Reliability polynomials crossing more than twice

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Abstract—**In this paper we study all-terminal reliability polynomials of networks having the same number of nodes and the same number of links. First we show that the smallest possible size for a pair of networks that allows for two crossings of their reliability polynomials have seven nodes and fifteen edges. Then we propose a construction of pairs of graphs whose reliability polynomials exhibit an arbitrary number of crossings. The construction does not depend on multigraphs. We also give concrete examples of pairs of graphs whose reliability polynomials have three, four and five crossings, respectively, and provide the first example of a graph with more than one point of inflection in (0,1).**

Keywords-component; probabilistic graph, reliability polynomial, edge connectivity

I. INTRODUCTION

In all networks that provide a service to a consumer, one of the main performance indicators is reliability [1]. The consumer, the user of the service, wants to be able to use the service for at least X% of the time. For instance, for the traditional telephony service often a five nines (99.999%) reliability is guaranteed. In order to be able to make such guarantees and commit to them in Service Level Agreements, network operators need to know their network's reliability. The reliability of a network generically depends on two factors: the availability of the individual network components and the topology of the network. During the lifetime of a network element, it may endure periods in which it is out of service either because of a malfunctioning, maintenance or repair work. If we denote the mean time to failure by MTBF and the mean repair time by MTTR, the availability of a network element (or more formally, the probability that the element is working properly) is defined as A = MTBF/(MTBF + MTTR).

A second factor affecting the global reliability of the network is the topology of the network. Obviously, higher redundancy in the network (e.g. more links connecting network switches) will lead to a higher reliability, but in general also to higher investment costs. Several ways to measure the robustness of network arising from a variety of transportation systems are presented in [2,3], including an indicator involving the cyclomatic number of a graph [2], and the algebraic connectivity of a graph [3]. All of these are measures of a static network. Ref. [4] discusses the network reliability for a number of simple network topologies and then applies this to determine the reliability of a real-life telecommunication network.

In this paper we will model a network subject to component failures as a finite, undirected probabilistic graph G (V, E) which consists of a set V of N nodes and collection E of L edges. Nodes represent communication centers in the network while edges correspond to bidirectional communication links. In this paper we assume that nodes are always operational while the edges are subjected to failures. More precisely we assume that each edge operates with link probability p_i independently (the probability p_i can be interpreted as the availability of the link). The assumption that the nodes never fail is based upon the fact that node failures occur much less frequent than link failures, which occur, for example, when fibers are unintentionally broken by means of shovels.

We further restrict our model here to simple graphs (no loops and no multiple edges) whose links all operate independently of each other with the same link probability p. The reliability of a network is concerned with the ability of a network to carry out a desired operation. One of the most common operations in a network is broadcasting which requires so-called all-terminal reliability in the network (see [5]). *All-terminal reliability* of a network is defined as the probability that any pair of nodes in the network can communicate with each other. The reliability polynomial $R_G(p)$ is a polynomial in p indicating the probability that graph G contains at least one spanning tree – in other words, that the network is connected. The reliability polynomial determines a variety of robustness features of a graph, including the number of spanning trees, the edge connectivity, and the number of connected subgraphs (see, for example, [5]). The reliability polynomial is a function of both p and the topology G. Reliability polynomials can be defined for two- terminal or k-terminal applications, depending on the type of operation crucial for a network's robustness, and indeed recent work has focused as well on extensions to directed graphs [6,7]. In this paper though, we focus on all-terminal reliability polynomials, which we refer to simply as reliability polynomials.

Moore and Shannon [8] were the first who expressed reliability of a network in the terms of link probability and they

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stated that for a network G (V, E) with N nodes and L edges, the reliability polynomial R(p) can be expressed as:

$$R_G(p) = \sum_{i=0}^{L} F_i (1-p)^i p^{L-i} , \quad (1)$$

where F_i corresponds to the number of sets of i links whose removal leaves G operational. Moore and Shannon proved that for every C > 0, every reliability polynomial crosses the curve

$$\frac{Cp}{1-p(1-C)}$$

at most once as p ranges from 0 to 1. Reliability polynomials are non-decreasing functions on [0,1] that are S-shaped (see Fig. 1, which shows the plot of the reliability polynomial of a complete graph on 5 nodes.)

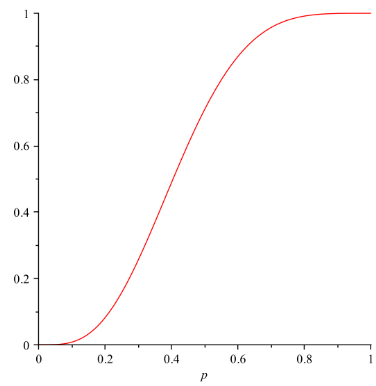


Figure 1: Reliability polynomial of the complete graph on 5 nodes

While reliability functions might seem simple as they are polynomials, their analytic properties are quite subtle. One might think that given a fixed number of nodes and links, there might exist an optimal network, independent of the value of p. While such has been shown to be false in the case of simple graphs, for multigraphs, it is not known whether such optimal networks exist [9]. A further question is whether reliability polynomials of two network topologies, having the same number of nodes and links, can cross more than once. This would indicate how subtle the notion of reliability is, and that how sensitive network reliability can be to link probability p. In [10] it was observed that reliability polynomials of simple graphs with the same size can cross, by constructing pairs of graphs where one graph is more reliable close to p = 0 and the other graph more reliable close to p = 1. Subsequently, Colbourn et al. [9] answered the naturally arising question if reliability polynomials of a pair of simple graphs with the same size could cross more than once, and provided an example of a pair of graphs, with ten nodes and twenty edges, whose reliability polynomials cross twice.

In [11] it was proved that for every integer k ≥ 1 and every k-tuple (m_1, m_2, …, m_k) of positive integers, infinitely many pairs of graphs of the same size exist, such that the difference of their reliability polynomials has exactly k roots $x_1 < \ldots < x_k$ in (0,1) such that x_i is of multiplicity m_i, i = 1..k. The argument utilizes multigraphs, though the multigraphs can also be converted to simple graphs with the same crossings properties. Also, [11] does not present concrete examples of graphs with more than two crossings.

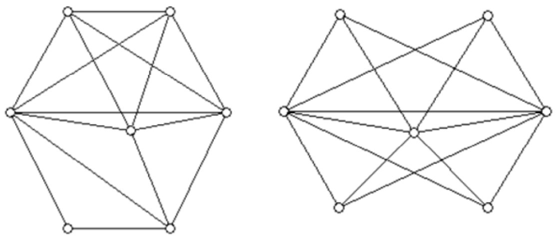
The aim of this paper is twofold. First we want to find the smallest possible size for a pair of networks that allows for two crossings of their reliability polynomials. If we denote this size by $N_{min}(2)$ then according to [9] it holds that $N_{min}(2) \leq 10$. Secondly, we want to construct examples of pairs of graphs whose reliability polynomials exhibit an arbitrary number of crossings, such that the construction does not depend on multigraphs.

The remainder of this paper is organized as follows. Section 2 contains the results of a brute force search for all pairs of graphs of size up till N = 8. The search method is implemented in the computer algebra program Maple. It is shown that $N_{min}(2) = 7$, because there is a pair of graphs with N = 7 and M = 15 whose reliability polynomials cross twice, while no such example exists for N < 7. In Section 3 a construction method is presented, depending only on simple graphs, that, for every integer k ≥ 1, leads to pairs of graphs with k crossings. Concrete examples are given with 3, 4 and 5 crossings. We conclude in Section 4 with some additional observations.

II. THE SMALLEST EXAMPLE OF SIMPLE GRAPHS WITH TRIPLE CROSSINGS

Using the computer algebra system Maple and Brendan McKay's lists of all non-isomorphic graphs of small order (http://cs.anu.edu.au/~bdm/data/graphs.html), we calculated the all-terminal reliability polynomials of all simple graphs on at most 8 vertices (the polynomials were calculated, for a graph G on N nodes and L edges, as the evaluation $p^{N-1}(1-p)^{L-N+1}T(G;1,(1-p)^{-1})$, where T(G;x,y) is the well-known two variable *Tutte polynomial* of G (see [5]).

We determined that there are no double crossings on graphs of order at most 6, and a single pair of graphs with N = 7 nodes (and L = 15 edges) that have a double crossing, at approximately 0.0911 and 0.3229 (see Fig. 2).



G_1 G_2

Figure 2: Smallest simple graphs G_1 and G_2 with a double crossing

There are many more pairs of double crossings on 8 nodes, with anywhere from 12 to 22 edges (see Table 1), though none have more than two crossings. There are precisely 5 pairs of reliability polynomials on 8 nodes whose curves touch. In turns out that the touching point always turned out to be at p = ½. Fig. 3 shows two graphs F_1 and F_2, both with 8 nodes and 12 edges, whose reliability polynomials touch, with reliability polynomials

$$-60p^{12}+368p^{11}-921p^{10}+1178p^{9}-772p^{8}+208p^{7}$$

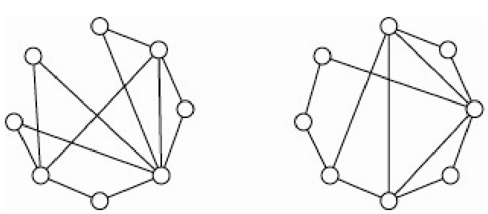
and

$$-56p^{12}+352p^{11}-896p^{10}+1159p^{9}-765p^{8}+207p^{7}$$

respectively. Fig. 4 depicts the difference of their reliabilities.

TABLE 1. NUMBER OF PAIRS OF RELIABILITY POLYNOMIALS OF GRAPHS OF ORDER 8 THAT HAVE A DOUBLE CROSSING, LISTED BY NUMBER OF EDGES

Number of Edges	12	13	14	15	16	17	18	19	20	21	22
Number of Pairs	1	6	12	42	83	39	13	3	5	2	4



F_1 F_2

Figure 3: Graphs whose reliability polynomials touch.

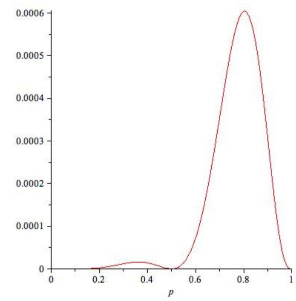


Figure 4: Difference between the reliabilities of the graphs of Fig. 3.

III. CONSTRUCTION METHOD FOR PAIRS OF GRAPHS WITH MORE THAN TWO CROSSINGS

In this section we will present a new construction method that, using only simple graphs, for every integer k ≥ 3, leads to pairs of graphs with k crossings. We will exemplify the method by first constructing an example of a pair of graphs with three crossings points. Then we generalize the method to k crossings. We end the section by giving concrete examples of graphs with four and five crossing points.

A. A pair of graphs whose reliablity polynomials cross three times

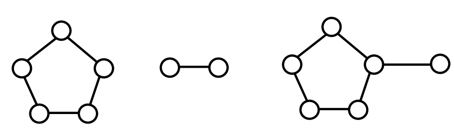
The general idea of our construction method is as follows: start with a pair of graphs with two crossings points and then adjust the graphs in such a way that the adjusted graphs also have the same size, have crossing points close to the original crossing points and in addition, an extra crossing point is generated. In order to prove that the construction method is valid, we will need the following well known results, see for instance [5].

Lemma 1: The reliability polynomial of the N-cycle graph C_N satisfies

$$R_{C_N}(p)=p^N+Np^{N-1}(1-p).$$

Lemma 2: Let G and H be two graphs with reliability polynomials $R_G(p)$ and $R_H(p)$, respectively. Denote by $G{\circ}H$ the concatenation of G and H, i.e. the graph obtained by identifying any node of G with any node of H. Then $R_{G\circ H}(p) = R_G(p)\, R_H(p)$.

As an example consider the graphs C_5 (the 5-cycle graph) and P_2 (path of length 1) depicted in Fig. 5.



C_5 P_2 $C_5 \circ P_2$

Figure 5: Concatenation of C_5 and P_2

Because the concatenation between cycle graphs and paths will be used later on, we adopt a notation for it:

$S_{N,M} = C_M \circ P_{N-M+1}$.

Thus, $S_{N,M}$ represents a graph on N nodes (and N edges), consisting of a M-cycle with a pendant path. For example, the graph depicted in Fig. 5 is denoted as $S_{6,5}$.

The following lemma relates the reliability polynomial to the edge connectivity, i.e. the minimum number of edges that need to be removed in order to disconnect the graph.

Lemma 3 [8]: Let G be a graph with edge connectivity $\lambda = \lambda(G)$ and denote by $r_\lambda(G)$ the number of subsets of λ edges whose removal disconnects G. Then, upon writing $p = 1 - q$, it holds $R_G(q) = 1 - r_\lambda(G)q^\lambda + o(q^\lambda)$.

Lemma 3 shows that the reliability polynomial close to p = 1 is determined by the edge connectivity and the number of minimum cut sets.

Lemma 4: Suppose that F_1 and F_2 are connected graphs, $k \geq 1$ and p in (0,1). For $s \geq 3$, set

$$H_{1,s} = F_1 \circ C_s \text{ and } H_{2,s} = F_2 \circ S_{s,s-k}.$$

Then provided s is large enough, the sign of $R_{H1,s}(p)- R_{H2,s}(p)$ is the same as the sign of $R_{F1}(p)- R_{F2}(p)$.

Proof. From Lemmas 1 and 2 it follows that $R_{Ss,s-k}(p)/ R_{Cs}(p) = (p + (s-k)(1-p))/ (p + s(1-p))$, which tends to 1 for any fixed p in (0,1). Now from Lemma 2 again,

$$\begin{aligned} R_{H1,s}(p)- R_{H2,s}(p) &= R_{F1}(p) R_{Cs}(p)- R_{F2}(p) R_{Ss,s-k}(p) \\ &= R_{Cs}(p) (R_{F1}(p) - (R_{Ss,s-k}(p)/ R_{Cs}(p))R_{F2}(p)) \end{aligned}$$

As $R_{Cs}(p) > 0$ and $R_{Ss,s-k}(p)/ R_{Cs}(p)$ tends to 1, we see that $R_{H1,s}(p)- R_{H2,s}(p)$ tends to $R_{F1}(p)- R_{F2}(p)$, and the first result now follows.

Now consider the graphs G_1 and G_2, with N = 7 and M = 15, given in Section 2, whose reliability polynomials have two crossing points, namely at p_1=0.0911 and at p_2=0.3229, with G_2 more reliable than G_1 near 1 (and near 0).

We now define $G_{1,s}^*$ as the concatenation of G_1 and C_s, i.e. $G_1^* = G_1 \circ C_s$ and $G_{2,s}^*$ as the concatenation of G_2 and $C_{s,s-1}$, i.e.$G_2^* = G_2 \circ S_{s,s-1}$.

Pick any points t_1, t_2, t_3, t_4 in (0,1) such that
$t_1 < p_1 < t_2, < t_3 < p_2 < t_4$.
As p_1 and p_2 are crossing points for G_1 and G_2, we see that $R_{G1}(t_i)- R_{G2}(t_i)$ (i = 1,2,3,4) alternate in sign. From Lemma 4, it follows that there is some S such that for all $s \geq S$, the same is true for $R_{G1,s^*}(t_i)- R_{G2,s^*}(t_i)$, and hence, by the Intermediate Value Theorem, $R_{G1,s^*}(p)- R_{G2,s^*}(p)$ has a crossing point in each of the intervals (t_1, t_2) and (t_3, t_4). Fix any $s \geq S$.

Lastly, we need to show that there is at least one more crossing point. The sign of $R_{G1,s^*}(t_4)- R_{G2,s^*}(t_4)$ is the same as the sign of $R_{G1}(t_4)- R_{G2}(t_4)$, which is negative, as G_2 is more reliable than G_1 to the right of $p_2 < t_4$. That is, G_2 is more reliable than G_1 at t_4. But both G_1 and G_2 are 2-connected, and as C_s has edge connectivity 2 while $S_{s,s-1}$ has edge connectivity 1, we find that $\lambda(G_{1,s}^*) = 2 > 1 = \lambda(G_{2,s}^*)$. From Lemma 3, it follows that close to 1, $G_{1,s}^*$ is more reliable than $G_{2,s}^*$, and hence there must be another crossing point for $G_{1,s}^*$ and $G_{2,s}^*$ (in (t_4,1)). It follows that $G_{1,s}^*$ and $G_{2,s}^*$ have at least 3 crossing points. (It turns out that for $s \geq 442$ G_1^* is less reliable than G_2^* around p = 0 and hence this is the onset for the regime with three crossing points.)

Table 2 shows the crossing points for G_1^* and G_2^* for some values of s.

TABLE 2. CROSSING POINTS FOR $G_{1,s}^*$ AND $G_{2,s}^*$ FOR SOME VALUES OF S

s	Crossing 1	Crossing 2	Crossing 3
442	0.0001766	0.3897	0.9480
1000	0.04679	0.3567	0.9666
100000	0.09061	0.3233	0.9968

Note that for s = 100000 the two smaller crossing points are indeed very close to p_1 and p_2.

B. *Construction method for k crossings*

The argument of the previous section can be extended to provide, for every $k \geq 3$, pairs of simple graphs with at least k crossing points. The general idea is to use the concatenation of graphs several times.

Specifically, the graphs $G_{1,s}^*$ and $G_{2,s}^*$ have 3 crossing points, with $G_{1,s}^*$ more reliable near 1 as it has edge connectivity 2 while $G_{2,s}^*$ has edge connectivity 1, with $r_\lambda(G_{2,s}^*) = 1$ (as $G_{2,s}^*$ is the concatenation of a 2-connected graph with a cycle and an path of length 1). Setting $G_1^3 = G_{1,s}^*$ and $G_2^3 = G_{2,s}^*$, we now consider $G_1^4 = G_2^3 \circ C_m$ and $G_2^4 = G_1^3 \circ C_{m,m-2}$ (note the changing places of G_1^3 and G_2^3). Then as before (using Lemma 4), for large enough m, G_1^4 and G_2^4 have 3 crossings near the crossings of G_1^3 and G_2^3, with G_2^4 more reliable than G_1^4 to the right of the third crossing.

To see that G_1^4 and G_2^4 have an extra crossing near 1, we observe that both will have edge connectivity 1 (as both have leaves), but $G_1^4 = G_2^3 \circ C_m$ has only 1 edge-cut of size 1 (that of G_2^3) while $G_2^4 = G_1^3 \circ C_{m,m-2}$ has 2 edge-cuts of size 1 (those in the path of length 2 in $C_{m,m-2}$). Thus $\lambda(G_1^4) = \lambda(G_2^4) = 1$ and $r_\lambda(G_1^4) = 1 < 2 = r_\lambda(G_2^4)$, so by Lemma 3, G_1^4 is more reliable than G_2^4 close to 1, and we get a fourth crossing.

More generally, for $k \geq 3$, we set $G_1^{k+1} = G_2^k \circ C_u$ and $G_2^4 = G_1^k \circ C_{u,u-2}$, and choose u large enough so that these pair of graphs have crossings near each of the crossings of G_1^k and G_2^k. Inductively we can prove that $\lambda(G_1^{k+1}) = \lambda(G_2^{k+1}) = 1$ and $r_\lambda(G_1^{k+1}) = k\text{-}2 < k\text{-}1 = r_\lambda(G_2^{k+1})$, so by Lemma 3, G_1^{k+1} is more reliable than G_2^{k+1} close to 1, and we get a $(k+1)^{th}$ crossing.

C. *Examples with four and five crossings*

In this subsection we will give examples of pairs of graphs with four and five crossings, where the size of the graphs is as small as possible, at least according to our construction method. Recall from subsection 3.A that for three crossings, we have an example with $s_1 = 442$, i.e. the size of the graphs in that case is N = 7 + 442 – 1 = 448 and L = 15 + 442 = 457.

Starting with the pair $(G_1 \circ C_{442}, G_2 \circ S_{442,441})$, it is easy to show that the onset for four crossings points occurs at $s_2 = 1043$. To be more precise, the reliability polynomials of the pair of graphs

$$(G_1 \circ C_{442} \circ S_{1043,1041}, G_2 \circ S_{442,441} \circ C_{1043})$$

have exactly four crossing points, namely at 0.0746, 0.3357, 0.9892 and at 0.9907. The graphs in this example have 1490 nodes and 1500 edges.

Continuing the construction, we find that five crossings occur from $s_3 = 5833$ onwards. The reliability polynomials of the pair of graphs

$$(G_1 \circ C_{442} \circ S_{1043,1041} \circ C_{5833}, G_2 \circ S_{442,441} \circ C_{1043} \circ S_{5833,5831})$$

have exactly five crossing points, namely at 0.05963, 0.3471, 0.9748, 0.999256 and at 0.999263. The graphs in this example have 7322 nodes and 7333 edges.

IV. Additional Observations

We also plotted all crossing points for simple graphs of order at most 8 (there are none of order smaller than 6), and found that they begin to fill out the interval from 0 to 1. In fact, we can prove that for any k ≥ 1, the closure of the crossings of all pairs of reliability polynomials that cross at least k times is the entire interval [0,1]. A sketch of the argument is as follows.

It is not hard to prove that, given graph G on n nodes and m edges, if we form graphs F and H by replacing each edge by a bundle of *i* edges in parallel, or by a path of length *j*, respectively, then

$$R_F(p) = R_G(1-(1-p)^i)$$

and

$$R_H(p) = p^{m(j-1)}(j(1-p)+p)^m R_G\left(\frac{p}{p+j(1-p)}\right).$$

For a value r in [0,1], $1-(1-p)^i = r$ iff $p = 1-(1-r)^{1/i}$, and $\frac{p}{p+j(1-p)} = r$ iff $p = \frac{jr}{1+(j-1)r}$. The functions $f_i(r) = 1-(1-r)^{1/i}$ and $g_j(r) = \frac{jr}{1+(j-1)r}$ are increasing functions from [0,1] onto [0,1]. It follows that both the parallel edge replacement and path replacement preserve crossings. Moreover, for any fixed *r*, as *j* tends to infinity, $f_i(r)$ tends to 1 while $g_j(r)$ tends to 0. Using this, the result follows once it is shown that for *any* x_0 in (0,1), the closure of the set $\{f_s(g_l(x_0)) : l, s \geq 2\}$ is all of [0,1]. To do so, we need to show that for any *n*, $f_s(g_l(x_0)) \in [r, r+1/n]$ for some $s, l \geq 2$. We observe that $f_{s+1}(r) - f_s(r) < \frac{1}{s}$. We then choose *l* large enough so that $g_j(x_0) < \frac{r}{n-rn+r}$ and apply f_s for large enough s until we land up in $[r, r+1/n]$ The details are omitted.

Finally, our investigation of reliability polynomials of small order showed something very interesting. In [12], it was conjectured that no reliability polynomial has more than one point of inflection. We have found that the conjecture is false, with examples starting on 7 nodes. One such example is the complete tripartite graph $K_{1,2,2}$ with two leaves attached (see Fig. 6). The corresponding reliability polynomial is

$$45p^6 - 128p^7 + 142p^8 - 72p^9 + 14p^{10}$$

with points of inflection at, approximately, 0.8190 and 0.8787 (the second derivative of the reliability polynomial is plotted in Fig. 7).

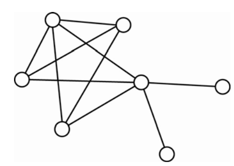


Figure 6: Smallest simple graph with 2 points of inflection

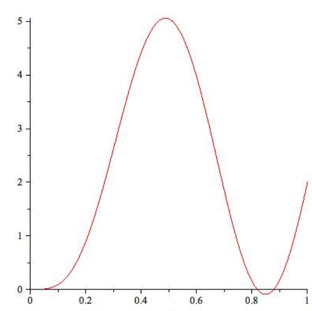


Figure 7: Plot of the second derivative of the reliability polynomial of the graph in Fig. 6

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