Epidemics on Networks

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in collaboration with
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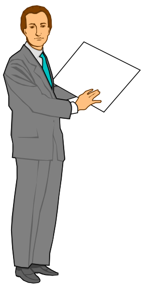
Outline



Exact SIS model

NIMFA: N-Intertwined MF Approximation

Recent developments

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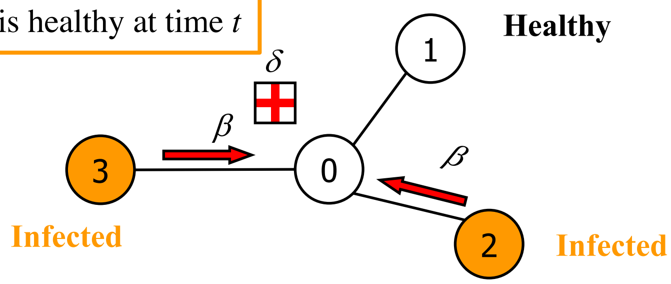
Continuous-time Markovian SIS epidemics on networks

- Constant infection rate β on all links
- Constant curing rate δ for all nodes

$\tau = \beta / \delta$: effective spreading or infection rate

$$\boxed{\begin{array}{ll} X_j(t) = 1 & \text{node } j \text{ is infected at time } t \\ X_j(t) = 0 & \text{node } j \text{ is healthy at time } t \end{array}}$$



Infection and curing are independent Poisson processes

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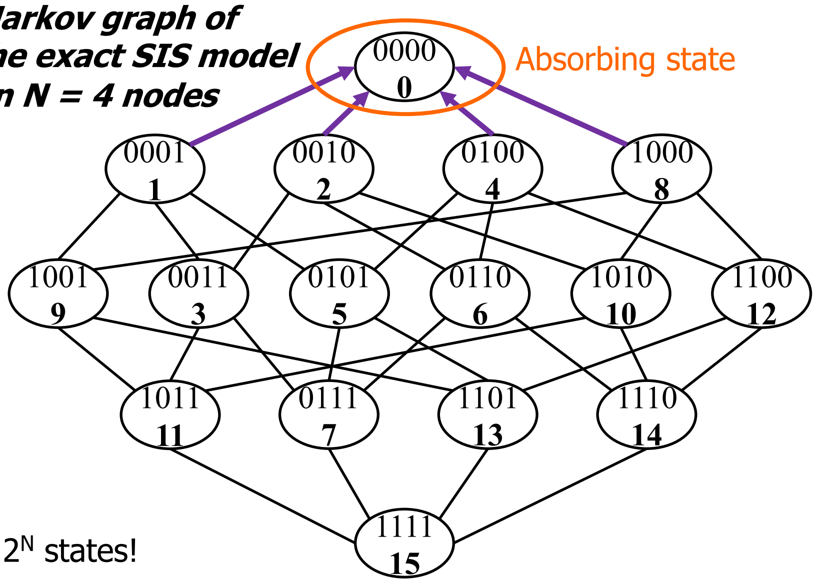
Markov graph of the exact SIS model on N = 4 nodes



2^N states!

P. Van Mieghem, J. Omic, R. E. Kooij, “Virus Spread in Networks”, IEEE/ACM Transaction on Networking, Vol. 17, No. 1, pp. 1-14, (2009).

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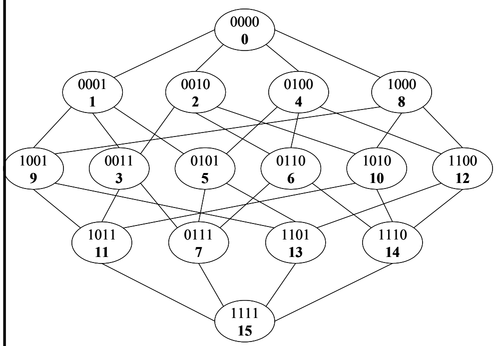
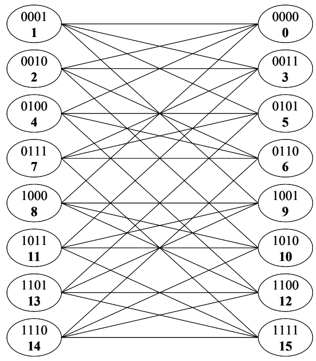
Markov theory



Regular bipartite Markov graph



Recursive structure of infinitesimal general Q_N

A. Economou, A. Gómez-Corral, M. López-García, A stochastic SIS epidemic model with heterogeneous contacts, Physica A, Volume 421, 1 March 2015, Pages 78-97

Simon, P., M. Taylor and I. Z. Kiss, Exact epidemic models on graphs using graph-automorphism driven lumping, Mathematical Biology, Vol. 62, pp. 479-508, 2011

Van Mieghem, P. and E. Cator, ε-SIS epidemics and the epidemic threshold, Physical Review E, vol. 86, No. 1, July, p. 016116, 2012

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Governing SIS equation for node *j*

$$\frac{dE[X_j]}{dt} = E\left[-\delta X_j + (1-X_j)\beta\sum_{k=1}^{N} a_{kj}X_k\right]$$

time-change of $E[X_j] = \Pr[X_j = 1]$, probability that node j is infected

if *infected*: probability of curing per unit time

if *not infected (healthy)*: probability of infection per unit time

$$\frac{dE[X_j]}{dt} = -\delta E\left[X_j\right] + \beta\sum_{k=1}^{N} a_{kj}E\left[X_k\right] - \beta\sum_{k=1}^{N} a_{kj}E\left[X_jX_k\right]$$

R. Pastor-Satorras, C. Castellano, P. Van Mieghem and A. Vespignani, "Epidemic processes in complex networks", Review of Modern Physics, Vol. 87, No. 3, pp. 925-979, 2015

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Outline

Exact SIS model

NIMFA: N-Intertwined MF Approximation

Recent developments

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NIMFA: replace rv by its mean

$$\frac{dE[X_j]}{dt} = E\left[-\delta X_j + (1 - X_j)\beta \sum_{k=1}^{N} a_{kj} X_k\right]$$

NIMFA $\Big\downarrow$ $\boxed{X_j \Rightarrow E\left[\breve{X}_j\right]}$

$$\frac{dE\left[E\left[\breve{X}_j\right]\right]}{dt} = E\left[-\delta E\left[\breve{X}_j\right] + (1 - E\left[\breve{X}_j\right])\beta \sum_{k=1}^{N} a_{kj} E\left[\breve{X}_k\right]\right]$$

Bernoulli rv $\Big\downarrow$ $\boxed{E\left[\breve{X}_j\right] = \Pr\left[\breve{X}_j = 1\right] = v_j}$

$$\boxed{\frac{dv_j}{dt} = -\delta v_j + (1 - v_j)\beta \sum_{k=1}^{N} a_{kj} v_k}$$

in normalized time $t^* = \delta t$:

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$$\frac{dv_j(t^*)}{dt^*} = -v_j(t^*) + \tau\left(1 - v_j(t^*)\right) \sum_{k=1}^{N} a_{kj}\, v_k(t^*) \qquad \tau = \frac{\beta}{\delta}$$

NIMFA: N-intertwined mean-field approxim.

$$\frac{dE[X_j]}{dt} = -\delta E[X_j] + \beta \sum_{k=1}^{N} a_{kj} E[X_k] - \beta \sum_{k=1}^{N} a_{kj} E[X_j X_k]$$

$$\Rightarrow \quad Cov[X_j X_k] = E[X_j X_k] - E[X_j] E[X_k] \geq 0$$

E. Cator and P. Van Mieghem, 2014, "Nodal infection in Markovian SIS and SIR epidemics on networks are non-negatively correlated," Physical Review E, Vol. 89, No. 5, p. 052802.
E. Cator, P. Donnelly and P. Van Mieghem, 2018, "Reply to "Comment on 'Nodal infection in Markovian SIS and SIR epidemics on networks are non-negatively correlated' " ", Physical Review E, Vol. 98, No. 2, August, p. 026302.

$$\frac{dE[X_j]}{dt} = \boxed{-\underbrace{\delta E[X_j] + \beta (1 - E[X_j]) \sum_{k=1}^{N} a_{kj} E[X_k]}_{\text{NIMFA: upper bounds SIS}}} - \underbrace{\beta \sum_{k=1}^{N} a_{kj} Cov[X_j X_k]}_{R_j > 0}$$

P. Van Mieghem and R. van de Bovenkamp, 2015, "Accuracy criterion for the mean-field approximation in SIS epidemics on networks," Physical Review E, Vol. 91, No. 3, p. 032812.

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Lower bound for the epidemic threshold

$$\frac{dv_j(t)}{dt} = -\delta v_j + \beta \sum_{k=1}^{N} a_{kj} v_k - \beta \sum_{k=1}^{N} a_{kj} E[X_i X_k] \qquad v_k(t) = E[X_k(t)]$$

Ignoring the last summation:

$$\frac{dV(t)}{dt} \leq (-\delta I + \beta A) V(t) \quad \Rightarrow \quad V(t) \leq e^{(-\delta I + \beta A)t} V(0)$$

If all eigenvalues of $\beta A - \delta I$ are negative, v_j tends exponentially fast to zero for sufficiently large time t. Hence, if

$$\beta \lambda_1(A) - \delta < 0 \quad \Rightarrow \quad \tau = \frac{\beta}{\delta} < \frac{1}{\lambda_1(A)} < \tau_c$$

The NIMFA epidemic threshold is precisely

$$\boxed{\tau_c^{(1)} = \frac{1}{\lambda_1(A)} < \tau_c} \qquad \boxed{\tau_c^{(1)} = \frac{1}{\lambda_1(A)} < \tau_c^{(2)} = \frac{1}{\lambda_1(H)} < \tau_c}$$

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What is so interesting about epidemics?

network protection

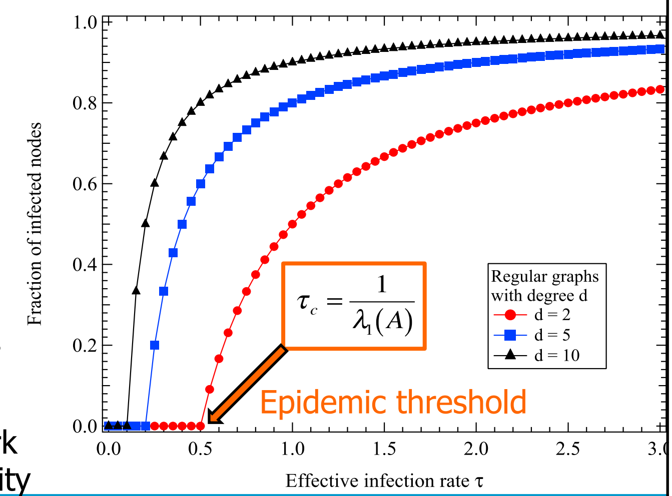
self-replicating objects (worms)

propagation errors

rumors (social nets)

epidemic algorithms (gossiping)

cybercrime : network robustness & security



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$$\max\left(E[D]\sqrt{1+\frac{Var[D]}{(E[D])^2}}, \sqrt{d_{\max}}\right) \le \lambda_1(A) \le d_{\max}$$

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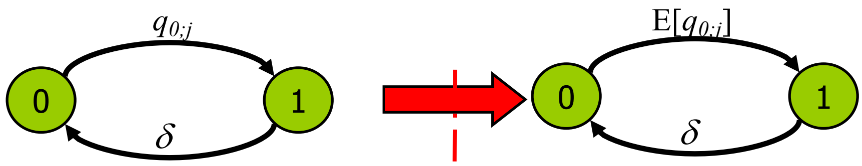
Exact vs. Mean-field approx. (NIMFA)



- 2^N linear equations
- Steady-state
  - absorbing (healthy) state
  - reached after unrealistically long time
- difficult to analyze
- **only** for exponential infection and curing times

- N non-linear equations
- Meta-stable state:
  - phase-transition
  - epidemic threshold
  - realistic
- analytically tractable
- lower bound epidemic threshold
- valid for **any** infection and curing time distribution

K. Devriendt and P. Van Mieghem, 2017, "Unified mean-field framework for SIS epidemics on networks, based on graph partitioning and the isoperimetric inequality", Physical Review E, Vol. 96, No. 5, p. 052314.

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Extensions of NIMFA

- **In-homogeneous**: each node i has own β_i and δ_i:
  P. Van Mieghem & J. Omic, 2008, "In-homogeneous Virus Spread in Networks", arXiv.1306.2588
- **SAIS** (Infected, Susceptible, Alert) and **SIR** instead of SIS:
  F. Darabi Sahneh & C. Scoglio, 2011,"Epidemic Spread in Human Networks", 50th IEEE Conf. Decision and Contol, Orlando, Florida.
  M. Youssef & C. Scoglio, 2011, "An individual-based approach to SIR epidemics in contact networks" Journal of Theoretical Biology 283, pp. 136-144.
- Generalized Epidemic mean-field model (**GEMF**): extension of NIMFA to m compartments (includes both SIS, SAIS, SIR,...):
  F. Darabi Sahneh, C. Scoglio, P. Van Mieghem, 2013, "Generalized Epidemic Mean-Field Model for Spreading Processes over Multi-Layer Complex Networks",, IEEE/ACM Transactions on Networking, Vol. 21, No. 5, pp. 1609-1620.
- NIMFA on **Interdependent networks**
  Wang, H., Q. Li, G. D'Agostino, S. Havlin, H. E. Stanley and P. Van Mieghem, 2013, "Effect of the Interconnected Network Structure on the Epidemic Threshold", Physical Review E, Vol. 88, No. 2, August, p. 022801.
- **Universal SIS mean-field framework**: NIMFA is close to optimal
  Devriendt, K. & P. Van Mieghem, 2017, "Universal mean-field framework for SIS epidemics on networks, based on graph partitioning and the isoperimetric inequality", Physical Review E, Vol. 96, No. 5, p. 052314.

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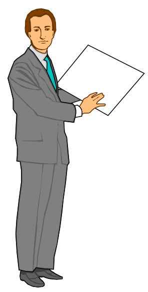
Outline



Exact SIS model

NIMFA: N-Intertwined MF Approximation

Recent developments

- **Non-Markovian epidemics**
- Tanh-approximation
- Inferring the graph

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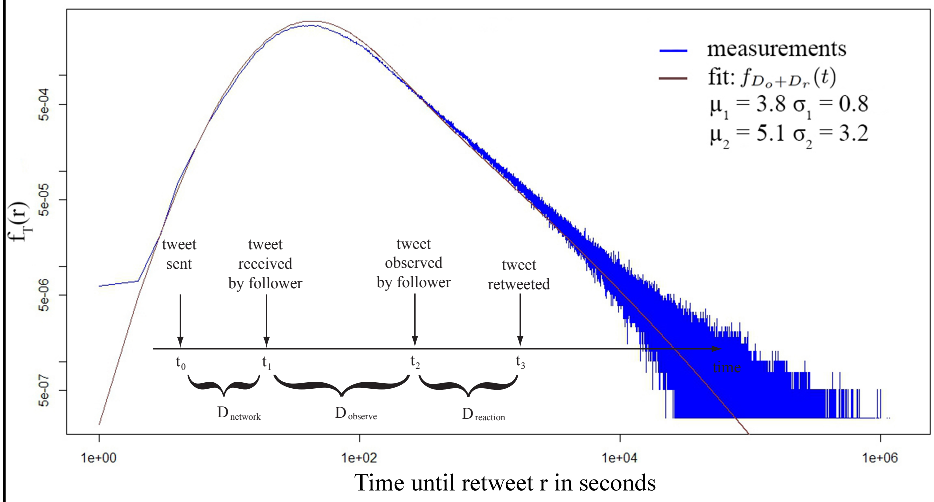
Epidemic times are not exponential



C. Doerr, N. Blenn and P. Van Mieghem, "Lognormal infection times of Online information spread", PLOS ONE, Vol. 8, No. 5, p. e64349, 2013

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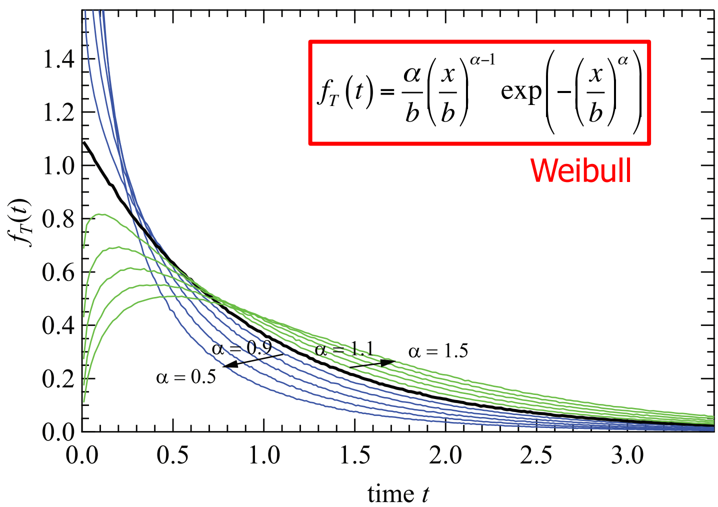
Non-Markovian infection times



Same mean E[T]:

$$b = \frac{1}{\beta\Gamma\left(1+\frac{1}{\alpha}\right)}$$

T is the time to infect a neighboring node

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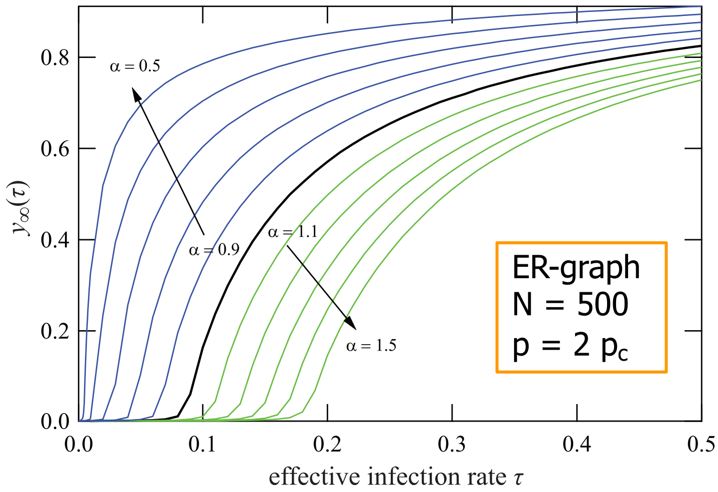
Non-Markovian epidemic threshold



Non-exponential infection time has a dramatic influence!

P. Van Mieghem and R. van de Bovenkamp, "Non-Markovian infection spread dramatically alters the SIS epidemic threshold", Physical Review Letters, vol. 110, No. 10, March 2013, p. 108701.

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GSIS: SIS with general infection times

NIMFA: valid provided the effective infection rate τ is replaced by the av. number $E[M]$ of infection events during a healthy period (via renewal theory assuming existence of metastable state):

$$\boxed{E[M] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi_T(z)\phi_R(-z)}{1-\phi_T(z)}\frac{dz}{z}} \qquad \phi_X(z) = E[e^{-zX}]$$

NIMFA steady state :

$$0 = -v_{j\infty} + \tau(1 - v_{j\infty})\sum_{k=1}^{N} a_{kj}\, v_{k\infty} \qquad \tau = \frac{\beta}{\delta}$$

$$\boxed{0 = -v_{j\infty} + E[M](1 - v_{j\infty})\sum_{k=1}^{N} a_{kj}\, v_{k\infty}}$$

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E. Cator, R. van de Bovenkamp and P. Van Mieghem, "SIS epidemics on networks with general infection and curing times", Physical Review E, Vol. 87, No. 6, p. 062816, 2013.

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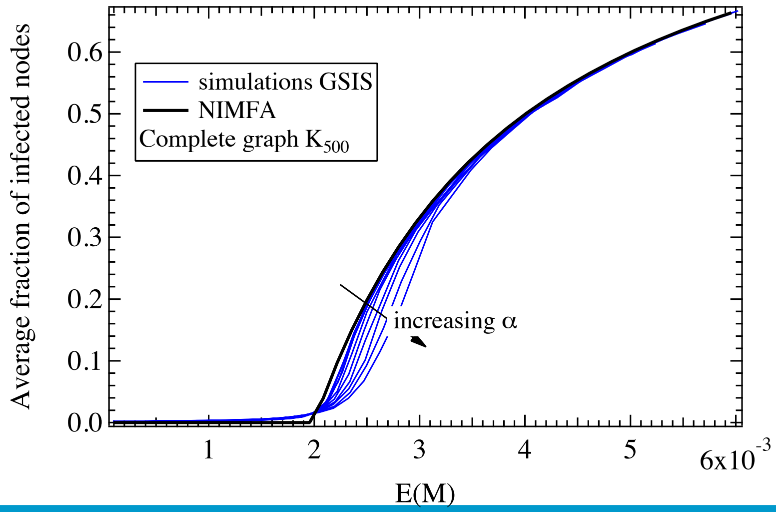
GSIS: E[M] gives the right scaling



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E. Cator, R. van de Bovenkamp and P. Van Mieghem, "SIS epidemics on networks with general infection and curing times", Physical Review E, Vol. 87, No. 6, p. 062816, 2013.

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GSIS: SIS with general infection times

Generalized criterion for the NIMFA epidemic threshold: $E[M_c] = \frac{1}{\lambda_1}$

If the recovery time R is exponential, then $E[M] = \frac{\phi_T(\delta)}{1-\phi_T(\delta)}$

and the epidemic threshold obeys: $\phi_T(\delta) = \frac{1}{1+\lambda_1}$

When the infection time T is Weibullian: $\phi_T\left(\frac{1}{\tau\Gamma\left(1+\frac{1}{\alpha}\right)};\alpha\right) = \frac{1}{1+\lambda_1}$

with pgf $\phi_T(w;\alpha) = \alpha\int_0^\infty e^{-wx-x^\alpha} x^{\alpha-1}dx$

Scaling law for large N

When infection time T is Weibullian: $\boxed{\tau_c^{(1)}(\alpha) = \frac{q(\alpha)}{\lambda_1^{1/\alpha}}}$ $q(\alpha) = O(1)$

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$$\boxed{E[M] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi_T(z)\phi_R(-z)}{1-\phi_T(z)}\frac{dz}{z}}$$

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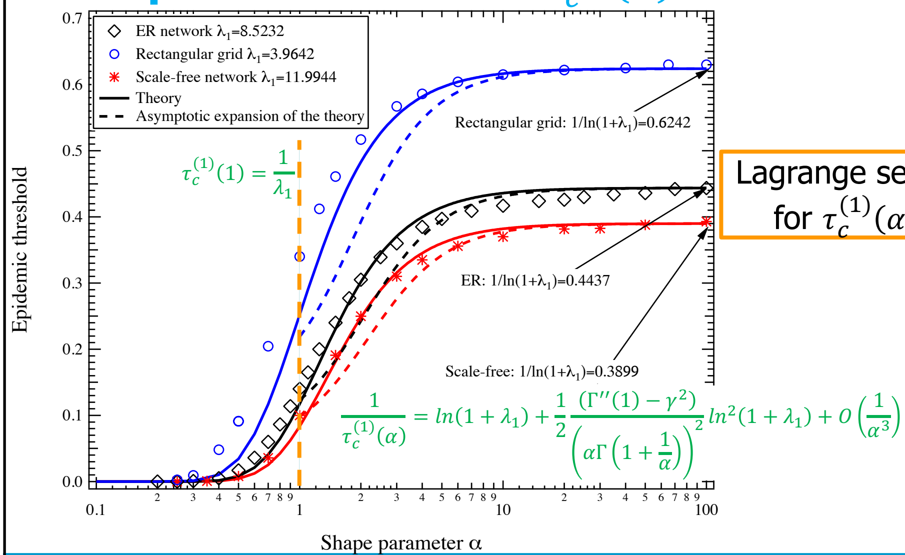
Epidemic threshold $\tau_c^{(1)}(\alpha)$ versus α



Lagrange series for $\tau_c^{(1)}(\alpha)$

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P. Van Mieghem and Q. Liu, 2019, "Explicit non-Markovian SIS mean-field epidemic threshold for Weibullian infections but Poisson curings", Physical Review E, vol. 100, No. 2, p. 022317.

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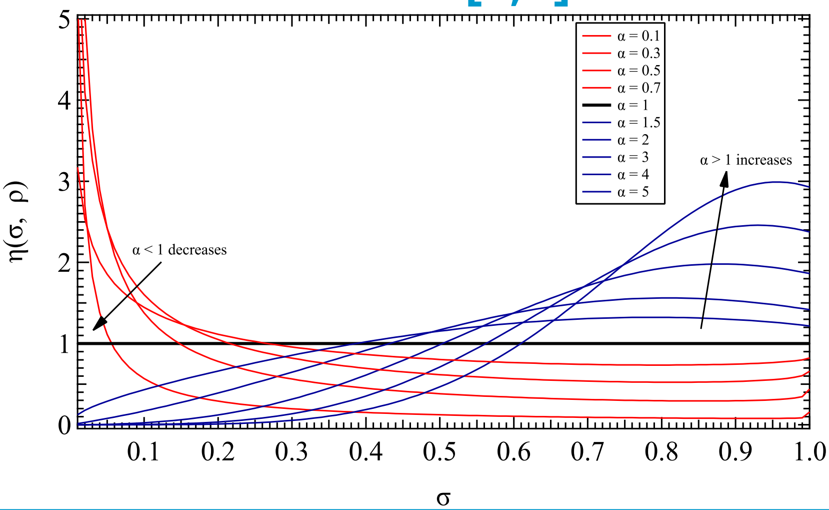
Occurrence of infection events in time interval [0,1]



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Given that 1 infection event has occurred in $[0,\rho]$, what is the prob. of its occurrence at time σ

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GSIS: infection time T is Gamma

The infection time T is Gamma distributed: $f_{T_\Gamma}(x;\xi) = \frac{\left(\frac{x}{b_\Gamma}\right)^{\xi-1}}{b_\Gamma \Gamma(\xi)} e^{-\frac{x}{b_\Gamma}}$

with mean $E[T_\Gamma] = b_\Gamma \xi$ and pgf $\phi_{T_\Gamma}(z;\xi) = (1 + b_\Gamma z)^{-\xi}$

Comparison requires $E[T_\Gamma] = \frac{1}{\beta}$ so that $b_\Gamma = 1/(\xi\beta)$

If the recovery time R is exponential, then $E[M] = \frac{\phi_T(\delta)}{1-\phi_T(\delta)}$

The NIMFA epidemic threshold obeys $E[M_c] = \frac{1}{\lambda_1}$, so $\phi_T(\delta) = \frac{1}{1+\lambda_1}$

$$\boxed{\tau_{c;\gamma}^{(1)}(\xi) = \frac{1}{\xi\left[(1+\lambda_1)^{\frac{1}{\xi}} - 1\right]} = \frac{\frac{1}{\xi}}{e^{\frac{\ln(1+\lambda_1)}{\xi}} - 1}}$$

$$\boxed{E[M] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi_T(z)\phi_R(-z)}{1-\phi_T(z)}\frac{dz}{z}}$$

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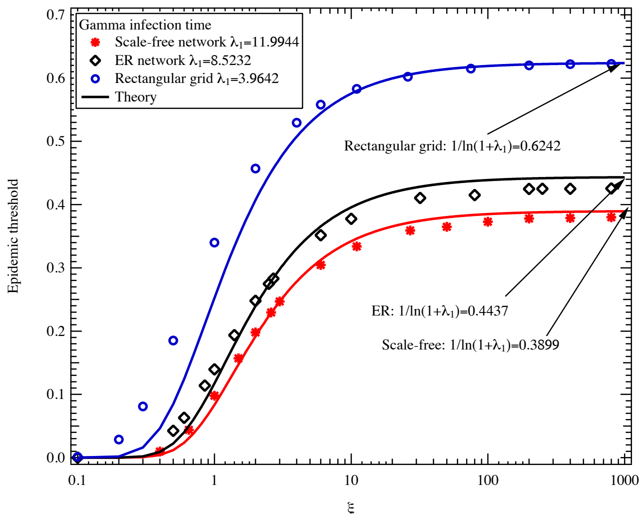
Epidemic threshold $\tau_{c;\Gamma}^{(1)}(\xi)$ versus ξ



$$\tau_{c;\gamma}^{(1)}(\xi) = \frac{\frac{1}{\xi}}{e^{\frac{\ln(1+\lambda_1)}{\xi}} - 1}$$

One-to-one relation infection time of viral item and its global endemic effect in a graph

P. Van Mieghem and Q. Liu, 2019, "Explicit non-Markovian SIS mean-field epidemic threshold for Weibullian infections but Poisson curings", Physical Review E, vol. 100, No. 2, p. 022317.

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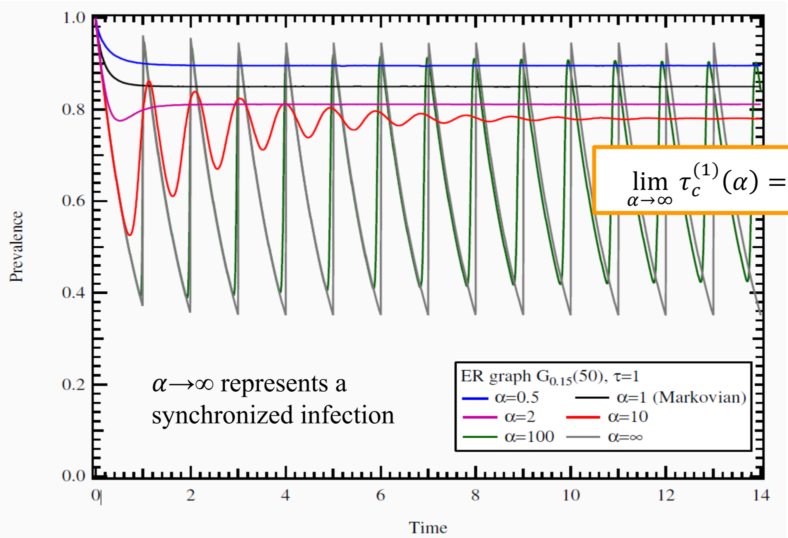
Time-dependent prevalence $\alpha \to \infty$



$$\lim_{\alpha\to\infty} \tau_c^{(1)}(\alpha) = \frac{1}{ln(1+\lambda_1)}$$

Liu, Q. and P. Van Mieghem, 2018, "Burst of virus infection and a possibly largest epidemic threshold of non-Markovian SIS processes on networks", Physical Review E, Vol. 97, No. 2, February, p. 022309 .

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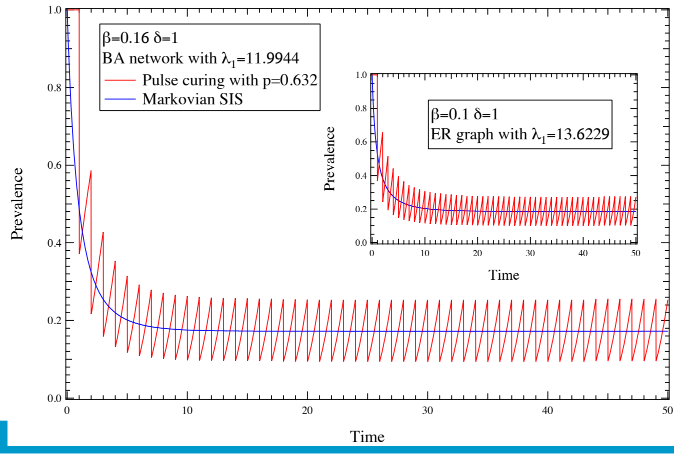
Pulse curing strategy on the SI process

Epidemic threshold: $\boxed{\frac{1}{\lambda_1}\ln\left(\frac{1}{1-p}\right)}$

p: fraction of nodes being cured

If $\ln\left(\frac{1}{1-p}\right) = 1$ *or* $p = 0.632$, then pulse curing is comparable with Poisson curing:



Liu, Q., X. Zhou and P. Van Mieghem, 2019, "Pulse strategy for suppressing spreading on networks", Europhysics Letters (EPL), Vol. 127, August, p. 38001. (also: arXiv:1904.10883)

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Challenges for epidemics on networks

- Tight upper bound of the epidemic threshold (for any graph), or near to exact determination of τ_c
- Time-dependent analysis of SIS epidemics: beyond the tanh-formula
- Non-Markovian epidemics
- Epidemics on *evolving, adaptive* and *temporal* networks
- Competing and mutating viruses on networks
- Modeling of social contagion
- Control of epidemics on networks
- **Measured data** of epidemics (e.g. fraction of infected nodes, the underlying topology of the 'contact' network) in real-world networks!

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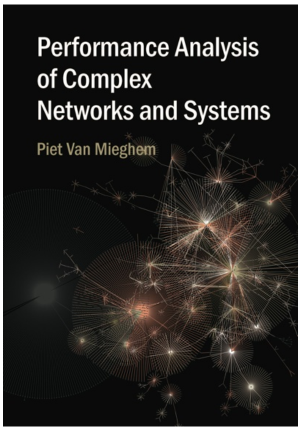
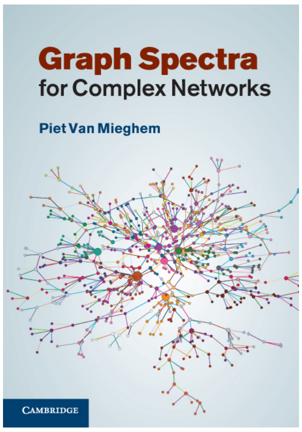
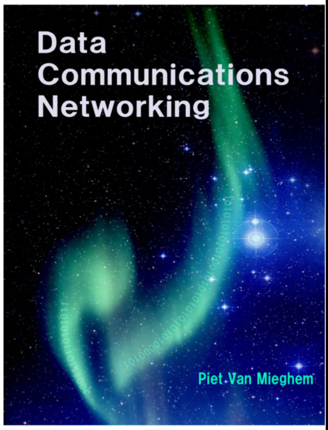
Books







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Thank You

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