Fractional Derivative in Markov Theory

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in collaboration with
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Outline

Markovian epidemics on networks

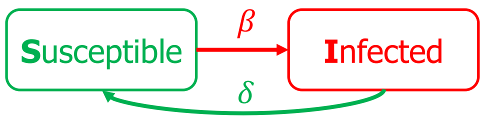
Fractional derivative

Non-Markovian epidemic on networks

Epidemic compartments

Single disease realization

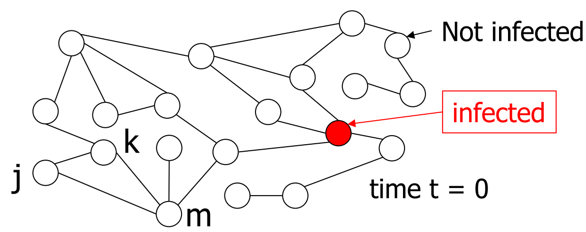
Diseases with re-infections



Essence: item can be only in 1 compartment at time t

SIS Virus spread in networks

Given:



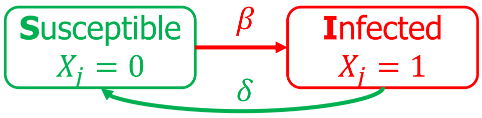
Infection process: Poisson with infection strength β_{jk}
Curing process: Poisson with curing strength δ_j

Compute: Probability that node j is infected at time $t > 0$

Assumptions:

1. SIS model: only 2 compartments: S & I
2. graph is static (not time-varying) and known
3. all processes are independent Poisson processes
4. infection and curing have constant strength (not time-varying, no mutations)

Markovian SIS epidemics in networks



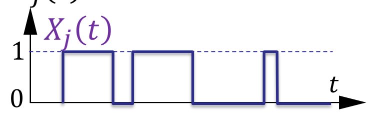
Markov state $X_j \in \{0,1\}$ of node j is a **Bernoulli random variable**

$$\Pr[X_j(t) = 1] = E[X_j(t)]$$

Each node j possesses a health state $X_j(t)$ at time t:

$X_j(t) = 0$: node j is not-infected at time t

$X_j(t) = 1$: node j is infected at time t



Infection probability of node j at time t: $v_j(t) = \Pr[X_j(t) = 1]$

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Infection process: Poisson with infection strength $\beta_{jk} = \beta$ (per link)

Curing process: Poisson with curing strength $\delta_j = \delta$ (per node)

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Continuous-time Markov process

Chapman-Kolmogorov governing equation of a continuous-time Markov process with m states is a linear differential equation:

$$\frac{ds(t)}{dt} = -Qs(t)$$

with solution $s(t) = e^{-Qt}\, s(0)$

where

$s(t)$: $m \times 1$ vector with $s_i(t) = \Pr[X(t) = i]$

Q: $m \times m$ weighted, directed Laplacian of the Markov graph

= - infinitesimal generator

Epidemics with c compartments in a graph with N nodes: $m = c^N$

All transition times (infection, curing, etc.) are exponential random variables

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Sahneh, F. D., C. Scoglio and P. Van Mieghem, 2013, "Generalized Epidemic Mean-Field Model for Spreading Processes over Multi-Layer Complex Networks", IEEE/ACM Transactions on Networking, Vol. 21, No. 5, pp. 1609-1620.

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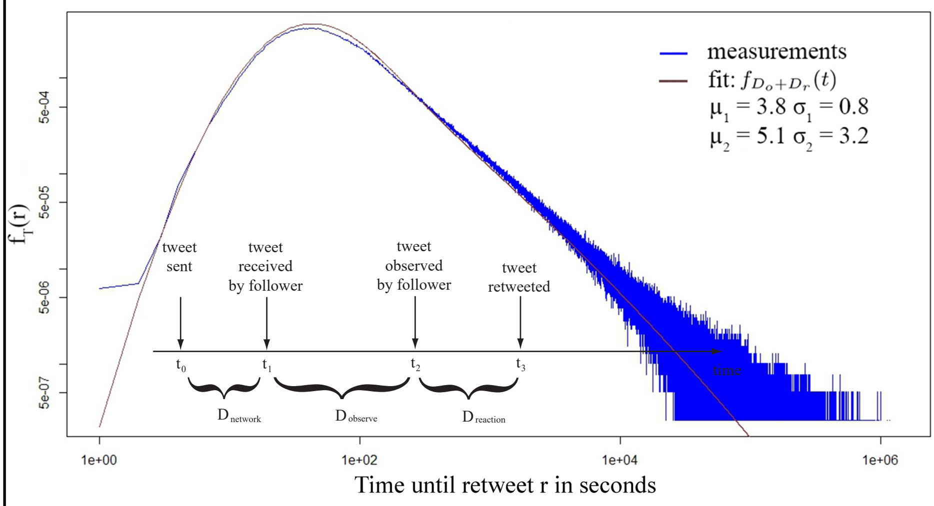
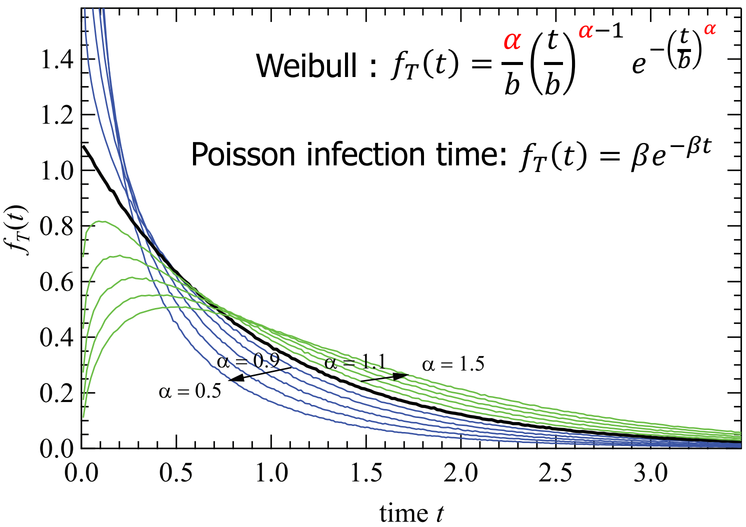
Twitter Epidemic times are not exponential



C. Doerr, N. Blenn and P. Van Mieghem, "Lognormal infection times of Online information spread", PLOS ONE, Vol. 8, No. 5, p. e64349, 2013

Non-Markovian infection times



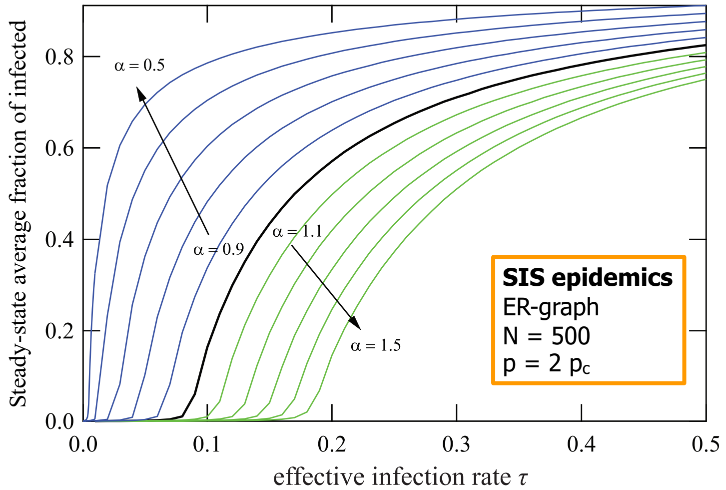
α: shape parameter

Each Weibull distribution has the same mean $E[T] = \frac{1}{\beta}$

$$\frac{1}{b} = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)$$

T is time to infect a neighboring node
f_T(t) is probability density of T: $\Pr[t \leq T \leq t + \Delta t]$

Non-Markovian epidemic threshold



Non-exponential infection time has a dramatic influence!

P. Van Mieghem and R. van de Bovenkamp, "Non-Markovian infection spread dramatically alters the SIS epidemic threshold", Physical Review Letters, vol. 110, No. 10, March 2013, p. 108701.

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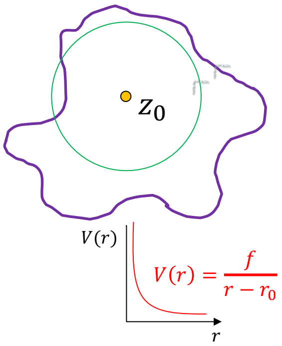
Cauchy Integral



$$f(z_0) = \frac{1}{2\pi i} \int_{C(z_0)} \frac{f(z)}{z - z_0} dz$$

Evaluation along a circle
With radius R centered at z_0

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta$$

Evaluation along a circle $f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{Re^{i\theta}} d(Re^{i\theta})$

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k-th derivate of a function

$$\left. \frac{d^k f(w)}{dw^k} \right|_{w=z_0} = \frac{k!}{2\pi i} \int_{C(z_0)} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

Formal extension of integer k to complex α

$$\left. \frac{d^\alpha f(w)}{dw^k} \right|_{w=z_0} = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{C(z_0)} \frac{f(z)}{(z - z_0)^{\alpha+1}} dz$$

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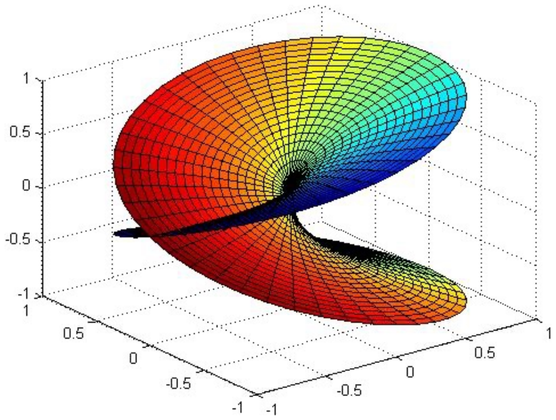
Riemann surface of $f(z) = z^{\alpha} = e^{\alpha \log(z)}$



If $\alpha \notin \mathbb{Z}$, then the negative real axis is a branch cut in complex plane

Wikipedia: Riemann surface for the function $f(z) = \sqrt{z}$, thus $\alpha = \frac{1}{2}$. The two horizontal axes represent the real and imaginary parts of z, while the vertical axis represents the real part of $\sqrt{z}$. The imaginary part of $\sqrt{z}$ is represented by the coloration of the points.

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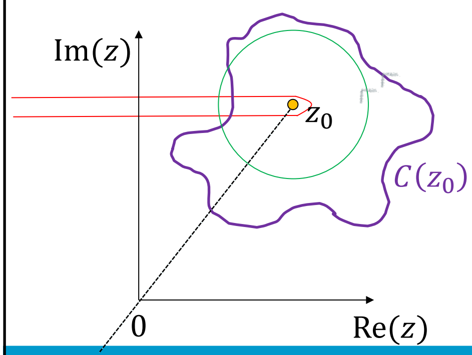
α-fractional derivate of a function

$$\left.\frac{d^{\alpha} f(w)}{dw^{k}}\right]_{w=z_0} = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C(z_0)} \frac{f(z)}{(z-z_0)^{\alpha+1}} dz$$



$(z - z_0)^{\alpha}$ is not defined if $z - z_0$ is a negative real number, which creates a branch cut

Area enclosed by the contour $C(z_0)$ is not analytic! Contour cannot be closed over branch cut

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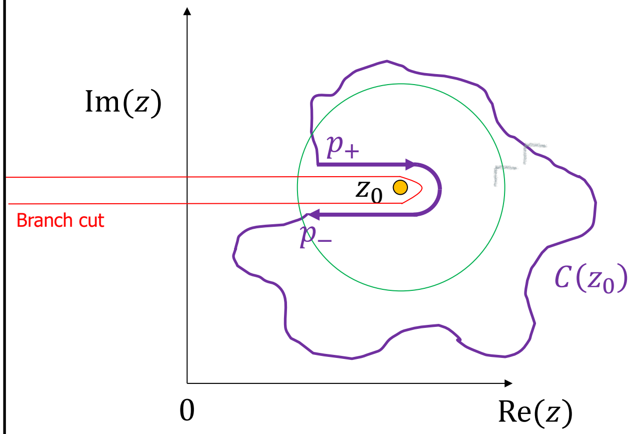
Deformation of the contour



Contour $C(z_0)$ depends upon the points p_+ and p_-

Infinitely many ways to define an $\boldsymbol{\alpha}$-fractional derivate !

$$D^{\alpha}_{p;m} f(t) = \left.\frac{d^{\alpha} f(z)}{dz^{\alpha}}\right|_{z=t} = \frac{1}{\Gamma(m-\alpha)}\int_p^t \frac{f^{(m)}(x)}{(t-x)^{\alpha+1-m}}dx \quad \text{for} \quad \Re(\alpha) < m$$

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Van Mieghem, P., "Origin of the fractional derivative and fractional non-Markovian continuous-time processes", Physical Review Research, Vol 4, No. 2, June 2022, p. 023242.

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Fractional Calculus for non-Markovian epidemics?

Chapman-Kolmogorov governing equation of a continuous-time Markov process with *N* states:

$$\frac{ds(t)}{dt} = -Qs(t)$$

with solution $s(t)=e^{-Qt}\, s(0)$

s(t): N x 1 vector with $Pr[X(t) = i]$

Q: N x N weighted Laplacian Markov graph = - infinitesimal generator

$0 < \alpha \leq 1$ Replace $\mathrm{D} = \frac{d}{dt}$ by Caputo fractional derivative D^{α}

$$D^{\alpha}\, s_{\alpha}(t) = -Q^{\alpha} s_{\alpha}(t)$$

with solution $s_{\alpha}(t)= E_{\alpha,1}(-(Qt)^{\alpha})\; s_{\alpha}(0)$

$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}$: Mittag-Leffler function

$E_{1,1}(z) = e^{z}$

Open problem: physical explanation of α-fractional non-Markovian process

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Van Mieghem, P., "Origin of the fractional derivative and fractional non-Markovian continuous-time processes", Physical Review Research, Vol 4, No. 2, June 2022, p. 023242.

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General *m* Caputo fractional derivative

$$D^{\alpha}_{p;m} f(t) = \left.\frac{d^{\alpha} f(z)}{dz^{\alpha}}\right|_{z=t} = \frac{1}{\Gamma(m-\alpha)} \int_p^t \frac{f^{(m)}(x)}{(t-x)^{\alpha+1-m}} dx \quad \text{for} \quad \Re(\alpha) < m$$

Fractional α-extension of classical Chapman-Kolmogorov governing equation of a continuous-time Markov process

$D^{\alpha}_{0;m}\, s(t) = -Q^{\alpha} s(t)$ Initial conditions: $\left\{s_{\alpha}^{(n)}(0)\right\}_{0 \leq n < m}$

Solution: $s_{\alpha}(t) = \sum_{n=0}^{m-1} t^n\, E_{\alpha,n+1}(-(Qt)^{\alpha}) s_{\alpha}^{(n)}(0)$ $\left(s_{\alpha}(t)\right)_i = \Pr[X_{\alpha}(t) = i]$

Axiom probability theory: $\sum_{i=1}^{N} \Pr[X_{\alpha}(t) = i] = 1$ ⟹ $u^T\; s_{\alpha}(t) = 1$

Necessary condition (with $Qu = 0$): $u^T s_{\alpha}\,(0) = 1$ and $u^T s_{\alpha}^{(n)}(0) = 0$ for $n > 0$

Necessary condition is insufficient for all α (proof by counter example)

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Van Mieghem, P., "Origin of the fractional derivative and fractional non-Markovian continuous-time processes", Physical Review Research, Vol 4, No. 2, June 2022, p. 023242.

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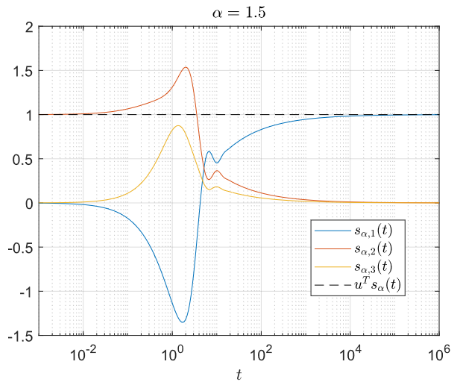
Range $0 < \alpha \leq 1$ for probability theory



The solution $s_\alpha(t)$ can only be a probability if $m = 1$ which implies $0 < \alpha \leq 1$

Mittag-Leffler function $E_{\alpha,1}(z)$ for real $z = -t$ is monotonic $E_{\alpha,1}(-t) > 0$ if $0 < \alpha \leq 1$

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Van Mieghem, P., 2020, "The Mittag-Leffler function", Delft University of Technology, report20200528 (http://arxiv.org/abs/2005.13330).

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Linear time-dependent first-order diff. equation

$\alpha -$ fractional (non-Markovian) process with N states:

$D^\alpha\, s_\alpha(t) = -Q^\alpha s_\alpha(t)$ with solution $s_\alpha(t) = E_{\alpha,1}(-(Qt)^\alpha)\, s_\alpha(0)$

equivalent

Assumption: Q is diagonalizable,
$Q^\alpha = \sum_{k=1}^{N} \mu_k^\alpha x_k y_k^T$

$$\frac{ds(t)}{dt} = -Q(t;\alpha)s(t)$$

$$Q(t;\alpha) = \sum_{k=1}^{N} \mu_k(t;\alpha) x_k y_k^T = XM(t;\alpha)Y^T$$

$$\mu_k(t;\alpha) = t^{\alpha-1}\frac{E_{\alpha,\alpha}(-\mu_k^\alpha t^\alpha)}{E_{\alpha,1}(-\mu_k^\alpha t^\alpha)}$$

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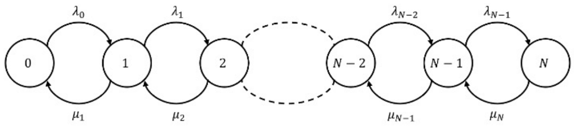
ε -SIS Epidemics on complete graph K_N

Birth-death process



$$\frac{ds(t)}{dt} = -Qs(t) \qquad Q = \begin{pmatrix} \lambda_0 & -\mu_1 & & & & \\ -\lambda_0 & \lambda_1+\mu_1 & -\mu_2 & & & \\ & -\lambda_1 & \lambda_2+\mu_2 & \ddots & & \\ & & \ddots & & & \\ & & \ddots & & -\mu_{N-1} & \\ & & & -\lambda_{N-2} & \lambda_{N-1}+\mu_{N-1} & -\mu_N \\ & & & & -\lambda_{N-1} & \mu_N \end{pmatrix}$$

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P. Van Mieghem, 2014, Performance Analysis of Complex Networks And Systems, Cambridge University Press; Sec. 17.6

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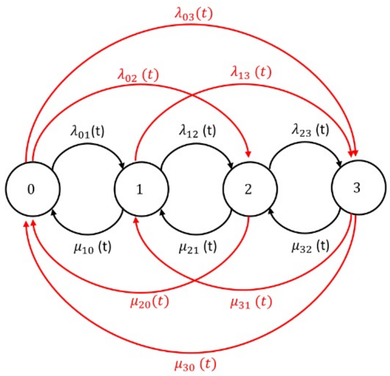
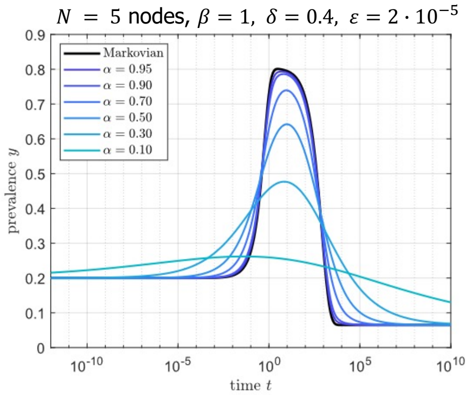
α –fractional ε -SIS Epidemics on K_N



N = 5 nodes, $\beta = 1$, $\delta = 0.4$, $\varepsilon = 2 \cdot 10^{-5}$



$$\frac{d}{dt} s_\alpha(t) = -Q(t;\alpha) s_\alpha(t) \qquad \begin{cases} Q(t;\alpha)_{ii} = \sum_{j<i} \mu_{ij}(t;\alpha) + \sum_{j>i} \lambda_{ij}(t;\alpha) & \text{for } i = 0,\dots,N \\ Q(t;\alpha)_{ji} = -\lambda_{ij}(t;\alpha) & \text{for } i < j = 0,\dots,N \\ Q(t;\alpha)_{ji} = -\mu_{ij}(t;\alpha) & \text{for } i > j = 0,\dots,N \end{cases}$$

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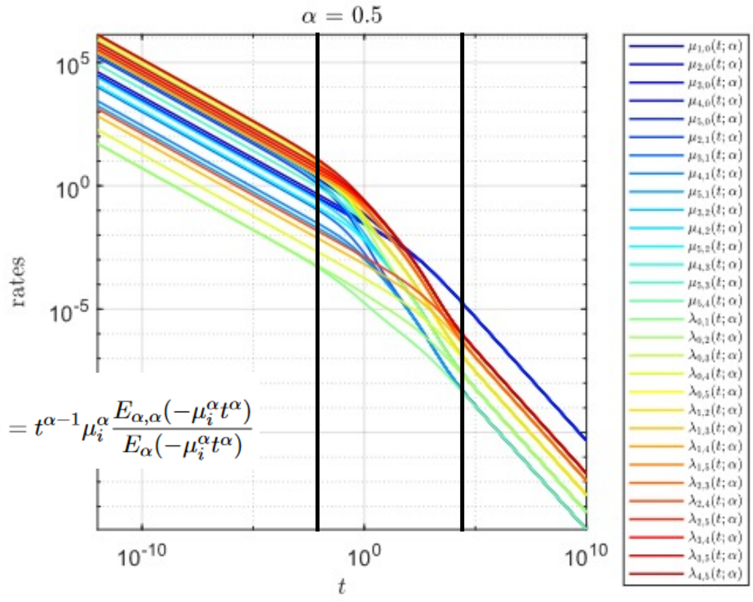
Time-dependent matrix elements



$$\mu_i(t;\alpha) = t^{\alpha-1}\mu_i^\alpha \frac{E_{\alpha,\alpha}(-\mu_i^\alpha t^\alpha)}{E_\alpha(-\mu_i^\alpha t^\alpha)}$$

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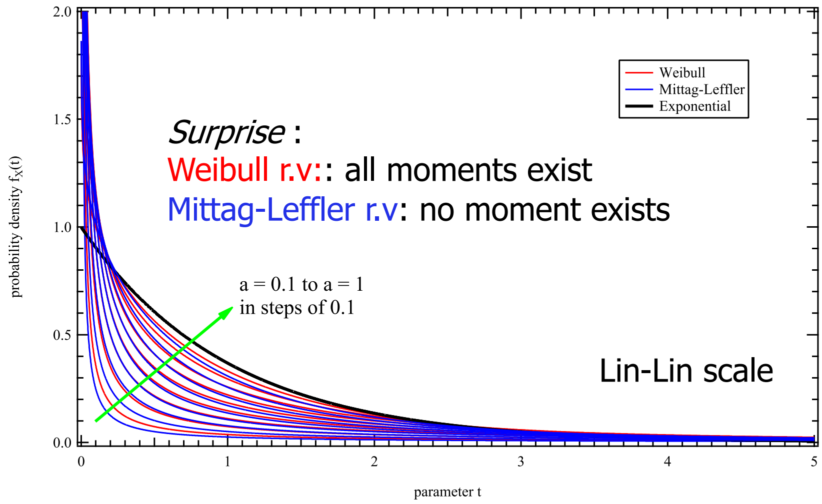
Weibull and Mittag-Leffler pdf



Surprise :

Weibull r.v:: all moments exist

Mittag-Leffler r.v: no moment exists

Lin-Lin scale

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Van Mieghem, P., 2020, "The Mittag-Leffler function", Delft University of Technology, report20200528 (http://arxiv.org/abs/2005.13330).

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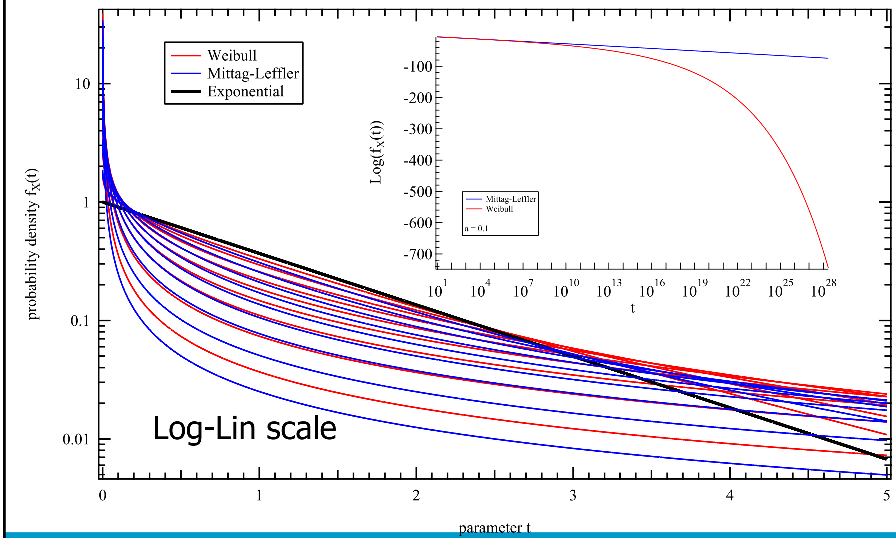
Weibull and Mittag-Leffler pdf



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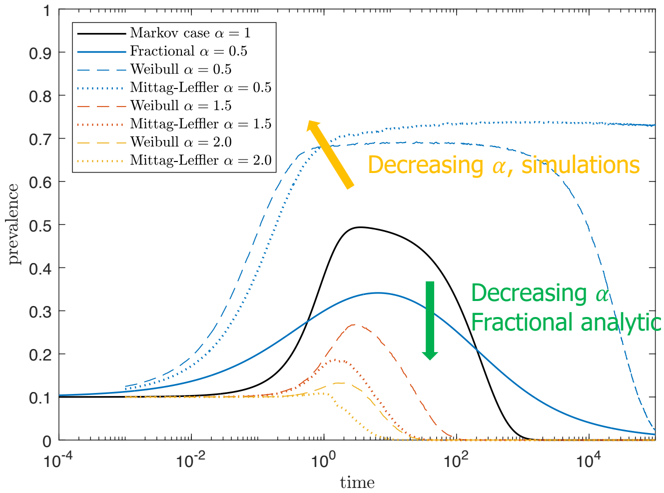
Fractional theory & SIS simulations



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Conclusions

- Real epidemics are (very) likely non-Markovian
- We do not understand Non-Markovian fractional epidemics:
  - Exact mathematical theory is attractive
  - However, the "physics" is lagging behind

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