A complex variant of the Kuramoto model

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Abstract

For any network topology, we investigate a complex variant of the Kuramoto model, whose real part formally reduces to the original Kuramoto model for coupled oscillators when the coupling strength κ is imaginary. The major attraction of the complex variant lies in its exact solution, because it obeys a linear set of differential equations. The entire spectrum (eigenvalues and eigenvectors) can be deduced from a generalized Laplacian. We show that the formal resemblence to the original Kuramoto model is deceptive and that forcing the complex variant to coincide with the original one, leads to inconsistencies. Moreover, the complex variant is unstable for most graphs (except for trees) when κ is imaginary. The positive news is that the linear complex variant for real κ is expected to be close "on average" to the repulsive, non-linear cosine-variant of the Kuramoto model.

1 Introduction

We consider a dynamic process on a connected graph G with N nodes and L links that is described by the set of differential equations

$$\frac{dz_k(t)}{dt} = \zeta_k - \kappa \sum_{j=1}^N a_{kj} e^{i(z_j(t) - z_k(t))} \qquad 1 \le k \le N$$

$$\tag{1}$$

where $z_k(t)$ and $\zeta_k = \omega_k + i\eta_k$ are a complex function of the real parameter t (time) and a complex number, respectively, associated to node k, where the adjacency matrix element¹ $a_{kj} = 1_{\{\text{link } k-j \text{ exists}\}}$ and where $\kappa = ge^{i\gamma}$ is a complex number. An important assumption is that the adjacency matrix A is symmetric, i.e. $a_{ij} = a_{ji}$ and $A = A^T$, which means that the graph G is undirected. We use the following notation². If x is a vector with components x_1, x_2, \ldots, x_n , then f(x) is the vector with components $f(x_1), f(x_2), \ldots, f(x_n)$, for any function f. The matrix representation of this z-process is

$$\frac{dz(t)}{dt} = \zeta - \kappa \operatorname{diag}\left(e^{-\mathrm{i}(z_k(t))}\right) A e^{\mathrm{i}(z(t))} \tag{2}$$

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¹The indicator function 1_x equals 1 if the event or condition x is true, otherwise it is zero.

²We remark that this notation is unambiguous only for vectors, but it does not apply to matrices.

where $\operatorname{diag}(y_j)$ is a diagonal matrix with diagonal elements y_1, y_2, \ldots, y_N . An attractive feature of the *z*-process is that this system (2) is exactly solvable as shown in Section 2.

The motivation to study this dynamic process arises from the observation that, for $\kappa = ig$ and g real, the real part of (1) formally equals the Kuramoto equations for the phases of coupled oscillators in a network. The governing equations of the Kuramoto model – presented in [2, Section 5.4], but we follow the notation of [7] – are

$$\dot{\theta}_k = \omega_k + g \sum_{j=1}^N a_{kj} \sin\left(\theta_j - \theta_k\right) \qquad 1 \le k \le N$$
(3)

where θ_k denotes the phase of oscillator k and ω_k its natural (time-independent) frequency and where $\dot{\theta}_k = \frac{d\theta_k(t)}{dt}$. The coupling strength g is a real number. Usually, one defines the mean of the natural frequencies as

$$\Omega = \frac{1}{N} \sum_{k=1}^{N} \omega_k$$

The "original" Kuramoto model was proposed for the complete graph K_N , where $a_{kj} = 1_{\{k \neq j\}}$. Many collective synchronization processes (see e.g. [7],[8] for a rather impressive list of examples) can be studied by the Kuramoto model. The cosine-variant of the Kuramoto model,

$$\dot{\theta}_k = \omega_k - g \sum_{j=1}^N a_{kj} \cos\left(\theta_j - \theta_k\right) \qquad 1 \le k \le N$$
(4)

is formally recognized as the real part of (1) for $\kappa = g$. Physically, the cosine-variant (4) is repulsive in contrast to the Kuramoto model because phase differences that approach zero are decreased (if g > 0).

Unfortunately, the formal resemblance of the complex process (1) with the Kuramoto model (3) is deceptive as we will show in Section 5, that explores the relation between the complex and original Kuramoto model in more depth. Only for real κ and $\zeta_k = \omega_k$, the linear, complex z-process may serve as a reasonable, in some mean sense as explained in Section 3, approximation to the cosine-variant (4) of the Kuramoto model.

2 Exact solution of the *z*-process

Since the inverse of the matrix diag (e^{-iz_k}) always exists (for finite z_k) and equals diag (e^{iz_k}) , after left-multiplying both sides of (2) by this inverse matrix, we obtain

diag
$$(e^{iz_k}) \frac{dz}{dt} = diag (e^{iz_k}) \zeta - \kappa A e^{iz_k}$$

Further, using diag $(e^{iz_k})\frac{dz}{dt} = \frac{1}{i}\frac{d}{dt}e^{iz}$ and diag $(e^{iz_k})\zeta = \text{diag}(\zeta_k)e^{iz}$, we have

$$\frac{d}{dt}e^{iz} = i\left(\operatorname{diag}\left(\zeta_k\right) - \kappa A\right)e^{iz}$$

Introducing the generalized Laplacian, defined earlier in [10],

$$Q(q_k) = \operatorname{diag}(q_k) - A$$

$$= \operatorname{diag}(q_k - d_k) + Q$$
(5)

where Q is the Laplacian of the graph and d_k is the degree of node k and letting $s = e^{iz}$, we arrive at

$$\frac{ds}{dt} = i\kappa \left(\operatorname{diag}\left(\frac{\zeta_k}{\kappa}\right) - A \right) s = i\kappa \mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) s \tag{6}$$

The general solution is

$$s(t) = e^{i\kappa t \mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)} s(0)$$

where $s(0) = e^{iz(0)}$ is the initial vector at t = 0.

Since $\mathcal{Q}(q_k)$ is symmetric by our assumption that $A = A^T$, there holds [3] that

$$\mathcal{Q} = X \operatorname{diag}(\mu_k) X^T$$

where the matrix X is orthogonal, i.e. $X^T X = X X^T = I$ and X has as k-th column the eigenvector x_k of \mathcal{Q} belonging to the eigenvalue μ_k . For any function f(x) with a converging Taylor series within a disk around zero with radius larger than the largest absolute value of the eigenvalues, we know that

$$f(\mathcal{Q}) = X \operatorname{diag} \left(f(\mu_k) \right) X^T = \sum_{k=1}^N f(\mu_k) x_k x_k^T$$

Applied to the matrix $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$ and transforming back to $s = e^{iz}$ yields the general solution

$$e^{iz(t)} = X \operatorname{diag}\left(e^{i\kappa t\mu_k}\right) X^T e^{iz(0)} \tag{7}$$

When working out the matrix multiplications in (7), we find

$$e^{iz(t)} = \sum_{k=1}^{N} e^{i\kappa t\mu_{k}} x_{k} x_{k}^{T} e^{iz(0)} = \sum_{k=1}^{N} e^{-(\sin\gamma \operatorname{Re}\mu_{k} + \cos\gamma \operatorname{Im}\mu_{k})gt} \left(e^{i(\cos\gamma \operatorname{Re}\mu_{k} - \sin\gamma \operatorname{Im}\mu_{k})gt} x_{k} x_{k}^{T} e^{iz(0)} \right)$$
(8)

Of course, all the details of the linear z-process lie hidden in the generalized Laplacian $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$ and its spectrum, which needs to be investigated in detail. If κ and $\zeta_k = \omega_k$ are all real, then $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) = \mathcal{Q}\left(\frac{\omega_k}{g}\right)$ is a real symmetric matrix, whose eigenvalues and eigenvectors are real. If $\frac{\zeta_k}{\kappa}$ is complex, then $\mathcal{Q}\left(\frac{\omega_k}{\kappa}\right)$ is a complex symmetric matrix, whose eigenstructure is considerably more complex.

2.1 Properties of the *z*-process

We write (8) for one function z_j , $1 \le j \le N$,

$$e^{iz_j(t)} = \sum_{k=1}^{N} e^{-(\sin\gamma\operatorname{Re}\mu_k + \cos\gamma\operatorname{Im}\mu_k)gt} \left(e^{i(\cos\gamma\operatorname{Re}\mu_k - \sin\gamma\operatorname{Im}\mu_k)gt} \sum_{l=1}^{N} \left(x_k x_k^T \right)_{jl} e^{iz_l(0)} \right)$$

Since the term between brackets is bounded for any t, we observe that, in order to have a finite solution for the vector function $e^{iz(t)}$ for all $t \ge 0$, there must hold that $\xi_k = \sin \gamma \operatorname{Re} \mu_k + \cos \gamma \operatorname{Im} \mu_k \ge 0$ for all $k \in [1, N]$, otherwise the complex z-process is called unstable. The case where $\kappa = 0$ is obvious from (1): $z(t) = z(0) + \zeta t$. At $\kappa = g = 0$, the z-process is considered as stable for all $\eta_k \ge 0$, because $e^{iz(t)}$ is bounded (although $\lim_{t\to\infty} z(t) = \infty$). When $\frac{\zeta_k}{\kappa}$ is real, which is equivalent to $\tan \gamma = \frac{\eta_k}{\omega_k}$ for all $k \in [1, N]$ (see Appendix A), the z-process is always bounded, hence, stable. Only, if there are eigenvalues satisfying $\xi_k = 0$, then the solution for $t \to \infty$ is a non-zero. The scalar product $v(t) = (e^{iz(t)})^T e^{iz(-t)} = (e^{iz(-t)})^T e^{iz(t)}$ equals, using (7) and the orthogonality relation $X^T X = X X^T = I$,

$$v(t) = \left(X \operatorname{diag}\left(e^{i\kappa t\mu_{k}}\right) X^{T} e^{iz(0)}\right)^{T} X \operatorname{diag}\left(e^{-i\kappa t\mu_{k}}\right) X^{T} e^{iz(0)}$$
$$= \left(e^{i\theta(0)}\right)^{T} X \operatorname{diag}\left(e^{i\kappa t\mu_{k}}\right) X^{T} X \operatorname{diag}\left(e^{-i\kappa t\mu_{k}}\right) X^{T} e^{iz(0)} = \left(e^{iz(0)}\right)^{T} e^{iz(0)} = v(0)$$

which shows that v(t) is a constant of motion (i.e. independent of time t). If the initial vector z(0) = 0, then v(t) = N and, thus, real.

The steady-state, that is usually defined as $\lim_{t\to\infty} s(t)$ and for which $\frac{ds}{dt} = 0$, does generally not exist. Indeed, the requirement that $\frac{ds}{dt} = 0$ in (6) implies that $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)s = 0$. A non-zero solution $s \neq 0$ requires that $\det \mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) = 0$, which means that $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$ must have a zero eigenvalue for all coupling strengths κ . For a connected graph and provided not all ζ_k are zero, $\det \mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) = 0$ only if $\frac{\zeta_k}{\kappa} = d_k$ in case $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) = Q$, the Laplacian of the graph. Except for those cases, the matrix $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$ does not possess a zero eigenvalue and is always invertible. The non-existence of such defined steady-state is a property of oscillators; also the harmonic oscillator does not possess a steady-state because the phase portrait are circles (see [6]).

3 The coupling strength κ and $\zeta_k = \omega_k$ are all real

Since each eigenvalue μ_k and its corresponding eigenvector x_k is real, the general solution (8) consists of a linear combination of purely periodic functions with frequencies equal to $g\mu_k$ for $1 \le k \le N$ for any coupling strength g. Moreover, consider the norm $\|e^{iz(t)}\|_2^2 = \sum_{j=1}^N |e^{iz_j(t)}|^2 = \left(\left(e^{iz(t)}\right)^*\right)^T e^{iz(t)}$, then, using (7) and the orthogonality relation $X^T X = X X^T = I$, we have

$$\left\| e^{iz(t)} \right\|_{2}^{2} = \left(X \operatorname{diag} \left(e^{-igt\mu_{k}} \right) X^{T} e^{-iz(0)} \right)^{T} X \operatorname{diag} \left(e^{igt\mu_{k}} \right) X^{T} e^{iz(0)}$$

= $\left(e^{-iz(0)} \right)^{T} X \operatorname{diag} \left(e^{-igt\mu_{k}} \right) X^{T} X \operatorname{diag} \left(e^{igt\mu_{k}} \right) X^{T} e^{iz(0)} = \left\| e^{iz(0)} \right\|_{2}^{2}$

which shows that the norm $\|e^{iz(t)}\|_2$ is, beside³ v(t), also a constant of motion (i.e. independent of time t) for all g. When the initial vector z(0) is real, then $N = \|e^{iz(0)}\|_2^2 = \|e^{iz(t)}\|_2^2$ and, hence, each function $z_j(t)$ is real for all t.

Since all $z_j(t)$ and $\zeta_k = \omega_k$ and $\kappa = g$ are real, the real part of (1) leads to cosine-variant (4) of the Kuramoto model, while the imaginary part gives

$$\sum_{j=1}^{N} a_{kj} \sin\left(\theta_j - \theta_k\right) = 0$$

Although this additional constraint that appears in the z-process is not physically meaningful in the cosine-variant (4) of the Kuramoto model, it is correct "on average" as follows from (23), where the average is over the total ensemble of oscillators. In this sense, we expect that the complex z-process with real coupling strength $\kappa = g$ may be close to the original physical system (4), that is non-linear.

 $\overline{\left(e^{-igt\mu_{k}}\right)X^{T}e^{iz(0)}} = \left(e^{iz(-t)}\right)^{T}e^{iz(t)} \text{ is different from } \left\|e^{iz(t)}\right\|_{2}^{2} = \left(\left(e^{iz(t)}\right)^{*}\right)^{T}e^{iz(t)} \text{ because } e^{iz(-t)} = X \operatorname{diag}\left(e^{-igt\mu_{k}}\right)X^{T}e^{iz(0)} \text{ is different from } \left(e^{iz(t)}\right)^{*} = X \operatorname{diag}\left(e^{-igt\mu_{k}}\right)X^{T}e^{-iz(0)}, \text{ except if } z(0) = 0.$

4 At least one $\frac{\zeta_k}{\kappa}$ is complex

The critical coupling strength $g_{cc}(\gamma)$ of the complex *z*-process with complex coupling strength $\kappa = ge^{i\gamma}$ is the highest value of $g = |\kappa|$ (as a function of γ) for which the *z*-process is still stable.

Theorem 1 Provided $H = \frac{1}{N} \sum_{k=1}^{N} \eta_k = 0$ and $\gamma \neq 0$ nor $\gamma \neq \pi$, the critical coupling strength $g_{cc}(\gamma)$ is bounded by

$$0 \le g_{cc}(\gamma) \le \sqrt{\frac{Var[\omega] - Var[\eta]}{E[d]} - 2\cot\gamma \frac{Cov[\omega,\eta]}{E[d]}}$$
(9)

At $g_{cc}(\gamma) + \varepsilon$ for arbitrary small $\varepsilon > 0$, eigenvalues of $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$ occur that satisfy the instability condition $\sin \gamma \operatorname{Re} \mu_k + \cos \gamma \operatorname{Im} \mu_k < 0.$

Proof: The exact solution (8) shows that instability occurs when there are eigenvalues of the generalized Laplacian $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$ that satisfy $\xi_k = \sin \gamma \operatorname{Re} \mu_k + \cos \gamma \operatorname{Im} \mu_k < 0$. Now,

$$\sum_{k=1}^{N} \xi_k^2 = \sin^2 \gamma \sum_{k=1}^{N} \left\{ (\operatorname{Re} \mu_k)^2 - (\operatorname{Im} \mu_k)^2 \right\} + \sum_{k=1}^{N} (\operatorname{Im} \mu_k)^2 + \sin 2\gamma \sum_{k=1}^{N} \operatorname{Re} \mu_k \operatorname{Im} \mu_k$$

from which

$$\sum_{k=1}^{N} \xi_k^2 - \sum_{k=1}^{N} (\operatorname{Im} \mu_k)^2 = \sin^2 \gamma \sum_{k=1}^{N} \left\{ (\operatorname{Re} \mu_k)^2 - (\operatorname{Im} \mu_k)^2 \right\} + \sin 2\gamma \sum_{k=1}^{N} \operatorname{Re} \mu_k \operatorname{Im} \mu_k$$

and, after substituting (19) and (20),

$$\sum_{k=1}^{N} \xi_k^2 - \sum_{k=1}^{N} \left(\operatorname{Im} \mu_k \right)^2 = \sin^2 \gamma \left\{ 2L - \frac{N}{g^2} \left(\operatorname{Var} \left[\omega \right] - \operatorname{Var} \left[\eta \right] + \Omega^2 - H^2 \right) \right\} + \frac{N \sin 2\gamma}{g^2} E\left[\omega \eta \right]$$
(10)

For sufficiently large g, (10) indicates that there are non-zero ξ_k . It follows from (17) and (18) that

$$\sum_{k=1}^{N} \xi_k = \frac{N}{g} H \tag{11}$$

Since H = 0 by assumption, the sum (11) tells us that both positive and negative ξ_k must occur.

We will now show that there exists a coupling strength g-interval $[0, g_{cc}(\gamma)]$ in which all $\xi_k = 0$ for $1 \le k \le 0$. If all $\xi_k = 0$ in (10) and $\sin \gamma \ne 0$, then

$$g = \sqrt{\frac{\sin^2 \gamma \left(\operatorname{Var} \left[\omega \right] - \operatorname{Var} \left[\eta \right] + \Omega^2 - H^2 \right) - \sin 2\gamma E \left[\omega \eta \right]}{\frac{1}{N} \sum_{k=1}^{N} \left(\operatorname{Im} \mu_k \right)^2 + E \left[d \right] \sin^2 \gamma}}$$

where $E[d] = \frac{2L}{N}$ is the average degree in G. Invoking the Cauchy-Schwarz inequality $N \sum_{k=1}^{N} (\operatorname{Im} \mu_k)^2 \ge \left(\sum_{k=1}^{N} |\operatorname{Im} \mu_k|\right)^2$ together with $\sum_{k=1}^{N} |\operatorname{Im} \mu_k| \ge \sum_{k=1}^{N} \operatorname{Im} \mu_k$ and (18) demonstrates that

$$\frac{1}{N}\sum_{k=1}^{N}\left(\operatorname{Im}\mu_{k}\right)^{2} \geq \frac{1}{g^{2}}\left(H\cos\gamma - \Omega\sin\gamma\right)^{2}$$

Using also $\sin \gamma \neq 0$ and H = 0, such that $E[\omega \eta] = \text{Cov}[\omega, \eta]$ (see e.g. [9]), yields

$$g^{2} \leq \frac{\left(\operatorname{Var}\left[\omega\right] - \operatorname{Var}\left[\eta\right] + \Omega^{2}\right) - 2\cot\gamma \operatorname{Cov}\left[\omega,\eta\right]}{\frac{1}{q^{2}}\Omega^{2} + E\left[d\right]}$$

Solving the inequality for g gives the tight upper bound in (9).

The maximum $g_{cc}(\gamma)$ occurs when $\gamma = \pi - \varepsilon$ (if $\operatorname{Cov}[\omega, \eta] > 0$) or $\gamma = \varepsilon$ (if $\operatorname{Cov}[\omega, \eta] < 0$) and when $\operatorname{Var}[\eta] = 0$, given that $E[\eta] = H = 0$. Theorem 1 underlines the necessity of "complex" natural frequencies η_k to enhance the stability of the z-process for non-real κ (i.e. $\gamma \neq 0$ nor $\gamma \neq \pi$). For, if all $\eta_k = 0$, then $g_{cc}(\gamma) \leq \sqrt{\frac{\operatorname{Var}[\omega]}{E[d]}}$ for all $\gamma \in (0, \pi) \cup (\pi, 2\pi)$.

4.1 The coupling strength $\kappa = ig$ and $\zeta_k = \omega_k$ are all real

With the settings $\kappa = ig$ and $\zeta_k = \omega_k$, the z-process is formally most close to the Kuramoto model (3) as shown in Section 1.

Theorem 2 The critical coupling strength $g_{cc}\left(\frac{\pi}{2}\right)$ in the complex z-process with imaginary coupling strength $\kappa = ig$ and all $\eta_k = 0$ for $1 \le k \le N$ is zero when the graph has a triangle.

Proof: When $\blacktriangle_G > 0$, $\gamma = \frac{\pi}{2}$ and all $\eta_k = 0$, we have that $\xi_k = \sin \gamma \operatorname{Re} \mu_k + \cos \gamma \operatorname{Im} \mu_k = \operatorname{Re} \mu_k$. Relation (21) then demonstrates that not all eigenvalues can have a zero real part. This means that $g_{cc}\left(\frac{\pi}{2}\right) = 0$, i.e. there must be eigenvalues with non-zero real part for g > 0.

Since most graphs, except for trees, have triangles, Theorem 2 implies that the complex z-process with imaginary coupling strength $\kappa = ig$ and all $\eta_k = 0$ is unstable for most graphs. Only if $\blacktriangle_G = 0$, the relations (17) and (20) show, for the first pair (k, l) of eigenvalues with $\operatorname{Re} \mu_k = -\operatorname{Re} \mu_l \neq 0$, that $\operatorname{Im} \mu_k = \operatorname{Im} \mu_l$. Theorem 2 is deemed difficult to extend to other values of $\gamma \in (0, \pi)$.

Recall that the phase transition in the Kuramoto model occurs approximately, as shown by Restrepo et al. [4], at

$$g_c \simeq \frac{2}{\pi f_\omega\left(0\right)\lambda_{\max}\left(A\right)}$$

where $f_{\omega}(x)$ is the probability density function of the natural frequency distribution. This results assumes a rotating coordinate frame such that the mean frequency Ω is set to zero (see Appendix B). The general formula for $f_{\omega}(0) = \lim_{\Delta x \to 0} \frac{\Pr[-\frac{\Delta x}{2} \le \omega \le \frac{\Delta x}{2}]}{\Delta x}$ (see e.g. [9]) does not provide much physical insight. For a Gaussian random variable with zero mean, $f_{\omega}(x) = \frac{1}{\sqrt{2\pi\sigma_{\omega}}}e^{-\left(\frac{x^2}{2\sigma_{\omega}^2}\right)}$, the quantity $f_{\omega}(0) = \frac{1}{\sqrt{2\pi\sigma_{\omega}}}$ is related to the square root of the variance and

$$g_c \simeq \frac{2\sqrt{2}\sigma_\omega}{\sqrt{\pi}\lambda_{\max}(A)} \simeq 1.6 \frac{\sqrt{\operatorname{Var}[\omega]}}{\lambda_{\max}(A)}$$

Since the largest eigenvalue of the adjacency matrix obeys $\lambda_{\max}(A) \geq E[d]$, it seems likely from (9) that the critical value of g at which instability occurs in the complex z-process with imaginary coupling strength $\kappa = ig$ is given by $g_{cc}\left(\frac{\pi}{2}\right) \simeq \sqrt{\frac{\operatorname{Var}[\omega]}{\lambda_{\max}(A)}}$. Comparison – for trees only by Theorem 2 – then would suggest that $g_c \simeq \sqrt{\lambda_{\max}(A)}g_{cc}$. For the case of N = 2 oscillators that can be computed exactly for the Kuramoto model, we find indeed that $g_{cc}\left(\frac{\pi}{2}\right) = g_c$ (because $\lambda_{\max}(A) = 1$).

5 Relation between the *z*-process and the Kuramoto model

The set (1) of the complex z-process is rewritten by expliciting the complex nature of the "phases" as

$$\operatorname{Re}\left(\dot{z}_{k}\right) + \operatorname{i}\operatorname{Im}\left(\dot{z}_{k}\right) = \omega_{k} + \operatorname{i}\eta_{k} - g\sum_{j=1}^{N} a_{kj} e^{\operatorname{i}\left\{\operatorname{Re}\left(z_{j}-z_{k}\right)+\gamma\right\}} e^{-\operatorname{Im}\left(z_{j}-z_{k}\right)}$$

Taking the real and imaginary part yields

$$\operatorname{Re}\left(\overset{\cdot}{z_k}\right) = \omega_k - g \sum_{j=1}^N a_{kj} e^{-\operatorname{Im}(z_j - z_k)} \cos\left(\operatorname{Re}\left(z_j - z_k\right) + \gamma\right)$$
$$\operatorname{Im}\left(\overset{\cdot}{z_k}\right) = \eta_k - g \sum_{j=1}^N a_{kj} e^{-\operatorname{Im}(z_j - z_k)} \sin\left(\operatorname{Re}\left(z_j - z_k\right) + \gamma\right)$$

where the argument γ of the complex coupling strength $\kappa = g e^{i\gamma}$ introduces a constant phase shift.

Clearly, the Kuramoto equations (3) are retrieved in the first equation provided $\gamma = \frac{\pi}{2}$ and $\operatorname{Im}(z_j - z_k) = c_{kj}$ for all j and k, where c_{kj} is independent of time t. Moreover, $w_{kj} = gc_{kj}a_{kj}$ can be interpreted as the coupling strength of the link between the oscillators k and j. Substituting $z_k \to \theta_k$, we arrive at the governing equations

$$\operatorname{Im}\left(\stackrel{\cdot}{\theta}_{k}\right) = \eta_{k} - \sum_{j=1}^{N} w_{kj} \cos\left(\operatorname{Re}\left(\theta_{j} - \theta_{k}\right)\right)$$
(12)

$$\operatorname{Re}\left(\stackrel{\cdot}{\theta}_{k}\right) = \omega_{k} + \sum_{j=1}^{N} w_{kj} \sin\left(\operatorname{Re}\left(\theta_{j} - \theta_{k}\right)\right)$$
(13)

The requirement $\operatorname{Im}(\theta_j - \theta_k) = c_{kj}$ to map the complex z-process with imaginary coupling strength $\kappa = \operatorname{i} g$ to the Kuramoto model (3) implies that $\operatorname{Im}\left(\dot{\theta}_k\right) = \operatorname{Im}\left(\dot{\theta}_j\right)$ for any pair j and k. In fact, that mapping requirement introduces N additional constraints, that may lead to inconsistencies. For, letting $\operatorname{Im}\left(\dot{\theta}_k\right) = \operatorname{Im}\left(\dot{\theta}_1\right)$, the equations (12) become

$$\operatorname{Im}\left(\stackrel{\cdot}{\theta_{1}}\right) = \eta_{1} - \sum_{j=1}^{N} w_{1j} \cos\left(\operatorname{Re}\left(\theta_{j} - \theta_{1}\right)\right)$$

and for $2 \leq k \leq N$,

$$\operatorname{Im}\left(\overset{\cdot}{\theta}_{1}\right) = \eta_{k} - \sum_{j=1}^{N} w_{kj} \cos\left(\operatorname{Re}\left(\theta_{j} - \theta_{k}\right)\right)$$

When subtracting the first differential equation from all others, the last N - 1 equations become non-linear equations without derivatives

$$\sum_{j=1}^{N} w_{kj} \cos \left(\operatorname{Re} \left(\theta_j - \theta_k \right) \right) - \sum_{j=1}^{N} w_{1j} \cos \left(\operatorname{Re} \left(\theta_j - \theta_1 \right) \right) = \eta_1 - \eta_k$$

that specify $\operatorname{Re}(\theta_j(t))$ for j > 1 and any time t, while $\operatorname{Re}(\theta_1(t))$ can be determined from the first equation.

The above equation introduces, for each time t, a constraint for the phases θ_m for $1 \le m \le N$. This additional constraint is not part of the Kuramoto equations (3). Even worse, since also the Kuramoto equations (13) determine all N phases $\operatorname{Re}(\theta_j(t))$ for any time t, given an initial condition $\operatorname{Re}(\theta_j(0))$ for $1 \le j \le N$, the two sets of solutions may be inconsistent. The inconsistency is very likely to appear unless the set of additional constraints reduces to identities. Unfortunately, in general, the set of additional constraints do not lead to identities as readily verified for small N (or numerically).

The conclusion is that, although one set of the complex equations can be modified to appear formally identical to the Kuramoto equations as in (13), the dual set of imaginary parts (12) causes inconsistencies. Hence, beside its unstable nature by Theorem 2, the complex z-process with imaginary coupling strength $\kappa = ig$ can, in general, not describe Kuramoto's coupled oscillator model.

5.1 Short review of Roberts' approach

Roberts [5] has proposed an approach that linearizes the Kuramoto equations (by introducing some tuning parameter η), but his approach suffers (even in the steady-state for which he has introduced the tuning parameter) from the same additional algebraic constraint on the phases. He starts by stating the linear equations

$$\dot{\psi}_k = (\mathrm{i}\omega_k - \eta)\,\psi_k + g\sum_{j=1}^N a_{kj}\psi_j$$

After introducing the non-linear transform $\psi_k(t) = R_k(t) e^{i\theta_k(t)}$, we obtain

$$\dot{R}_{k}(t) e^{i\theta_{k}(t)} + R_{k}(t) e^{i\theta_{k}(t)} \dot{i\theta}_{k}(t) = (i\omega_{k} - \eta) R_{k}(t) e^{i\theta_{k}(t)} + g \sum_{j=1}^{N} a_{kj}R_{j}(t) e^{i\theta_{j}(t)}$$

Roberts divides both sides by $R_k(t) e^{i\theta_k(t)}$, thereby implicitly assuming that $\psi_k(t)$ or $R_k(t)$ is never zero, and finds, after taking the real and imaginary part,

$$\frac{\dot{R}_{k}(t)}{R_{k}(t)} = -\eta + g \sum_{j=1}^{N} a_{kj} \frac{R_{j}(t)}{R_{k}(t)} \cos\left(\theta_{j}(t) - \theta_{k}(t)\right)$$
$$\dot{\theta}_{k}(t) = \omega_{k} + g \sum_{j=1}^{N} a_{kj} \frac{R_{j}(t)}{R_{k}(t)} \sin\left(\theta_{j}(t) - \theta_{k}(t)\right)$$

He now chooses η such that $R_k(t)$, for each k, tends to a steady-state where $\lim_{t\to\infty} R_k(t) = 0$ and $\lim_{t\to\infty} \frac{R_j(t)}{R_k(t)} = c_{jk}$. Thus, using $w_{jk} = ga_{jk}c_{jk}$,

$$0 = -\eta + \sum_{j=1}^{N} w_{kj} \cos \left(\theta_j - \theta_k\right)$$

$$\dot{\theta}_k(t) = \omega_k + \sum_{j=1}^{N} w_{kj} \sin \left(\theta_j - \theta_k\right)$$

These equations are special cases of (12) and (13) where all θ_j are purely real. In [5], Roberts neglects the first relations that expresses the additional constraint imposed to the phases, but dwells on his tuning parameter η needed for stability reasons as shown in Lemma 1. In summary, his linear formulation of the Kuramoto model is, as shown above by forcing $\lim_{t\to\infty} \frac{R_j(t)}{R_k(t)} = c_{jk}$ for all k, defective.

6 Conclusions

The proposed, complex but linear z-process (1) is shown to be only stable for relatively small values of the coupling strength $g = |\kappa|$, except when κ is real. In that case, the z-process seems a promising linearization of the cosine-variant (4) of the Kuramoto model. We conjecture that this cosine-variant (4) does not possess a phase transition, as opposed to the original Kuramoto model (3) and that it, therefore, describes another, though related, physical phenomenon. In the other cases where $\gamma \in (0, \pi)$, the formal resemblence of the z-process with the Kuramoto model, in particular for the almost always unstable $\gamma = \frac{\pi}{2}$ case, is deceptive. We explain why and show where former work [5] is erroneous.

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A Spectrum of the generalized Laplacian $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right)$

u

Here, we present some general properties of the spectrum of $\mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) = \mathcal{Q}\left(u_k + iv_k\right)$, where

$$_{k} = \frac{\operatorname{Re}\zeta_{k}\operatorname{Re}\kappa + \operatorname{Im}\zeta_{k}\operatorname{Im}\kappa}{g^{2}} = \frac{\omega_{k}\cos\gamma + \eta_{k}\sin\gamma}{g}$$
(14)

and

$$v_k = \frac{\operatorname{Im} \zeta_k \operatorname{Re} \kappa - \operatorname{Re} \zeta_k \operatorname{Im} \kappa}{g^2} = \frac{\eta_k \cos \gamma - \omega_k \sin \gamma}{g}$$
(15)

Gerschgorin's Theorem [11, p. 71-75] indicates that the eigenvalues of $\mathcal{Q}(u_k + iv_k) = \operatorname{diag}(u_k + iv_k) - A$ are centered around $u_k + iv_k$ with radius equal to the degree d_k , i.e. an eigenvalue μ of $\mathcal{Q}(u_k + iv_k)$ lies in a disk $|\mu - u_j + iv_j| \leq d_j$ for some $1 \leq j \leq N$.

Lemma 1 The z-process is surely stable if

$$\frac{\min_{1 \le j \le N} (\eta_j)}{g} - d_{\max} \left| \sin \gamma + \cos \gamma \right| \ge 0$$

and definitely unstable if

$$\frac{\max_{1 \le j \le N} (\eta_j)}{g} + d_{\max} \left| \sin \gamma + \cos \gamma \right| < 0$$

Proof: The general solution (8) shows that a necessary condition for stability of the z(t)-process with $t \ge 0$ entails that $\sin \gamma \operatorname{Re} \mu_k + \cos \gamma \operatorname{Im} \mu_k \ge 0$ for all $1 \le k \le N$. Gerschgorin's Theorem provides the lower bound for any k,

$$\begin{aligned} \sin\gamma \operatorname{Re}\mu_k + \cos\gamma \operatorname{Im}\mu_k &\geq \min_{1 \leq j \leq N} \left(\sin\gamma u_j + \cos\gamma v_j - d_j \left(\sin\gamma + \cos\gamma \right) \right) \\ &= \min_{1 \leq j \leq N} \left(\frac{\eta_j}{g} - d_j \left(\sin\gamma + \cos\gamma \right) \right) \\ &\geq \frac{\min_{1 \leq j \leq N} \left(\eta_j \right)}{g} - d_{\max} \left| \sin\gamma + \cos\gamma \right| \end{aligned}$$

and, similarly, the upperbound for any k,

$$\sin \gamma \operatorname{Re} \mu_k + \cos \gamma \operatorname{Im} \mu_k \leq \max_{1 \leq j \leq N} \left(\sin \gamma u_j + \cos \gamma v_j + d_j \left(\sin \gamma + \cos \gamma \right) \right)$$
$$\leq \frac{\max_{1 \leq j \leq N} \left(\eta_j \right)}{g} + d_{\max} \left| \sin \gamma + \cos \gamma \right|$$

This demonstrates Lemma 1. \Box

Since $|\sin \gamma + \cos \gamma| \le \sqrt{2}$, Lemma 1 states that we can always make the z-process stable by choosing $\frac{\min_{1\le j\le N}(\eta_j)}{g} \ge \sqrt{2}d_{\max}$.

In the $\kappa = 0$ case, the eigenvalues of $\lim_{\kappa \to 0} \kappa \mathcal{Q}\left(\frac{\zeta_k}{\kappa}\right) = \operatorname{diag}(\zeta_k)$ are equal to $\mu_k = \zeta_k$ for $1 \le k \le N$.

A.1 The sum of powers of eigenvalues

For each integer value of m, we invoke the general relation (see e.g. [9, Appendix A])

trace
$$\left(\mathcal{Q}^m\left(\frac{\zeta_k}{\kappa}\right)\right) = \sum_{k=1}^N \mu_k^m$$
 (16)

For m = 1 in (16), we obtain

$$\sum_{k=1}^{N} \mu_k = \sum_{k=1}^{N} u_k + i \sum_{k=1}^{N} v_k$$

which shows, in terms of the mean $\Omega = \frac{1}{N} \sum_{k=1}^{N} \omega_k$ and $H = \frac{1}{N} \sum_{k=1}^{N} \eta_k$, that

$$\sum_{k=1}^{N} \operatorname{Re} \mu_{k} = \frac{N \cos \gamma}{g} \Omega + \frac{N \sin \gamma}{g} H$$
(17)

and

$$\sum_{k=1}^{N} \operatorname{Im} \mu_{k} = \frac{N \cos \gamma}{g} H - \frac{N \sin \gamma}{g} \Omega$$
(18)

Using

trace
$$\left(\mathcal{Q}^{2}\left(q_{i}\right)\right)$$
 = trace $\left(A^{2}\right)$ + trace $\left(\operatorname{diag}\left(q_{i}^{2}\right)\right)$

we have for m = 2 in (16) that

$$\sum_{k=1}^{N} \mu_k^2 = 2L + \sum_{k=1}^{N} u_k^2 - \sum_{k=1}^{N} v_k^2 + 2i \sum_{k=1}^{N} u_k v_k$$

Substituting (14) and (15), we obtain, after taking real and imaginary parts,

$$\sum_{k=1}^{N} (\operatorname{Re} \mu_{k})^{2} - \sum_{k=1}^{N} (\operatorname{Im} \mu_{k})^{2} = 2L + \frac{N \cos 2\gamma}{g^{2}} \left(\operatorname{Var} [\omega] - \operatorname{Var} [\eta] + \Omega^{2} - H^{2} \right) + \frac{2N \sin 2\gamma}{g^{2}} E[\omega\eta] \quad (19)$$

and

$$\sum_{k=1}^{N} \operatorname{Re} \mu_{k} \operatorname{Im} \mu_{k} = \frac{N \sin 2\gamma}{2g^{2}} \left(\operatorname{Var} \left[\eta \right] - \operatorname{Var} \left[\omega \right] + H^{2} - \Omega^{2} \right) + \frac{N \cos 2\gamma}{g^{2}} E \left[\omega \eta \right]$$
(20)

One computational step further for m = 3 in (16) gives

trace
$$(\mathcal{Q}^3(q_i)) = \text{trace}(\text{diag}(q_i^3)) + 2\text{trace}(\text{diag}(q_iA^2)) + \text{trace}(A\text{diag}(q_i)A) - \text{trace}(A^3)$$

where

$$\operatorname{trace}\left(\operatorname{diag}\left(q_{i}^{3}\right)\right) = \sum_{k=1}^{N} q_{k}^{3}$$
$$\operatorname{trace}\left(\operatorname{diag}\left(q_{i}A^{2}\right)\right) = \operatorname{trace}\left(A\operatorname{diag}\left(q_{i}\right)A\right) = \sum_{k=1}^{N} q_{k}d_{k}$$
$$\operatorname{trace}\left(A^{3}\right) = \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{s=1}^{N} a_{kl}a_{ls}a_{sk}$$

In particular, trace (A^3) is the number of closed walks of length 3 and equals 6 times the number of triangles in the graph, which we denote here by \blacktriangle_G . Thus, we obtain

$$\sum_{k=1}^{N} \mu_k^3 = -6 \blacktriangle_G + \sum_{k=1}^{N} u_k^3 - 3 \sum_{k=1}^{N} u_k v_k^2 + 3 \sum_{k=1}^{N} u_k d_k + 3i \sum_{k=1}^{N} v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^2 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^3 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^3 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^3 v_k d_k - i \sum_{k=1}^{N} v_k^3 + 3i \sum_{k=1}^{N} u_k^3 v_k d_k - i \sum_{k=1}^{N} v_k^3 v_k d_k - i$$

After introducing (14) and (15) and taking the real and imaginary part, we find

$$\sum_{k=1}^{N} \operatorname{Re} \mu_{k} \left((\operatorname{Re} \mu_{k})^{2} - 3 \left(\operatorname{Im} \mu_{k} \right)^{2} \right) = -6 \blacktriangle_{G} + \frac{\cos 3\gamma}{g^{3}} \sum_{k=1}^{N} \omega_{k}^{3} - \frac{\sin 3\gamma}{g^{3}} \sum_{k=1}^{N} \eta_{k}^{3} + \frac{3 \sin 3\gamma}{g^{3}} \sum_{k=1}^{N} \omega_{k}^{2} \eta_{k} - \frac{3 \cos 3\gamma}{g^{3}} \sum_{k=1}^{N} \omega_{k} \eta_{k}^{2} + \frac{3 \cos \gamma}{g} \sum_{k=1}^{N} \omega_{k} d_{k} + \frac{3 \sin \gamma}{g} \sum_{k=1}^{N} \eta_{k} d_{k} \quad (21)$$

and

$$\sum_{k=1}^{N} \operatorname{Im} \mu_{k} \left(3 \left(\operatorname{Re} \mu_{k} \right)^{2} - \left(\operatorname{Im} \mu_{k} \right)^{2} \right) = -\frac{\sin 3\gamma}{g^{3}} \sum_{k=1}^{N} \omega_{k}^{3} - \frac{\cos 3\gamma}{g^{3}} \sum_{k=1}^{N} \eta_{k}^{3} + \frac{3\cos 3\gamma}{g^{3}} \sum_{k=1}^{N} \omega_{k}^{2} \eta_{k} + \frac{3\sin 3\gamma}{g^{3}} \sum_{k=1}^{N} \omega_{k} \eta_{k}^{2} - \frac{3\sin \gamma}{g} \sum_{k=1}^{N} \omega_{k} d_{k} + \frac{3\cos \gamma}{g} \sum_{k=1}^{N} \eta_{k} d_{k} \quad (22)$$

B The constant of motion in the Kuramoto model

We compute the constant of motion in the Kuramoto model (3) for any undirected graph. Summing the first $m \leq N$ equations in (3) yields

$$\sum_{k=1}^{m} \dot{\theta}_{k} = \sum_{k=1}^{m} \omega_{k} + g \sum_{k=1}^{m} \sum_{j=1}^{N} a_{kj} \sin(\theta_{j} - \theta_{k})$$
$$= \sum_{k=1}^{m} \omega_{k} + g \sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} \sin(\theta_{j} - \theta_{k}) + \sum_{k=1}^{m} \sum_{j=1+m}^{N} a_{jk} \sin(\theta_{k} - \theta_{j})$$

Let us now change $k \to j$ and $j \to k$ in the first double sum, then

$$\sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} \sin(\theta_j - \theta_k) = \sum_{j=1}^{m} \sum_{k=1}^{m} a_{jk} \sin(\theta_k - \theta_j)$$

We invoke the symmetry in $a_{kj} = a_{jk}$, but the oddness of $\sin(\theta_k - \theta_j) = -\sin(\theta_j - \theta_k)$, and reverse the order of summation such that

$$\sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} \sin(\theta_j - \theta_k) = -\sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} \sin(\theta_j - \theta_k)$$

and conclude that this sum vanishes (because a number that obeys x = -x can only be zero). The total mutual interaction between a subset of m nodes (oscillators) in the network precisely cancels. The arguments show that this total mutual cancellation holds for any odd coupling function f(x) = -f(-x), and not only for the sinus. Thus, provided $A = A^T$ and $1 \le m \le N$, there holds for any odd coupling function that

$$\sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} f\left(\theta_{j} - \theta_{k}\right) = 0$$
(23)

Hence, for any odd coupling function f, we arrive at

$$\sum_{k=1}^{m} \dot{\theta}_{k} = \sum_{k=1}^{m} \omega_{k} + \sum_{k=1}^{m} \sum_{j=1+m}^{N} a_{jk} f \left(\theta_{k} - \theta_{j}\right)$$

where the last sum reflects the interactions that the group of m oscillators experience from the other oscillators in the network. When m = N, we deduce that

$$\sum_{k=1}^{N} \dot{\theta}_k = \sum_{k=1}^{N} \omega_k \tag{24}$$

which leads to

$$\sum_{k=1}^{N} \theta_k(t) = N\Omega t + c \tag{25}$$

where the constant $c = \sum_{k=1}^{N} \theta_k(0)$ is the initial sum of all phases. The transform $\tilde{\theta}_k = \theta_k + \Omega t$ yields

$$\frac{d}{dt}\left(\sum_{k=1}^{N}\widetilde{\theta}_{k}\right) = 0$$

This means that the aggregate of all oscillator phases with respect to the mean frequency Ω does not change over time. As Strogatz [7] remarks, we can set $\Omega = 0$ due to the rotational symmetry in the model and $\tilde{\theta}_k$ is the frequency of oscillator k in a rotating coordinate frame at frequency Ω .