

Network Science

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Meet IEEE Fellows
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Nokia, Antwerpen

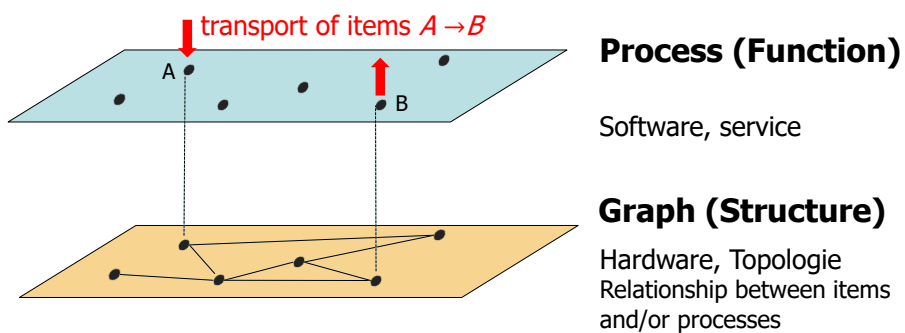
Network Architectures and Services

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Network = Process + Graph



Network Science: Theory of processes on/in graphs



Duality between **process** and **graph** is cornerstone

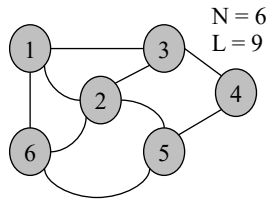
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Three equivalent representations of an undirected graph

Topology domain



$$A_{N \times N} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

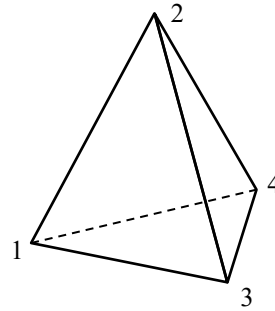
Spectral domain

$$A = A^T = X \Lambda X^T$$

$X_{N \times N}$: orthogonal
eigenvector matrix

$\Lambda_{N \times N}$: diagonal
eigenvalue matrix

Geometric domain



Each undirected graph
with N nodes
= a simplex in Euclidean
(N-1)-dimensional space

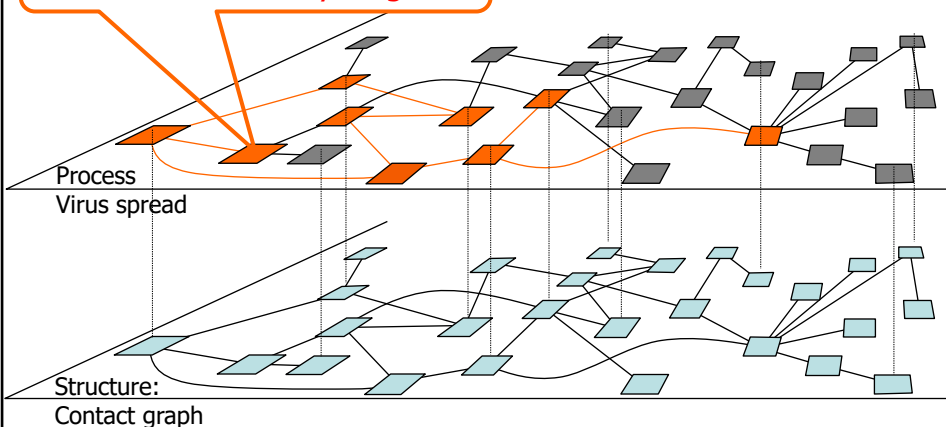
Devriendt, K. and P. Van Mieghem, "The Simplex Geometry of Graphs",
Journal of Complex Networks, Volume 7, Issue 4, pp. 469–49 August 2019.
(<http://arxiv.org/abs/1807.06475>).



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Local Rule – Global Emergent behavior

While infected until recovered
then do infect healthy neighbors



LRGE dynamics:

$$\frac{dx_i(t)}{dt} = f_i(x_i(t)) + \sum_{j=1}^N a_{ij} g(x_i(t), x_j(t))$$



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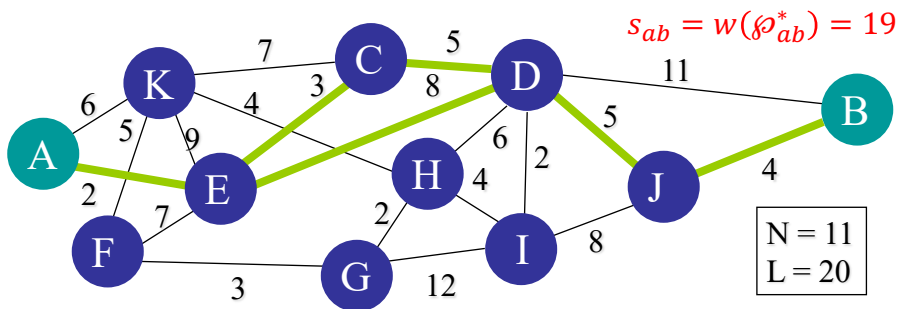
Outline



Inverse **linear** process:
Inverse all shortest path problem

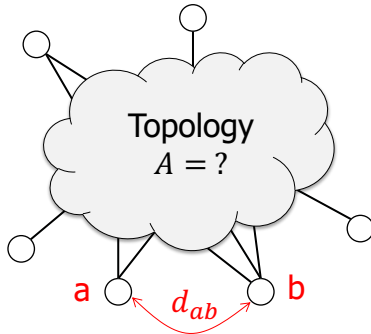
Inverse **non-linear** process:
Prediction of LRGE dynamics

Shortest Path Routing



- Link metric is non-negative and *additive* (e.g. delay, cost,...)
- Weight of path ϕ_{ij} is $w(\phi_{ij}) = \sum_{l \in \phi_{ij}} w_l$
- The shortest path ϕ_{ij}^* is minimizer of $s_{ij} = w(\phi_{ij}^*) \leq w(\phi_{ij})$

Inverse All Shortest Path Problem (IASPP)



Given an $N \times N$ demand matrix D
find the *weighted adjacency matrix* A
subject to

1) each shortest path weight obeys

$$s_{ab} \leq d_{ab}$$

for any node pairs (a, b) in the graph
with N nodes

2) $\|D - S\|$ is minimized

Motivation:

- end-to-end delay (QoS) in telecommunications
- travel times in transportation
- seismic tomograph (earthquakes) of geologic zones
- EEG/MEG in human brain

$$s_{ab} = w(\varphi_{ab}^*) = \sum_{l \in \varphi_{ab}^*} w_l$$

D: not necessarily a distance matrix
S: always a distance matrix

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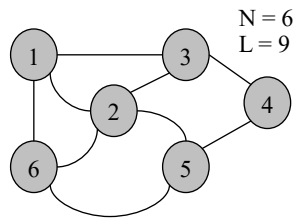
Function of network

- Usually, the function of a network is related to the *transport of items over its underlying graph*
- In man-made infrastructures: two major types of transport
 - Item is a **flow** (e.g. electrical current, water, gas,...)
 - Item is a **packet** (e.g. IP packet, car, container, postal letter,...)
- **Flow equations and physical laws** determine transport
Maxwell equations, Kirchhoff & Ohm, hydrodynamics, Navier-Stokes equation (turbulent, laminar flow equations, etc.), epidemic spread, ...
- **Protocols** determine transport of packets (IP protocols and IETF standards, car traffic rules, etc.)

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Laplacian matrix Q



$$Q_{N \times N} = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & -1 & 3 \end{bmatrix}$$

Degree of node $d_i = \sum_{k=1}^N a_{ki}$

$$Q = \Delta - A \quad A : \text{adjacency matrix and } \Delta = \text{diag}(d_1 \ d_2 \ \dots \ d_N)$$

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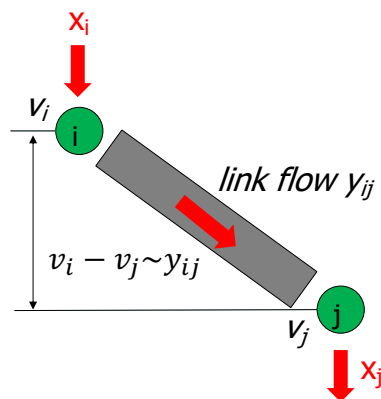
P. Van Mieghem, Graph Spectra of Complex Networks, 2nd edition, Cambridge University Press, 2023 (to appear)



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Linear dynamics on flow networks

Linear dynamic process: "proportional to" (\sim) graph of network



Examples:

- water (or gas) flow \sim pressure
- displacement (in spring) \sim force
- heat flow \sim temperature
- electrical current \sim voltage

$$x = Q \cdot v$$

injected nodal current vector	weighted Laplacian of the graph	nodal potential vector
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Inverse of $x = Qv$ is $v = Q^\dagger x$ subject to $u^T v = 0$ (average potential is zero)
 Q^\dagger is the pseudoinverse of the Laplacian matrix

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P. Van Mieghem, K. Devriendt and H. Cetinay, 2017, "Pseudoinverse of the Laplacian and best spreader node in a network", Physical Review E, vol. 96, No. 3, p 032311.



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The diagram shows a resistor network on a plane with nodes a , b , and k . Resistors are connected between nodes, with one resistor labeled r_{lk} . Currents I_c are shown entering node a and leaving node b . Below this, the effective resistance ω_{ab} is shown as a yellow cloud between nodes a and b . The voltage across it is $v_a - v_b = I_c \omega_{ab}$. The formula $\omega_{ab} = Q_{aa}^\dagger + Q_{bb}^\dagger - 2Q_{ab}^\dagger$ is also provided.

The effective resistance matrix Ω is the **flow analogon** of the shortest path weight matrix S

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The famous Fiedler's block inverse

$$\begin{bmatrix} 0 & u^T \\ u & \Omega \end{bmatrix}^{-1} = \begin{bmatrix} -2\sigma^2 & p^T \\ p & -\frac{1}{2}Q \end{bmatrix} \quad \Omega p = 2\sigma^2 u$$

there the Laplacian $Q = \Delta - A$ and $\Delta = \text{diag}(d)$
Applying inverse block matrix formulae,

$$p = \frac{1}{u^T \Omega^{-1} u} \Omega^{-1} u \quad 2\sigma^2 = \frac{1}{u^T \Omega^{-1} u}$$

$$A = \Delta + 2\Omega^{-1} - \frac{1}{\sigma^2} p p^T$$

Inverse **flow** problem is exactly solvable!

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Van Mieghem, P., 2021, "A tree realization of a distance matrix: the inverse shortest path problem with a demand matrix generated by a tree", Delft University of Technology, report20211012.

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From flow to path networks

Path networks are more confined than flow networks: $\omega_{ij} \leq s_{ij}$

IASPP: Demand (end-to-end delay) constraint: $s_{ij} \leq d_{ij}$

Wrong inequality!

(we want to replace ω_{ij} by d_{ij} ; in fact, $\omega_{ij} \leq d_{ij}$)

IASPP:

- is NP-complete
- Fiedler's block inverse does not seem applicable;
(except for trees: flow = path)
- Spectral **sparsification** (current research)

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Shortest path weight: $s_{ij} = w(\rho_{ij}^*) = \sum_{l \in \rho_{ij}^*} w_l$



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Outline



Inverse **linear** process:

Inverse all shortest path problem

Inverse **non-linear** process:

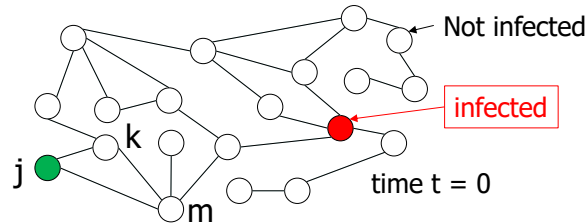
Prediction of LRGE dynamics



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SIS Virus spread in networks

Given:



Infection process: Poisson with infection strength β_{jk}
 Curing process: Poisson with curing strength δ_j

Compute: Probability that node j is infected at time $t > 0$

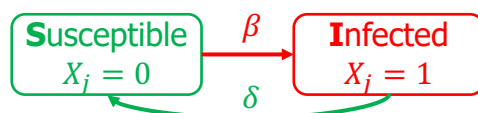
Assumptions:

1. SIS model: only 2 compartments: S & I
2. graph is static (not time-varying) and known
3. all processes are independent Poisson processes
4. infection and curing have constant strength (not time-varying, no mutations)

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Markovian SIS epidemics in networks



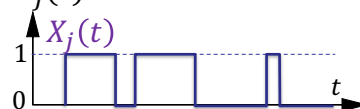
Markov state $X_j \in \{0,1\}$ of node j is a **Bernoulli random variable**

$$\Pr[X_j(t) = 1] = E[X_j(t)]$$

Each node j possesses a health state $X_j(t)$ at time t :

$X_j(t) = 0$: node j is not-infected at time t

$X_j(t) = 1$: node j is infected at time t

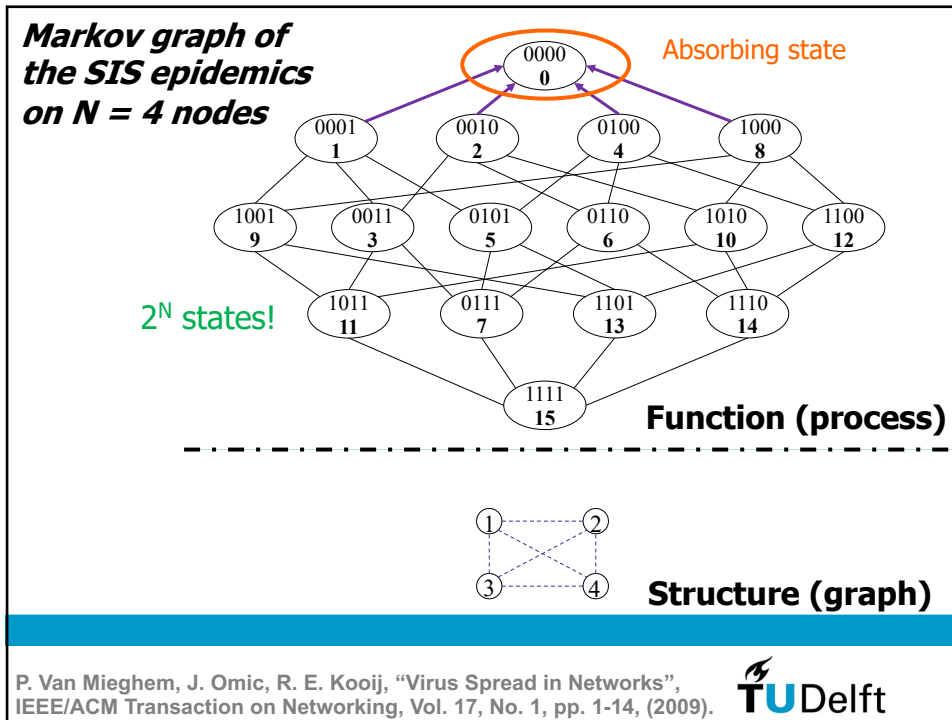


Infection probability of node j at time t : $v_j(t) = \Pr[X_j(t) = 1]$

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Infection process: Poisson with infection strength $\beta_{jk} = \beta$ (per link)
 Curing process: Poisson with curing strength $\delta_j = \delta$ (per node)

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Governing Markovian SIS equation for node j

$$\frac{dE[X_j(t)]}{dt} = E \left[-\delta X_j(t) + (1 - X_j(t)) \left\{ \beta \sum_{k \in \text{neighbor}(j)} X_k(t) \right\} \right]$$

time-change of
 $E[X_j] = \Pr[X_j = 1]$
probability that
node j is infected

if *infected* ($X_j = 1$):
probability of
curing per
unit time

if *not infected* ($X_j = 0$):
probability of infection per
unit time from
infected neighbors

$$\frac{dE[X_j]}{dt} = -\delta E[X_j] + \beta \sum_{k=1}^N a_{kj} E[X_k] - \beta \sum_{k=1}^N a_{kj} E[X_j X_k]$$

Complication
 $E[X_j X_k] = \Pr[X_j = 1, X_k = 1]$

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R. Pastor-Satorras, C. Castellano, P. Van Mieghem and A. Vespignani, "Epidemic processes in complex networks", Review of Modern Physics, Vol. 87, No. 3, pp. 925-979, 2015 **TU Delft**

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Mean-field approximation: replace random variable by its mean

NIMFA



$$\frac{dE[X_j]}{dt} = E \left[-\delta X_j + (1 - X_j)\beta \sum_{k=1}^N a_{kj} X_k \right]$$



$$X_j \Rightarrow E[\check{X}_j] = w_j$$

$$\frac{dw_j}{dt} = -\delta w_j + (1 - w_j)\beta \sum_{k=1}^N a_{kj} w_k$$

From 2^N linear Markov differential equations to
 N non-linear mean-field **approximating** diff. equations

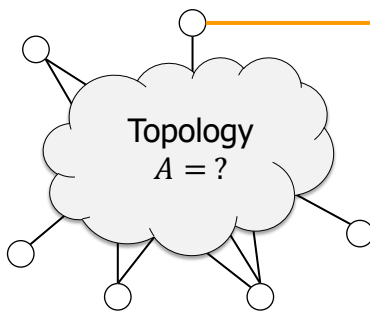
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P. Van Mieghem, "The N -Intertwined SIS epidemic network model",
Computing (Springer), Vol. 93, Issue 2, p. 147-169, 2011

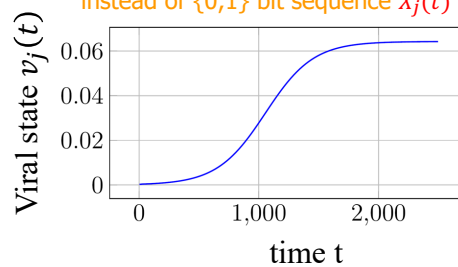


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Graph Reconstruction from epidemics



Viral state observation = infection
probability $v_j(t) = \Pr[X_j(t) = 1]$ over time,
instead of $\{0,1\}$ bit sequence $X_j(t)$



Aim: Determine the $N \times N$ adjacency matrix A of the contact graph
from a series of infection probabilities over time of all nodes

Solution: **only partially possible**

Prasse, B. and P. Van Mieghem, 2018, "Exact Network Reconstruction
from Complete SIS Nodal State Infection Information Seems Infeasible",
IEEE Transactions on Network Science and Engineering, Vol. 6, No. 4,
October-December, pp. 748-759.



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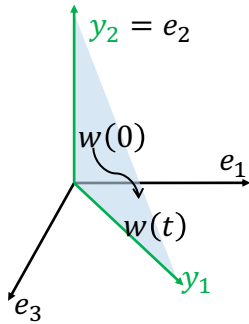
Agitation modes

Proper orthogonal decomposition (POD) of the viral state vector

$$w(t) \approx \sum_{i=1}^m c_i(t) y_i$$

y_1, \dots, y_m : orthonormal agitation modes

$$c_i(t) = y_i^T w(t): \text{scalar, projection of } w(t) \text{ on } y_i$$



If the POD is accurate, we do not need N differential equations:

$$N \text{ differential equations } \frac{dw_i(t)}{dt} = f_{\text{SIS},i}(w(t)), \quad i = 1, \dots, N$$

↓ Projection on agitation modes y_l

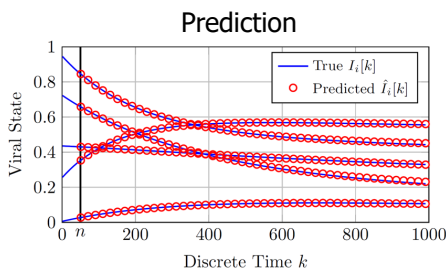
$$m < N \text{ differential equations } \frac{dc_l(t)}{dt} \approx y_l^T f_{\text{SIS}}(\sum_{i=1}^m c_i(t) y_i), \quad l = 1, \dots, m$$

Prasse, B. and P. Van Mieghem, 2022, "Predicting network dynamics without requiring the knowledge of the interaction graph", Proceedings of the National Academy of Sciences (PNAS), Vol. 119, No. 44, e2205517119

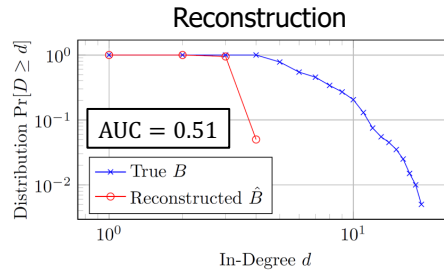


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Accurate prediction of epidemic outbreaks *without* accurate network reconstruction



Process: only a few agitation modes



Graph: nearly all eigenmodes

Basis of the **Network Inference Prediction Algorithm (NIPA)**

Real-time data loading from RIVM (Dutch ministry of health):

<https://www.nas.ewi.tudelft.nl/nipa/covid-prediction>

B. Prasse and P. Van Mieghem, 2020, "Network Reconstruction and Prediction of Epidemic Outbreaks for General Group-Based Compartmental Epidemic Models", IEEE Transactions on Network Science and Engineering, Vol. 7, No. 4, October-December, pp. 2755-2764



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Summary: Inverse problems

- Inverse all shortest path problem (IASPP):
 - Broad application range
 - Exact solution for flow networks
 - effective resistance matrix for any graph are one-to-one coupled with the adjacency matrix (Fiedler's block inverse)
 - Path networks are generally challenging (NP-complete)
 - *Current work* : construct *approximate* algorithms for IASSP
- Prediction of "local-rule, global emergent" dynamics:
 - Possible without knowing the (assumed fixed) interaction graph!
 - Explanation of success of "deep learning methods"
 - Autonomous dynamic in high dimensions only evolves in a small subspace

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Thank You

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