

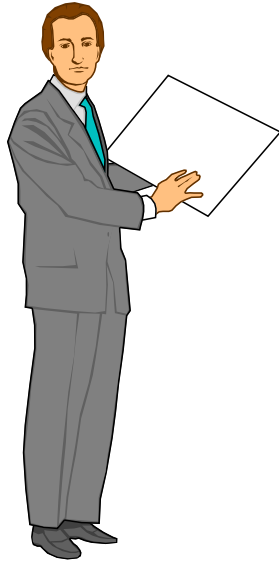
# Time-varying SIS prevalence in networks: theory and a new approximate formula

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# Outline



Exact SIS prevalence

Tanh-formula

Generalizations

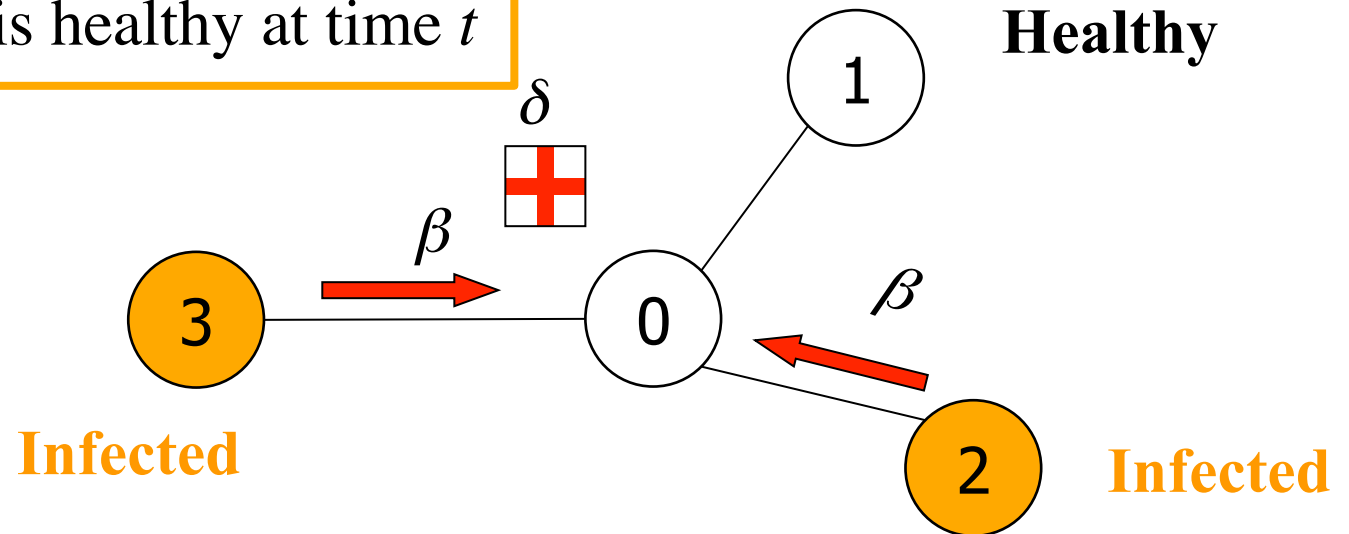
# Continuous-time SIS model on networks

- Constant infection rate  $\beta$  on all links
- Constant curing rate  $\delta$  for all nodes

$\tau = \beta / \delta$  : effective spreading rate

$X_j(t) = 1$  node  $j$  is infected at time  $t$

$X_j(t) = 0$  node  $j$  is healthy at time  $t$



**Infection and curing are independent Poisson processes**

## Governing SIS equation for node $j$

$$\frac{dE[X_j]}{dt} = E \left[ -\delta X_j + (1 - X_j) \beta \sum_{k=1}^N a_{kj} X_k \right]$$



time-change of  
 $E[X_j] = \Pr[X_j = 1]$ ,  
 probability that  
 node  $j$  is infected



if *infected*:  
 probability of  
 curing per  
 unit time



if *not infected (healthy)*:  
 probability of  
 infection per  
 unit time

$$\frac{dE[X_j]}{dt} = -\delta E[X_j] + \beta \sum_{k=1}^N a_{kj} E[X_k] - \beta \sum_{k=1}^N a_{kj} E[X_j X_k]$$



# N-Intertwined Mean-Field Approximation (NIMFA)

replace rv by its mean

$$\frac{dE[X_j]}{dt} = E \left[ -\delta X_j + (1 - X_j) \beta \sum_{k=1}^N a_{kj} X_k \right]$$

NIMFA



$$X_j \Rightarrow E[\tilde{X}_j]$$

$$\frac{dE[E[\tilde{X}_j]]}{dt} = E \left[ -\delta E[\tilde{X}_j] + (1 - E[\tilde{X}_j]) \beta \sum_{k=1}^N a_{kj} E[\tilde{X}_k] \right]$$

Bernoulli rv



$$E[\tilde{X}_j] = \Pr[\tilde{X}_j = 1] = v_j$$

$$\frac{dv_j}{dt} = -\delta v_j + (1 - v_j) \beta \sum_{k=1}^N a_{kj} v_k$$

# SIS Prevalence

- Fraction of infected nodes in the graph  $G$

$$S(t) = \frac{1}{N} \sum_{j=1}^N X_j(t) \quad (\text{random variable!})$$

- Prevalence: Expected fraction of infected nodes in  $G$

$$y(t) = E[S(t)] = \frac{1}{N} \sum_{j=1}^N \Pr[X_j(t) = 1]$$

# Differential equation prevalence (1)

$$\frac{dE[X_j]}{dt} = E \left[ -\delta X_j + (1 - X_j) \beta \sum_{k=1}^N a_{kj} X_k \right]$$

After summing over all nodes:

$$\frac{d}{dt} \left( \frac{1}{N} \sum_{j=1}^N E[X_j] \right) = E \left[ -\frac{1}{N} \sum_{j=1}^N X_j + \frac{\tau}{N} \sum_{j=1}^N \sum_{k=1}^N (1 - X_j) a_{kj} X_k \right]$$

Using the definition of prevalence:  $y(t) = \frac{1}{N} \sum_{j=1}^N E[X_j]$

$$\frac{dy(t^*)}{dt^*} = -y(t^*) + \frac{\tau}{N} E \left[ \sum_{j=1}^N \sum_{k=1}^N (1 - X_j) a_{kj} X_k \right]$$

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Normalized time  $t^* = \delta t$

## Differential equation prevalence (2)

$$\frac{dy(t^*)}{dt^*} = -y(t^*) + \frac{\tau}{N} E \left[ \sum_{j=1}^N \sum_{k=1}^N (1 - X_j) a_{kj} X_k \right]$$

Executing the double sum:

$$\sum_{j=1}^N \sum_{k=1}^N (1 - X_j) a_{kj} X_k = 2 \sum_{l \in L} (1 - X_{l^+}) X_{l^-}$$

For Bernoulli rv's:

$$\begin{aligned} (X_{l^-} - X_{l^+})^2 &= X_{l^-}^2 - 2X_{l^-}X_{l^+} + X_{l^+}^2 = X_{l^-} - 2X_{l^-}X_{l^+} + X_{l^+} \\ &= X_{l^-}(1 - X_{l^+}) + X_{l^+}(1 - X_{l^-}) \end{aligned}$$

Thus:

$$\sum_{j=1}^N \sum_{k=1}^N (1 - X_j) a_{kj} X_k = \sum_{l \in L} (X_{l^+} - X_{l^-})^2 = w^T Q w$$

where  $w$  is the nodal random Bernoulli vector

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$$w = (X_1, X_2, \dots, X_N)$$

## Differential equation prevalence (3)

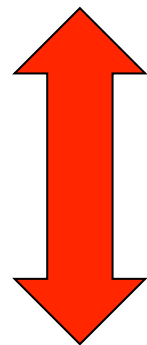
Finally, in terms of the Laplacian  $Q = \Delta - A$  and the normalized time  $t^* = \delta t$

$$\frac{dy(t^*)}{dt^*} = -y(t^*) + \frac{\tau}{N} E[w^T Q w]$$

P. Van Mieghem, F. Darabi Sahneh and C. Scoglio, 2014, "Exact Markovian SIR and SIS epidemics on networks and an upper bound for the epidemic threshold", Proceedings of the 53rd IEEE Conference on Decision and Control (CDC'14), December 15-17, Los Angeles, CA, USA (also on <http://arxiv.org/abs/1402.1731>).

# “Local rule - global emergent properties” class

$$\frac{dE[X_j(t)]}{dt} = E \left[ -\delta X_j(t) + (1 - X_j(t)) \beta \sum_{k=1}^N a_{kj} X_k(t) \right]$$



Local SIS rule

Global emergent SIS spread

$$\frac{dy(t^*)}{dt^*} = -y(t^*) + \frac{\tau}{N} E \left[ w^T(t^*) Q w(t^*) \right]$$

The Laplacian  $Q = \Delta - A$

The normalized time  $t^* = \delta t$

Bernoulli state vector

$$w(t^*) = (X_1(t^*), X_2(t^*), \dots, X_N(t^*))$$

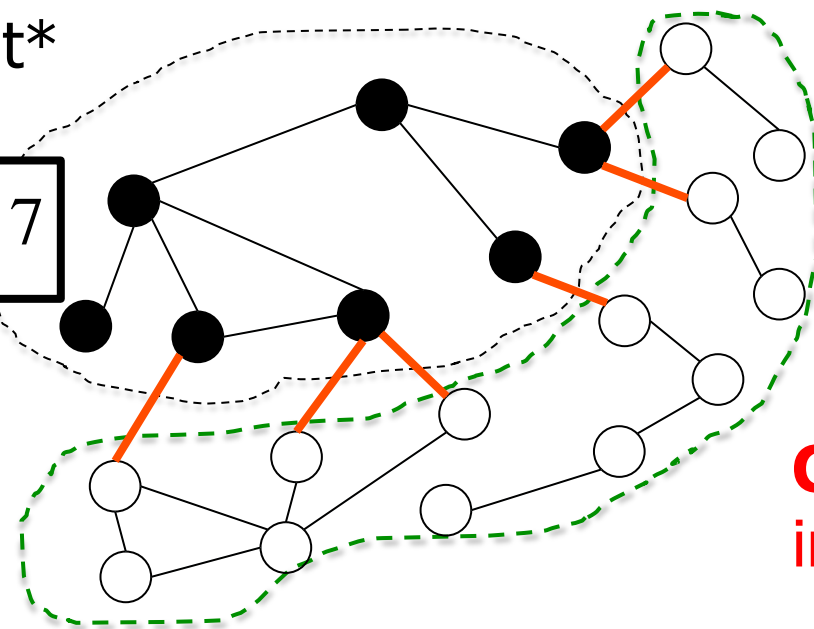
P. Van Mieghem, F. Darabi Sahneh and C. Scoglio, 2014, "Exact Markovian SIR and SIS epidemics on networks and an upper bound for the epidemic threshold", Proceedings of the 53rd IEEE Conference on Decision and Control (CDC'14), December 15-17, Los Angeles, CA, USA (also arxiv.1402.1731).

# SIS prevalence dynamics

$$\frac{dy(t^*)}{dt^*} = -y(t^*) + \frac{\tau}{N} E \left[ w^T(t^*) Q w(t^*) \right]$$

Set of infected nodes  
at time  $t^*$

$$NS(t^*) = 7$$

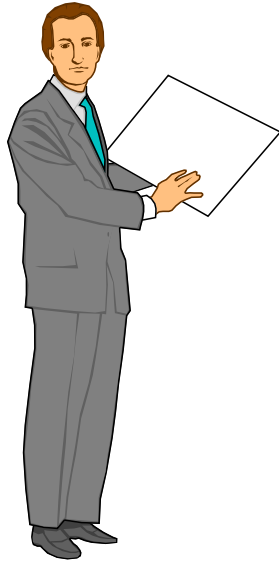


$$w^T(t^*) Q w(t^*) = 12$$

**Cut-Set:** set of links with 1  
infected node at time  $t^*$

Set of susceptible nodes  
at time  $t^*$

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# Spectral decomposition

Expanding  $w$  in terms of all eigenvectors of  $Q$ ;  $x_k$  belongs to  $\mu_k$

$$w = \sum_{k=1}^N \xi_k x_k \quad \longrightarrow \quad w^T Q w = \sum_{k=1}^N \mu_k \xi_k^2 \quad \text{with} \quad \xi_k = w^T x_k$$

Using  $Qu=0$  and Bernoulli rv properties lead to

$$w^T Q w = \mu_{N-1} N (S - S^2) + R$$

and

$$\frac{dy(t^*)}{dt^*} = (\tau \mu_{N-1} - 1) y(t^*) - \tau \mu_{N-1} y^2(t^*) - \Psi$$

with

$$\Psi = \tau \mu_{N-1} \left( \text{Var}[S] - \frac{E[R]}{N \mu_{N-1}} \right)$$

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$$R = \sum_{k=1}^{N-2} (\mu_k - \mu_{N-1}) \xi_k^2(t) \geq 0$$

$$R_{K_N} = 0$$

# Spectral decomposition: tanh-formula

Bounding  $c_L \leq \Psi \leq c_U$  yields a Riccati differential equation

$$\frac{d\tilde{y}(t^*)}{dt^*} = (\tau\mu_{N-1} - 1)\tilde{y}(t^*) - \tau\mu_{N-1}\tilde{y}^2(t^*) - c$$

with solution

$$\tilde{y}(t^* | c) = \frac{1}{2} \left( 1 - \frac{1}{\tau\mu_{N-1}} \right) + \frac{\Xi}{2} \tanh \left( \frac{\tau\mu_{N-1}\Xi}{2} t^* + \Omega \right)$$

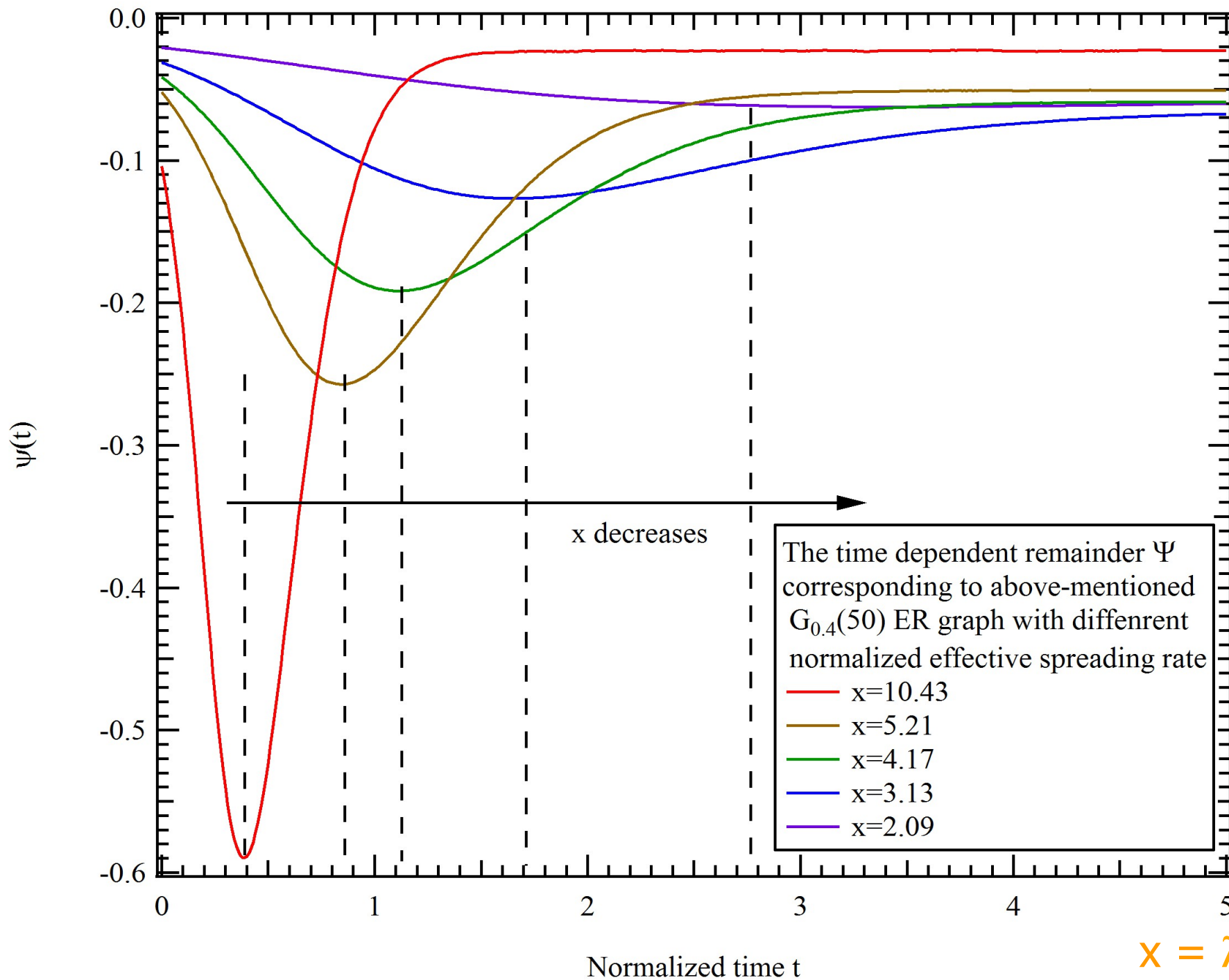
where

$$\Xi = \sqrt{\left( 1 - \frac{1}{\tau\mu_{N-1}} \right)^2 - \frac{4c}{\tau\mu_{N-1}}}$$

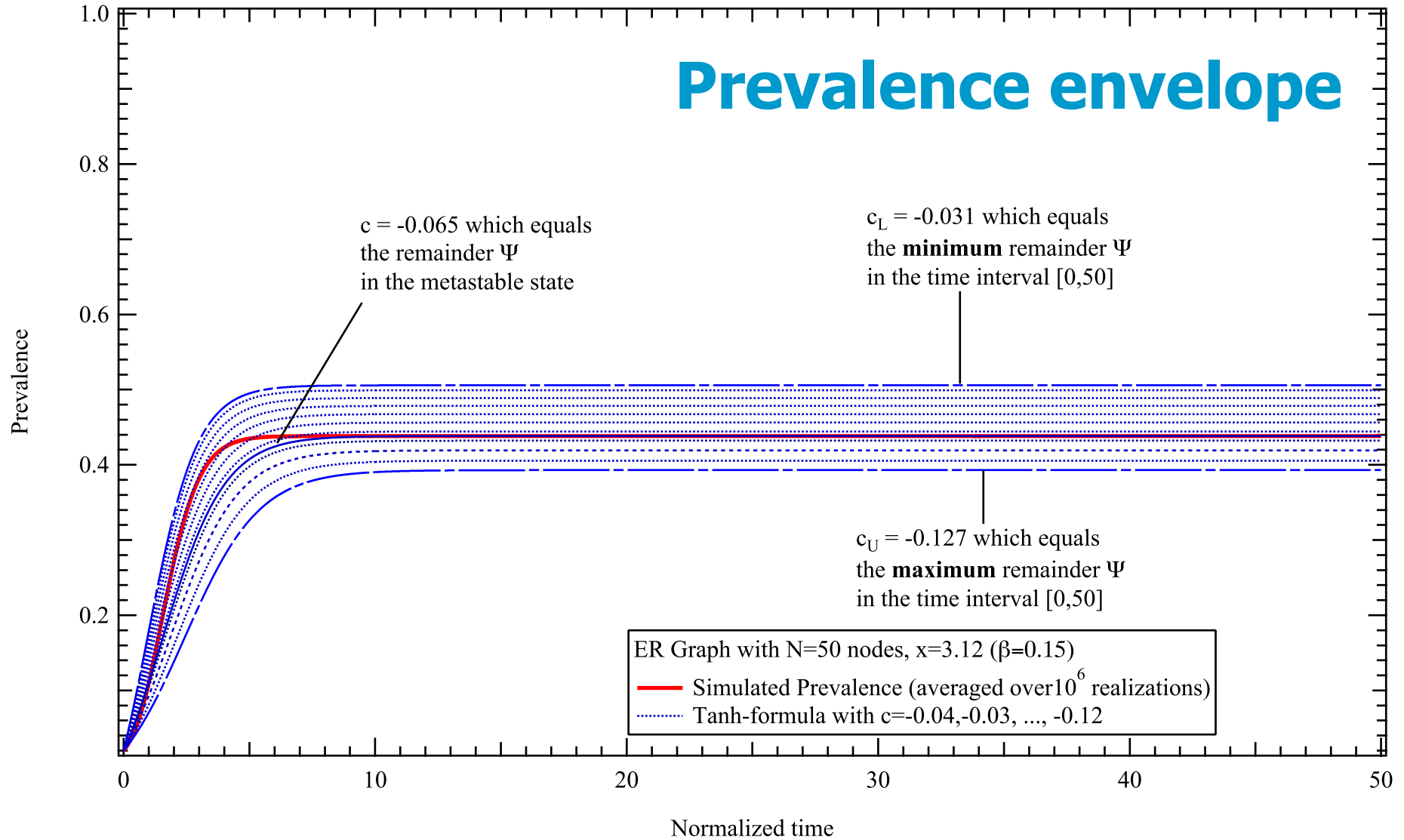
$$\Omega = \operatorname{arctanh} \left( \frac{1}{\Xi} \left( 2y_0 - \left( 1 - \frac{1}{\tau\mu_{N-1}} \right) \right) \right)$$

The prevalence is upper and lower bounded by the tanh-formula:

$$\tilde{y}(t^* | c_U) < y(t^*) < \tilde{y}(t^* | c_L)$$

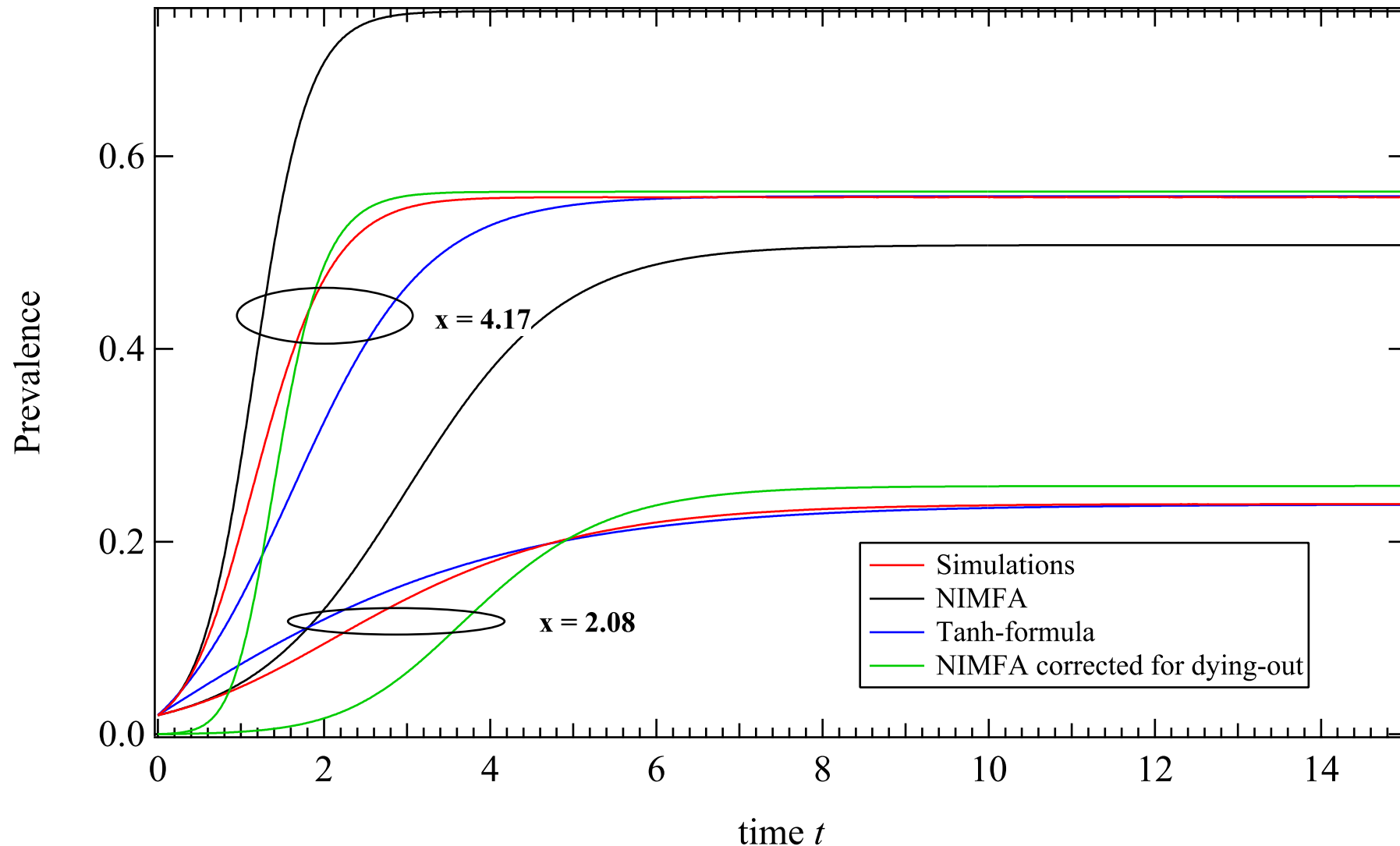


# Prevalence envelope

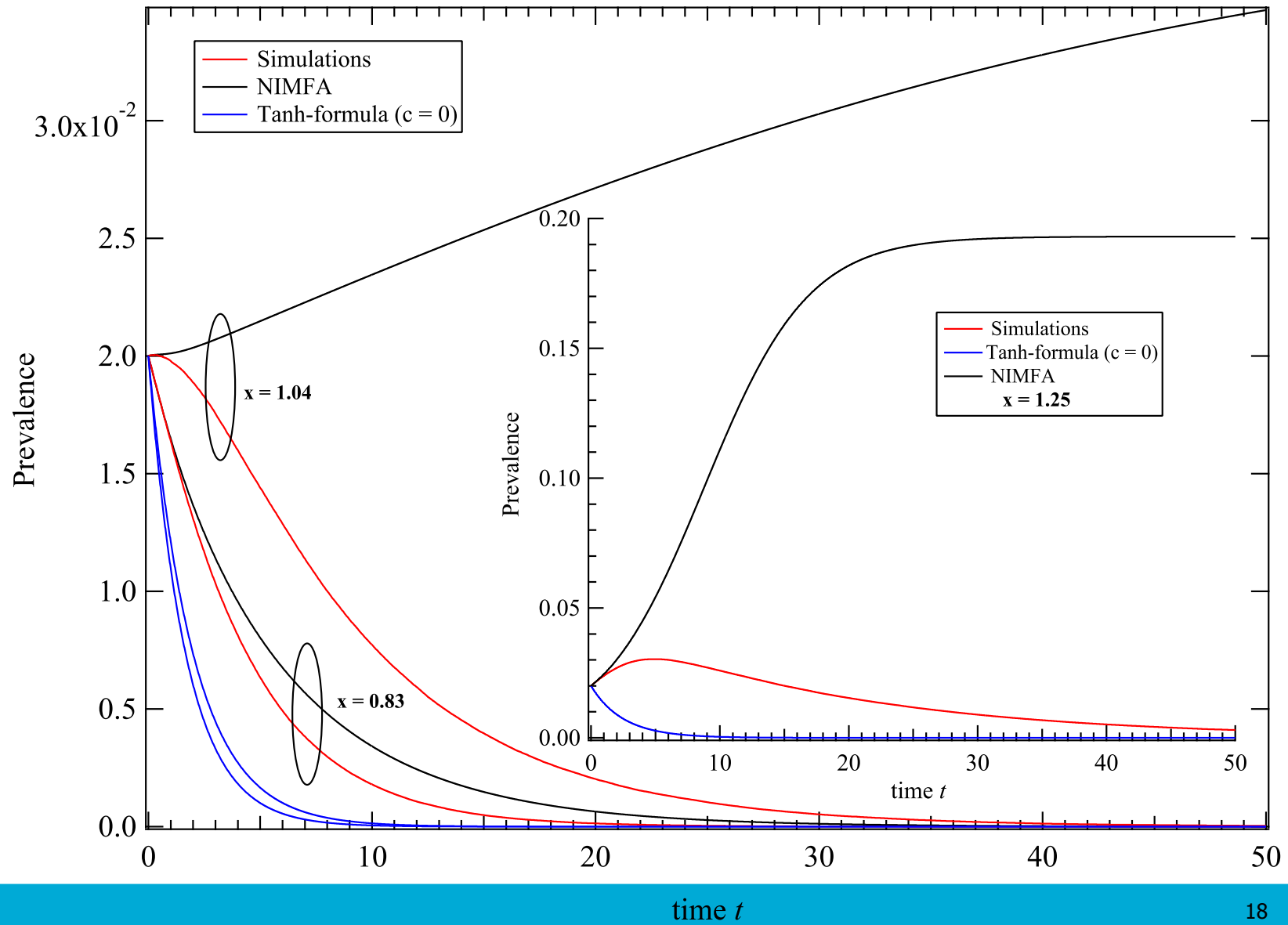


$$\bar{y}(t^* | c_U) < y(t^*) < \bar{y}(t^* | c_L)$$

# Comparison in ER $G_{0.4}(50)$

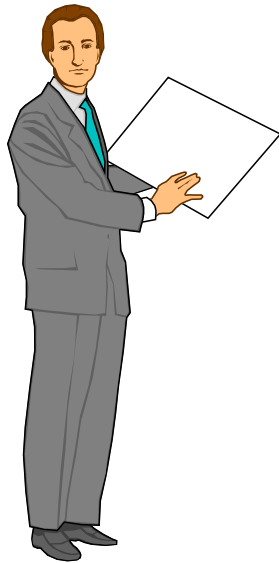


$\lambda_1 = 20.85$  and  $\mu_{N-1} = 10.11$  and  $x = \lambda_1 \tau$



ER  $G_{0.4}(50)$

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# Heterogeneous SIS prevalence

- Curing rate for node  $i$ :  $\delta_i$
- Infection rate for link  $(i,j)$ :  $\beta_{ij}$
- Nodal curing rate vector:  $c = (\delta_1, \delta_2, \dots, \delta_N)$
- Weighted adjacency matrix:  $\check{a}_{ij} = a_{ij}\beta_{ij}$

$$N \frac{dy(t)}{dt} = -E[c^T w(t)] + E[(u - w(t))^T \check{A}w(t)]$$

Only if  $\beta_{ij} = \beta_{ji}$

$$N \frac{dy(t)}{dt} = -E[c^T w(t)] + E[w^T(t) \check{Q}w(t)]$$



# Prevalence in temporal networks

$$\frac{dy(t^*)}{dt^*} = -y(t^*) + \frac{\tau}{N} E \left[ w^T(t^*) Q(t^*) w(t^*) \right]$$

Changing topology:  $\lim_{\varepsilon \rightarrow 0} A(t^* - \varepsilon) = A_1$   $\lim_{\varepsilon \rightarrow 0} A(t^* + \varepsilon) = A_2$

Continuity of  $w(t)$  (Markov property) implies continuity of  $y(t)$

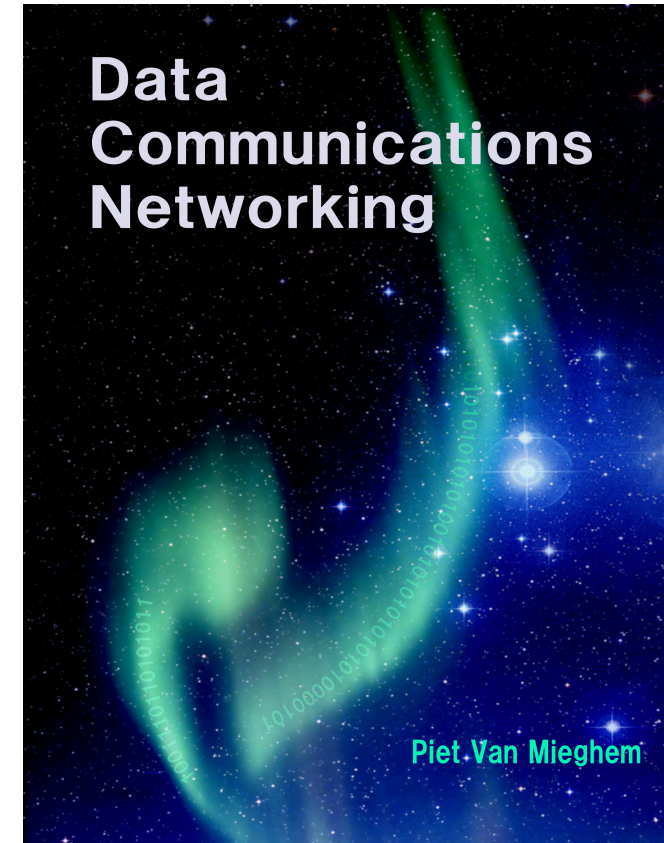
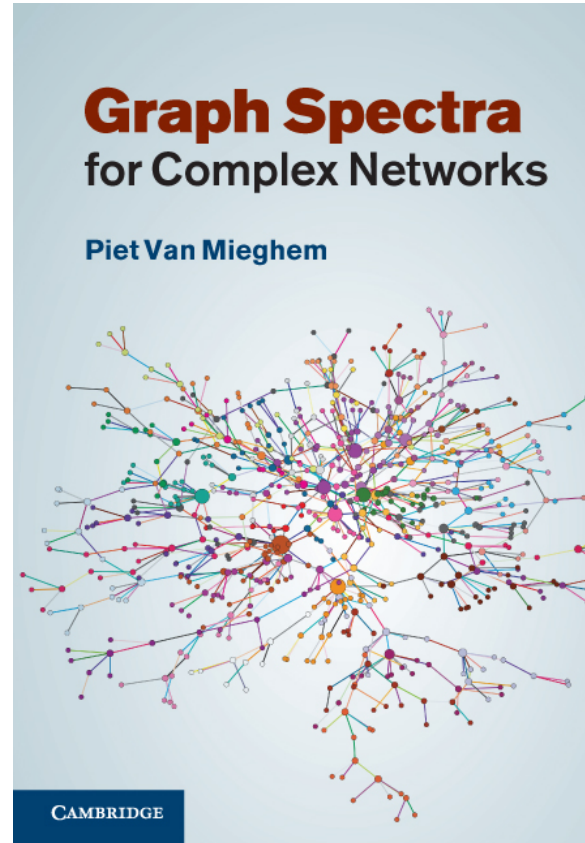
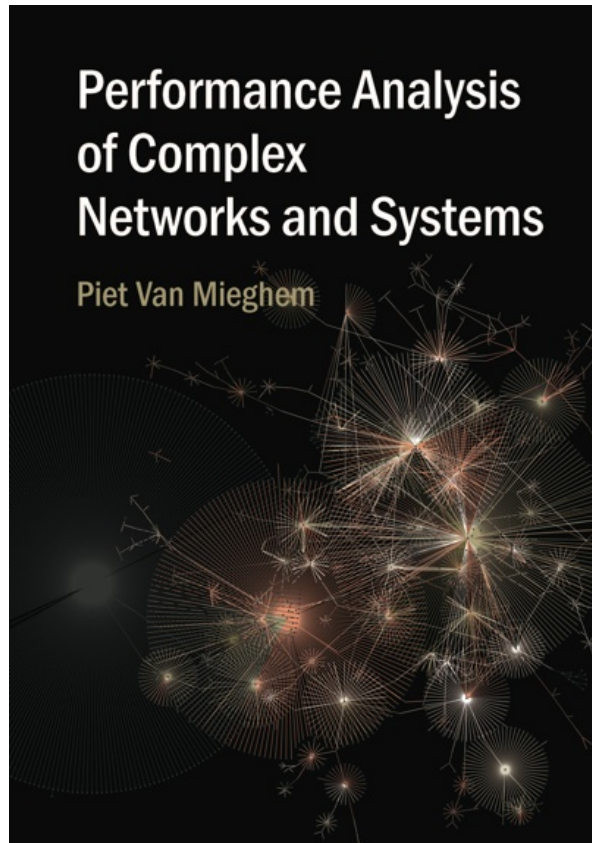
$$\left. \frac{dy(t^*)}{dt^*} \right|_{t^* + \varepsilon} - \left. \frac{dy(t^*)}{dt^*} \right|_{t^* - \varepsilon} = \frac{\tau}{N} E \left[ w^T(t^*) \{Q_2 - Q_1\} w(t^*) \right]$$

Discontinuity in slope of prevalence at topology changes!

# Conclusion

- The prevalence  $y(t)$  is a premier indicator of epidemic spread in networks
- Time-dependent evolution is hardly studied
- **Tanh-formula:**
  - Extends to networks the seminal 1927 formula of Kermack and McKendrick (first proposal + solution of SIR on  $K_N$ )
  - Analytic: simple and fast computable
  - Prevalence envelope: fit of real data
  - Room to improve the tanh-formula's accuracy

# Books



Articles: <http://www.nas.ewi.tudelft.nl>

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A photograph of a modern architectural structure, likely a stadium or arena, featuring a prominent conical roof with a metal framework. The structure is situated on a green hillside with a paved walkway and a person walking. The sky is blue with scattered white clouds.

**Thank You**

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