cdot \sum_{j=0}^{k} \frac{\ln^j N}{j!} c_{k+1-j}. \tag{24}$$

This procedure has two drawbacks: (i) the value of N is unknown; (ii) the distribution of T is only known asymptotically.

Evidently we have to estimate the number of routers N, from the available data, assuming that the model is true. For this we use the asymptotic expression for the expectation of the hopcount given by (2) as $E[h_N] \approx -0.42 + \log N$. Another complicating matter is that the measurement of the hopcount performed in Demeester's laboratory (the sender side) includes 4 proprietary hops of the private university network that are not part of the (public) Internet. However, the same situation holds for the receiver side. This inherent disadvantage of trace-route measurements performed from a computer attached to a private network limits to compute precise numbers of $E[h_N]$, $var[h_N]$ and

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[5] For any exponential variable, $Exp(\lambda a)$, with parameter λa holds in distribution that $Exp(\lambda a) \overset{\mathcal{D}}{=} \frac{1}{\lambda} Exp(a)$ for any $a \geq 0$.

$\frac{E[h_N]}{var[h_N]}$, and is only sufficient to study tendencies as in this paper. Let us denote x as the average number of proprietary hops at the receiver side. By inspecting the IP-addresses, the measurements indicated that the average $x = 2$. Therefore, the average hopcount of 17.84, reported in [23], is diminished with a total of $4 + x = 6$ hops resulting in an estimated number of routers,

$$\hat{N} = \lfloor \exp\{17.84 - 6 + 1 - \gamma\} \rfloor = 211,670. \tag{25}$$

Thus, we estimate the number of public Internet routers in the whole Internet (medio 1998) at about $N \approx 211,670$. As reported in [15], the estimated number of Internet *hosts* was 6.6 10^6 in July 1995, but so far, we did not find reported values on the number of nodes (routers) in the Internet.

To avoid inaccuracies from the assumed normality of the cell quantities $(o_k - e_k)/\sqrt{e_k}$, we use the bootstrap, a computing intensive method which is by now well established [12]. The bootstrap is a method to obtain the distribution of the test-statistic T by simulation. Given the hypothesis H_0: "*the asymptotic model is correct*", the measurements are obtained from sampling from the distribution (4). More specifically, the bootstrap generates 1000 samples of size 217 (the size of the data set measured by Demeester) from the distribution (4), with the value of $N = 211,670$ as estimated in (25). For each sample we compute

$$T_m^* = \sum_{k=1}^{m} (o_k^* - e_k^*)^2 / e_k^*,$$

where o_k^* is the observed frequency of the value k in the bootstrap sample and e_k^* equals the expected frequency calculated from the distribution (4) with $N = N^*$, the estimated number of routers obtained from the sample $o_1^*, \ldots, o_m^*$, i.e.

$$N^* = \exp\left(\sum_{j=1}^{m} j o_j^* + 1 - \gamma \right).$$

In this way we obtain 1000 bootstrap values:

$$t_1^*, \ldots, t_{1000}^*$$

of the test statistic T_m^*. It is well known that the distribution of T_m^* gives an adequate representation of the distribution of T_m^* under H_0. Hence we propose to reject H_0 when

$$\frac{|\{j : t_j^* > 22.55\}|}{1000} < \alpha,$$

with $\alpha = 0.05$ the significance level of the test, and $T_m^* = 22.55$ is the realization of the test statistic (23) with $m = 25$. It turns out that $t_{(951)}^* = 79.55$ and that the median of the bootstrap samples $t_1^*, \ldots, t_{1000}^*$ is $m^* = 26.19$. Hence, we do *not* reject H_0. Indeed the fit is good as can also be observed in the table below that shows the *shifted* observations o_k together with the theoretical cell frequencies e_k given through (24) with $N = 211670$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
o_k	0	0	2	2	2	6	11	12	22	26	19	28	20
e_k	0	.1	.4	1.1	2.7	5.6	9.7	14.8	19.8	23.8	25.8	25.5	23.2

k	14	15	16	17	18	19	20	21	22	23	24	25
o_k	18	15	15	6	3	3	4	3	0	0	0	0
e_k	19.5	15.2	11.1	7.6	4.9	2.9	1.7	.9	.5	.2	.1	.1

8 Discussion and Conclusions.

We have presented an asymptotic expression (4) for the pdf of the hopcount of the shortest path derived from r.g.'s of the class $G_p(N)$ with exponentially or uniformly distributed link weights. As important properties, we have shown that, for large N, the hopcount is independent of the link density p, insensitive to the precise details of the topology (as the results also hold for Waxman graphs) but it varies with the link weight distribution. The asymptotic expression (4) has been compared with Internet measurements and the agreement was surprisingly good.

Although the Internet is not a random graph of the class $G_p(N)$, the results deduced in the way presented here can be understood as follows. First of all, any graph is a subgraph of the complete graph, also the graph of the Internet. Next, we have randomly thinned the complete graph in two different ways: by altering the structure via p and by super-imposing weights on the links. Any graph where communication takes place between arbitrary nodes using a shortest path algorithm can be obtained in this way, by erasing from the complete graph the appropriate links that are not present in the real graph, and putting the right weights on the available links. Next, the focus on the shortest paths starting from a destination node A towards an arbitrary node B in the network leads us to consider a shortest path tree. Only links of this shortest path tree matter for the hopcount and a large number of links in the topology seems superfluous. Thus, by confining to the shortest path, we filter the actual topology to a tree rooted at A that is dependent on the link weights. This explains the apparent negligible influence on the details of the topology and underlines the importance of the link weight distribution. It also shows that information about the hopcount alone is insufficient to construct the Internet topology: not the number of links from a given node matters but the number of links *with small weights*. Hence, previous observations of power laws in Internet [7, 2] cannot be verified from our analysis.

We conclude that either the limiting distribution (4) is robust under minor changes to the model detailed above, and that the Internet is a minor change to that model, or that the model we have chosen describes the hopcount of the shortest path in Internet well. Aside from the simplicity of our model, the parsimonious parameters involved and the elegant, analytical result may be useful to enhance our understanding of the complex routing phenomena in the Internet.

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A The hopcount in the tree associated to the discovery Markov process in the complete graph K_N.

Lemma 1 *For $N \geq 1$ and $1 \leq k \leq N-1$,*

$$p_N^{(k)} = \frac{1}{N} \sum_{m=k}^{N-1} p_m^{(k-1)}, \tag{26}$$

and the corresponding generating function satisfies

$$\varphi_N(x) = \sum_{k=0}^{N-1} p_N^{(k)} x^k = \prod_{i=2}^{N} \left(\frac{i-1+x}{i} \right) = \frac{\Gamma(N+x)}{\Gamma(N+1)\Gamma(x+1)}. \tag{27}$$

PROOF: Denote by $X_N^{(k)}$ the number of nodes with hopcount k in the associated tree of size N and by y_N the hopcount of a randomly chosen node, possibly equal to the root. Since each vertex has probability $p_N^{(k)}$ of having hopcount k we obtain

$$\Pr[y_N = k] = p_N^{(k)} = \frac{EX_N^{(k)}}{N}. \tag{28}$$

If the size of the associated tree grows from m to $m+1$, each node at hopcount k from the source node generates a node at hopcount $k+1$ with probability $1/m$. Hence, for $k \geq 1$,

$$EX_N^{(k)} = \sum_{m=k}^{N-1} \frac{EX_m^{(k-1)}}{m},$$

which, together with (28), proves (26).

Since only the root can be accessed in zero steps, $p_N^{(0)} = \frac{1}{N}$ and

$$\begin{aligned} \varphi_N(x) &= \frac{1}{N} + \sum_{k=1}^{N-1} p_N^{(k)} x^k = \frac{1}{N} + \frac{1}{N} \sum_{k=1}^{N-1} \sum_{m=k}^{N-1} p_m^{(k-1)} x^k \\ &= \frac{1}{N} + \frac{1}{N} \sum_{m=1}^{N-1} \sum_{k=1}^{m} p_m^{(k-1)} x^k = \frac{1}{N} + \frac{x}{N} \sum_{m=1}^{N-1} \varphi_m(x). \end{aligned}$$

Taking the difference of $(N+1)\varphi_{N+1}(x)$ and $N\varphi_N(x)$ yields the recursion

$$(N+1)\varphi_{N+1}(x) = (N+x)\varphi_N(x). \tag{29}$$

Generating function (27) is easily verified from (29) and $\varphi_1(x) = 1$.

The explicit form of the generating function shows that the average hopcount y_N in a tree of size N equals

$$E[y_N] = \varphi_N'(1) = \sum_{l=2}^{N} \frac{1}{l}. \tag{30}$$

Also note from (21) that

$$p_N^{(k)} = \frac{(-1)^{n-k} S_N^{(k)}}{N!},$$

where $S_N^{(k)}$ denotes the Stirling number of the first kind [1, 24.1.3]. The above results are known [16], but derived in a different way.

Finally, we are interested in the hopcount y_N excluding the event $y_N = 0$. Thus, for $1 \leq k \leq N-1$,

$$\begin{aligned}
\Pr[h_N = k] &= \Pr[y_N = k | y_N \neq 0] = \frac{\Pr[y_N = k, y_N \neq 0]}{\Pr[y_N \neq 0]} \\
&= \frac{N}{N-1} \Pr[y_N = k] = \frac{N}{N-1} \frac{(-1)^{n-k} S_N^{(k)}}{N!},
\end{aligned}$$

with corresponding generating function,

$$\begin{aligned}
H_N(x) &= \sum_{k=1}^{N-1} \Pr[h_N = k]\ x^k = \frac{N}{N-1} \sum_{k=0}^{N-1} \Pr[y_N = k]\ x^k - \frac{N}{N-1} \Pr[y_N = 0] \\
&= \frac{N}{N-1} \left(\varphi_N(x) - \frac{1}{N} \right).
\end{aligned}$$