Shifting the Link Weights in Networks

Huijuan Wang and Piet Van Mieghem

***Abstract*— Transport in large networks follows near to shortest paths. A shortest path depends on the topology as well as on the link weight structure. While much effort has been devoted to understand the properties of the topology of large networks, the influence of link weights on the shortest path received considerably less attention. Here, we first compute analytically and by simulation the effect of shifting the uniform distribution for the link weights from $[0,1]$ to $(a,1]$ where $a>0$. When $a>0$, the properties of the shortest path (hopcount and weight) differ from those of small a close to zero. The difference is shown to be highly related to the topology. Furthermore, by tuning the link weight parimeter a, the traffic can be controlled. When a is large, the traffic is more likely to follow the minimum hopcount shortest path, which leads to more balanced traffic traversing the network.**

***Index Terms*— Link weight, graph, network, shortest path, network simulations.**

I. Introduction

While in most practical networks, there exist many possible paths between a source and a destination, usually the path that minimizes some link weight (e.g. the hopcount, the delay, the monetary cost, etc.) is preferred. Paths in the Internet can be observed via the trace-route utility. About 70-80% of the Internet paths are shortest paths and we expect that this portion will still increase, because of efficiency and cost reduction reasons: the more the path deviates from a shortest, the larger the loss in resources or, equivalently, the higher its cost. In this article we concentrate on properties of shortest paths, in particular, the influence of the link weights on the shortest path.

Although link weights are obviously needed to compute a shortest path in a graph, in practice, little is known about the link weights. We will not infer link weights from the shortest path measurements [2]. Instead, we are interested in the combined modeling of the topology of the network and the link weights. It is natural to first investigate the effect of the link weight structure on the resulting routes with given topology. In fixed networks, link weights are usually chosen as part of an optimization process with given togology and traffic characteristics, which is also termed as traffic engineering [1]. Our work provides insights on how the link weight structure affects the traffic.

Partial studies of effects of link weights on the optimal or shortest path in complex networks are found in [3][4] which characterize many biological, social and communication systems [5][6][7].

We confine ourselves to additive and strict positive link weights such that the shortest path is the minimizer of the sum of the link weights of any path between those two nodes. We investigated the influence of shifting the uniform distribution of link weights. Any other link weight distribution can be generated as a function of the shifted uniform distribution. Therefore, our work contributes to the selection of link weight structure for network simulations.

In this paper, we will briefly review theory on the weight and hopcount of the shortest path in Section II. And relate this with weak and strong disorder regimes studied in physical complex system. The motivation to investigate the shifted uniform distribution in different classes of graphs is also explained. The shifted uniform distribution is defined in Section III. In the next Section IV, we show by simulation and by analytic computation how the characteristics of the shortest path change when the link weight distribution is shifted away from zero in random graphs. In Section V, the simulation and analytic results are presented for the square lattice. The results are summarized in Section VI.

II. The shortest path

In large networks, the link weights are hardly correlated and can be considered as independent to a good approximation. A notable exception are wireless ad-hoc networks in which the "air-capacity" correlates all links in a certain (small) area. A focal point is the importance of the distribution of the link weights around zero.

Since the shortest path (SP) is mainly sensitive to the smaller, non-negative link weights, the probability distribution of the link weights around zero will dominantly influence the properties of the resulting shortest path tree. A *regular* link weight distribution $F_w(x)=\Pr[w\le x]$ has a Taylor series expansion around $x=0$,

$$F_w(x)=f_w(0)\,x+O\left(x^2\right)$$

since $F_w(0)=0$ and $F'_w(0)=f_w(0)$ exists. A regular link weight distribution is thus linear around zero. The factor $f_w(0)$ only scales all link weights, but it does not influence the shortest path. However, adding a constant to all link weights can change the shortest paths. Indeed, suppose that the shortest path contains many hops and the second shortest path only a few. In that case, there always exists a positive constant that, after added to all link weights, dethrones the initial shortest path. As further illustrated in Section IV, the probability distribution of the weight of the shortest path can change dramatically when adding a constant to all link weights in the network.

The simplest distribution of the link weight w with a distinct different behavior for small values than a regular distribution is the polynomial distribution,

$$F_w(x)=x^{\alpha}1_{x\in[0,1]}+1_{x\in[1,\infty)},\quad \alpha>0, \tag{1}$$

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where the indicator function 1_x is one if x is true else it is zero. The corresponding density is $f_w(x) = \alpha x^{\alpha-1}$, $0 < x < 1$. The exponent

$$\alpha = \lim_{x \downarrow 0} \frac{\log F_w(x)}{\log x}$$

is called the *extreme value index* of the probability distribution of w and $\alpha = 1$ for regular distributions. By varying the exponent α over all non-negative real values, any extreme value index can be attained and a large class of corresponding shortest path trees (SPT), in short α-trees, can be generated.

Let us consider a connected graph $G(N, L)$ with N nodes and L links and with independent polynomial link weights specified by (1). We briefly present three special α-trees for $\alpha = 1, \infty$ and 0, respectively and then limit ourselves to the range $\alpha \in [1, \infty]$ in Section IV and Section V.

Link weights with $\alpha = 1$ are e.g. those that are uniformly or exponentially distributed. All links contribute to the sum, the weight of the shortest path and this case corresponds to weak disorder. Earlier in [16], it was shown that the SPT in the complete graph with uniform (or exponential) link weights is precisely a Uniform Recursive Tree (URT). A URT is grown by sequentially attaching a new node uniformly to a node that is already in the URT. A URT is asymptotically the shortest path tree in the Erdös-Rényi random graph $G_p(N)$ (see e.g. [10]) with link density p above the disconnectivity threshold $p_c \sim \frac{\ln N}{N}$. The interest of the URT is that analytic modeling is possible (see e.g. [15, Part III]) and that it serves as a reasonable first order model to explain measurements in the Internet.

If $\alpha \to \infty$, it follows from (1) that $w = 1$ almost surely for all links. Since all links have unit weight, the $\alpha \to \infty$ regime reduces to the computation of the SPT in the underlying graph. The $\alpha \to \infty$ regime is thus entirely determined by the topology of the graph because the link weight does not differentiate between links. This also corresponds to the week disorder regime. In the Erdös-Rényi random graphs [10] with unit link weight and also in a small world graph, the average hopcount of the shortest path $E[H_N]$ scales as $\log N$.

If $\alpha \to 0$, the ratio $\frac{\sqrt{\text{Var}[w]}}{\mathbb{E}[w]} \sim \frac{1}{\sqrt{\alpha}}$ diverges which means that, in this limit, the link weights possess strong fluctuations. The $\alpha \to 0$ regime corresponds to a strong disorder regime where one link, the one with the maximum link weight in a path, controls the selection of the shortest path. This regime exhibits a quite eccentric behavior as shown in [18] and [17]. Equipping such link weight structure to the Erdös-Rényi random graphs, the average hopcount of the shortest path $E[H_N]$ scales as $N^{1/3}$.

Figure 1 illustrates schematically the probability distribution of the link weights around zero $(0, \epsilon]$, where $\epsilon > 0$ is an arbitrarily small, positive real number. The larger link weights in the network will hardly appear in a shortest path provided the network possesses enough links. These larger link weights are drawn in Figure 1 from the double dotted line to the right. The nice advantage that only small link weights dominantly influence the property of the resulting shortest path tree lies in that the remainder of the link weight distribution (denoted by the arrow with larger scale in Figure 1) only plays a second order role. However, the properties of the shortest path will change when the link weights are shifted away from zero. Perhaps the simplest case is the uniform distribution between $0 < a < b$, which we call the *shifted uniform distribution*, and which is studied in this paper. Apart from being attractive in a theoretical analysis, the uniform distribution is the underlying distribution to generate an arbitrary other distribution and is especially interesting for computer simulations. Hence, this distribution appears most often in network simulations and deserves – for this reason alone perhaps – to be studied.

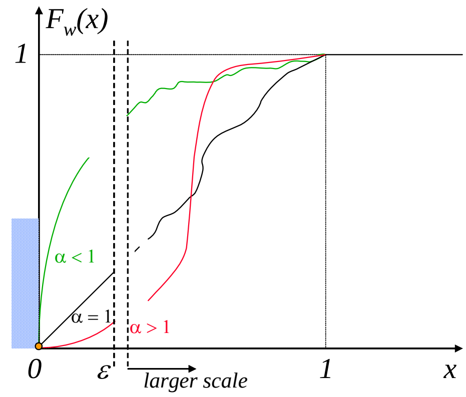

Fig. 1. A schematic drawing of the distribution of the link weights for the three different α-regimes. The shortest path problem is mainly sensitive to the small region around zero. The scaling invariant property of the shortest path allows us to divide all link weights by the largest possible such that $F_w(1) = 1$ for all link weight distributions.

The understanding of the shortest path with shifted uniformly distributed link weights will also give more insights into the stability of paths. For instance, the changes in the shortest path due to the adding of constant noise to all link weights. The interest in understanding the stability of paths lies in the fact that it could direct efficient triggers for network updates. For example, when does a node decide to inform the rest of the network about the changes in the state of one or more of its links.

As shown earlier in [17], by tuning the *extreme value index* α in Equation (1), all traffic can flow over a critical backbone, the minimum spanning tree. In contrast, by tuning the parameter a in the shifted uniform distribution as defined by (3) below, we are able to force more balanced traffic in the network. When a is large, as will be shown in Section IV and Section V, the weight of paths with more hops is more likely higher than that of paths with less hops. Hence, resources in the network are used more efficiently and the total traffic in the network is likely more balanced.

We study the following complex network models: the Erdös-Rényi random graph $G_p(N)$, the square lattice and the scale-free graph. Traditionally, the complex networks have been modeled as Erdös-Rényi random graphs. Besides that, the Erdös-Rényi random graphs are reasonably accurate models for peer-to-peer networks [20] and ad-hoc networks [11]. The square lattice, in which each node has four neighbors, is the basic model of a transport network as well as in percolation theory [14]. It is also frequently used to study the network traffic (see e.g. [21][22][23]). The scale-free graph [24] is proposed as model for complex networks that have a power-law degree distribution [8], such as the World Wide Web and the Internet.

III. Notations of the shifted uniformly distributed link weights

Any shifted uniformly distributed link weights w can be specified by

$$f_w(x) = \frac{1_{a<x\leq b}}{b-a} \quad (2)$$

The shifted link weight probability density function (2) can be considered as a result from adding a constant a to a uniform link weight in $[0,1]$ when $b = 1 + a$.

Figure 2 shows the three possible cases. The scaling of all

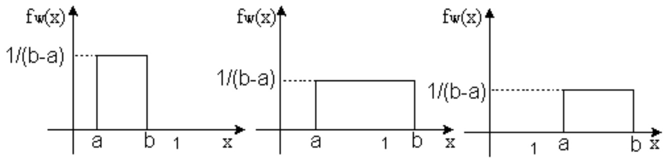

Fig. 2. Shifted uniform distribution. (left) $0 < a < b \leq 1$, (middle) $0 < a \leq 1 < b$ and (right) $1 \leq a < b$.

link weights in the graph by a positive number does not change the shortest path. Multiplying all weights w by $\frac{1}{b}$ reduces all three cases to the left one in Figure 2 where b then equals 1. If (capital) W denotes the weight of the shortest path, the scaling of the link weights w by $\frac{1}{b}$, results in a weight $\frac{W}{b}$ of the shortest path with probability density function (*pdf*)

$$\begin{aligned} f_{\frac{W}{b}}(x) &= \frac{d}{dx}\Pr\left[\frac{W}{b} \leq x\right] \\ &= \frac{d}{dy}\Pr[W \leq y] \cdot \left.\frac{dy}{dx}\right|_{y=bx} \\ &= bf_W(bx) \end{aligned}$$

After scaling by $\frac{1}{b}$, the only specifier of the link weight is the parameter a with $0 < a < 1$ and (2) reduces to

$$f_w(x) = \frac{1_{a<x\leq 1}}{1-a}, \quad 0 \leq a < 1 \quad (3)$$

If h denotes the hopcount of a path P in a graph with link weights specified by (3), then the weight of P is bounded by

$$ah \leq w(P) \leq h$$

Since we assume that all link weights are i.i.d. and additive such that $w(P) = \sum_{j=1}^{h} w(n_j \rightarrow n_{j+1})$, we observe that for large h the weight of an arbitrary path tends to a Gaussian

$$w(P) \xrightarrow{d} N(hE[w], hvar[w])$$

by the Central Limit Theorem. Interestingly, this holds for any distribution of i.i.d.. link weights. Thus, for large h, the shortest path P^* between two nodes is the minimum of all Gaussian random variables that represent a path between those two nodes. For realistic networks, this number of possible paths n_P is huge such that, to a good approximation, we can assume that $n_P \rightarrow \infty$. Now, the complicating factor is that most of these paths have links in common, in other words, they are not independent. When we assume that these paths are independent, the limit distribution of $w(P^*)$ can be computed and will lead, after appropriate scaling, to a Gumbel distribution (see e.g. [9]). However, the assumption that paths are independent is in most cases not realistic. Simulation on a lattice, for example, where the source and the destination node are placed on the corner of the diagonal to assure a large hopcount h, show that $w(P^*)$ is almost Gaussian which means that the minimum of those dependent Gaussian random variables is again a Gaussian random variable. Unfortunately, the mean $E[w(P^*)]$ and variance $var[w(P^*)]$ are generally difficult to compute. In the sequel, we confine the discussion therefore to small world graphs (see e.g. [19]) for which h is small with high probability.

IV. The Shortest Path in $G_p(N)$ with Shifted Uniformly Distributed Link Weights

In this Section, we compute the weight and the hopcount of the shortest path in Erdös-Rényi random graphs $G_p(N)$ with shifted uniform link weights.

A. *The complete graph* ($p = 1$)

Let us first confine to the complete graph of which any other graph is a subgraph. The complete graph K_N with N nodes can be regarded as the random graph $G_1(N)$ with link density $p = 1$. Figure 3 shows the *pdf* $f_W(x)$ of the weight of the shortest path for different values of $a \leq \frac{1}{2}$.

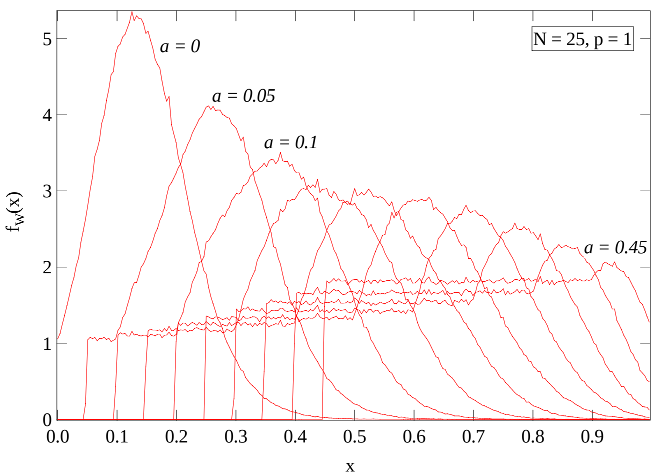

Fig. 3. The pdf of the weight of the shortest path in the complete graph with $N = 25$ nodes and with link weights given by (3) for various $a \in [0, \frac{1}{2}]$ with step $\Delta a = 0.05$.

The sequel is devoted to explain the curious behavior of the *pdf* of the weight of the shortest path in the complete graph with shifted uniform link weights specified by (3). The main interest here lies in $a > 0$ because the case $a = 0$ is known in detail. As mentioned in Section II, when $a = 0$, the shortest path tree in the complete graph with uniform links (or equivalently with exponential links since the extreme value index for both distributions is the same) is a URT. The probability generating function of the shortest path is derived in [15, Section 16.3] as

$$\varphi_{W_N}(z) = \mathbb{E}\left[e^{-W_N z}\right] = \frac{1}{N-1}\sum_{k=1}^{N-1}\prod_{n=1}^{k}\frac{n(N-n)}{z+n(N-n)} \tag{4}$$

Although the inverse Laplace transform can be computed analytically, the resulting expression for $f_{W_N}(x)$ is not quite insightful. Fortunately, a nice closed form asymptotic expression exists.

1) The case $\frac{1}{2} \leq a < 1$: When $a = \frac{1}{2}$, the link weights are uniformly distributed within $(\frac{1}{2}, 1]$. The weight of a path $P_{k-1} = n_1 \to n_2 \cdots \to n_k$ consisting of $k-1$ hops or links equals the sum of the weights of its constituent links

$$w(P_{h=k-1}) = \sum_{j=1}^{k-1} w(n_j \to n_{j+1})$$

where $w(P_{h=i})$ stands for the weight of a path with i hops. In the complete graph with link weights specified by (3) with $\frac{1}{2} \leq a < 1$, the shortest path must be the direct link, because the weight $w(P_{h>1})$ of any path with $h > 1$ hops and the weight $w(P_{h=1})$ of the direct link between the source and destination nodes obey

$$\begin{aligned} w(P_{h>1}) &= \sum_{j=1}^{h} w(n_j \to n_{j+1}) \geq \sum_{j=1}^{2} w(n_j \to n_{j+1}) \\ &> 1 \geq w(P_{h=1}) \end{aligned}$$

This argument shows that, when $\frac{1}{2} \leq a < 1$, the shortest path is the direct link (which always exists in the complete graph). The hopcount of the shortest path is always one and the weight of the shortest path is uniformly distributed within $(a, 1]$ which precisely explains the simulations in Figure 3 with $a = \frac{1}{2}$. This is the reason why the class of uniform distributions with $a > \frac{1}{2}$ are not simulated.

The same idea can be applied to explain why all the *pdf*s of the weight of the shortest paths in Figure 3 have certain uniformly distributed part. In a complete graph with uniformly distributed link weights specified by (3) with $0 < a < \frac{1}{2}$, the weight $w(P_{h>1})$ of any path with $h > 1$ hops is bounded by

$$w(P_{h>1}) = \sum_{j=1}^{h} w(n_j \to n_{j+1}) > a.h$$

From this relation we can draw two consequences. First, the direct link w is always the shortest path with $w(P_{h=1}) = w$ provided $w \in (a, 2a]$ because in the complete graph there are always $N-2$ paths with $h = 2$ hops. Thus, the probability density of the uniform part observed in Figure 3 is

$$f_W(x) = \frac{1}{1-a}, \quad x \in (a, 2a]$$

There are two extreme cases. When $a = 0$, the uniformly distributed area becomes a point with value 1, which corresponds to the point $f_W(0) = 1$ in Figure 3. When $a \geq \frac{1}{2}$, the *pdf* is uniformly distributed for $x \in (a, 1]$. Second, since the weight of the direct link $w(P_{h=1})$ is bounded by 1, the maximum possible number of hops in the shortest path P^* follows from $\min w(P_{h>1}) \leq 1$ as $h < \left[\frac{1}{a}\right]$ where $[x]$ denotes the integer part of the real number x. Hence, if $\frac{1}{k+1} \leq a < \frac{1}{k}$ for any integer $k \geq 1$, the shortest path has at most k hops.

2) The case $\frac{1}{3} \leq a < \frac{1}{2}$: This case corresponds to $k = 2$. Therefore, when the direct link weight lies in $(a, 2a]$, the weight of shortest path is uniformly distributed as explained above. When the direct link weight lies in $(2a, 1]$, the path with one hop and the $N-2$ paths with two hops compete to become the shortest path P^*. Any pair of paths between the source and destination node with one or two hops is independent, because these two hop paths do not have links in common and link weights are assumed to be independent.

Theorem 1: In the complete graph K_N equipped with link weights uniformly distributed within $(a, 1]$ and $\frac{1}{3} \leq a < \frac{1}{2}$, the *pdf* of the weight of the shortest path is

$$\begin{aligned} f_W(x) &= \frac{1_{a<x\leq 2a}}{1-a} + \frac{1_{2a<x\leq 1}}{1-a}\left(1 - \frac{1}{2}\left(\frac{x-2a}{1-a}\right)^2\right)^{N-2} \\ &\quad + \frac{(N-2)(1-x)(x-2a)}{(1-a)^3} \\ &\quad \times \left(1 - \frac{1}{2}\left(\frac{x-2a}{1-a}\right)^2\right)^{N-3} \cdot 1_{2a<x\leq 1} \end{aligned} \tag{5}$$

Proof: See Section A. □

This analytic result (5) is verified by simulation in Figure 4 for $a = 0.4$

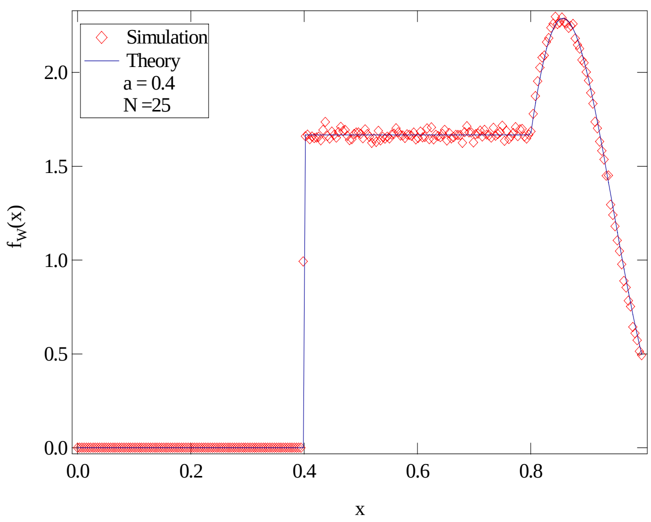

Fig. 4. The pdf of the weight of the shortest path in the complete graph with link weights specified by (3) with $a = 0.4$ both computed by (5) and simulated.

3) The case $a < \frac{1}{3}$: When $a < \frac{1}{3}$, the same reasoning as above shows that the shortest path can have three or more hops. In general, paths with three or more hops can be overlapping, which prevent simple analytic derivations as above and necessitates a combinatorial approach as shown in [13].

B. The Random Graph ($p < 1$)

We will extend the previous analysis to the broader class of Erdös-Rényi random graphs $G_p(N)$ which may serve as a first order graph model for e.g. wireless Ad-Hoc networks.

1) The case $a = \frac{1}{2}$*:* In contrast to the complete graph ($p = 1$), the weight of the shortest path in $G_p(N)$ can exceed unity if the direct link does not exist. By the law of total probability, we have

$$\begin{aligned} f_W(x) &= f_{W|P_{h=1}}(x)\Pr[P_{h=1} \text{ exists}] \\ &\quad + f_{W|P_{h>1}}(x)\Pr[P_{h=1} \text{ does not exist}] \\ &= 2p \cdot 1_{0.5<x\le 1} + (1-p) f_{W|P_{h>1}}(x) \qquad (6) \end{aligned}$$

because $f_{W|P_{h=1}}(x) = f_w(x)$ is uniformly distributed within $(0.5, 1]$. When $a = 0.5$, the weight of any path with h hops is bounded by

$$\frac{h}{2} < w(P_h) \le h$$

Suppose the shortest path P^* has k hops and suppose that there exists a path with h hops. Then $\frac{k}{2} < w(P^*) \le h$, which implies that the shortest path cannot have more than $2h$ hops. Since the probability that no path with 2 hops exists in $G_p(N)$,

$$\Pr[P_{h=2} \text{ does not exist}] = (1-p^2)^{N-2}$$

rapidly decrease for $p < 1$ and sufficiently large N, the weight in case the direct link does not exist, is mainly determined by the 2 hops and 3 hops paths.

Theorem 2: For large N, in the Erdös-Rényi random graphs $G_p(N)$ with link density $p > p_c \sim \frac{\ln N}{N}$, i.e. as long as the graph $G_p(N)$ is connected and equipped with link weights uniformly distributed within $(\frac{1}{2}, 1]$, the *pdf* of the weight of the shortest path is

$$\begin{aligned} f_W(x) &\simeq 2p \cdot 1_{0.5<x\le 1} + 4p^2(1-p)(N-2)(x-1) \\ &\quad \times (1 - 2p^2(x-1)^2)^{N-3} \cdot 1_{1<x\le 1.5} \qquad (7) \end{aligned}$$

Simulations in Figure 5 confirm the correctness of Theorem 2.

Proof: See Section B. □

2) The Case $\frac{1}{2} < a < 1$*:* Similar to the case of $a = 0.5$, the *pdf* of the weight of the shortest path as illustrated in Figure 6 consists of two parts: the uniform part when the direct link exists and the more complicated part when the direct link does not exist. For the second part, when the direct link does not exist, the *pdf* starts from $2a$, since

$$ah < w(P_h) \le h$$

The probability $\Pr[w(P^*_{h=2}) \le 3a]$ that the shortest path with two hops is smaller than $3a$ is similarly derived as (12) and equals

$$\Pr[w(P^*_{h=2}) \le 3a] = \begin{cases} 1 - (\frac{p^2}{2}(3 - \frac{1}{1-a})^2 + 1 - p^2)^{N-2} \\ \quad \text{for } \frac{1}{2} < a \le \frac{2}{3} \\ 1 - (1-p^2)^{N-2} \\ \quad \text{for } \frac{2}{3} < a \le 1 \end{cases}$$

which increases with a. Therefore, the probability for the shortest path to have more than 2 hops is even smaller than

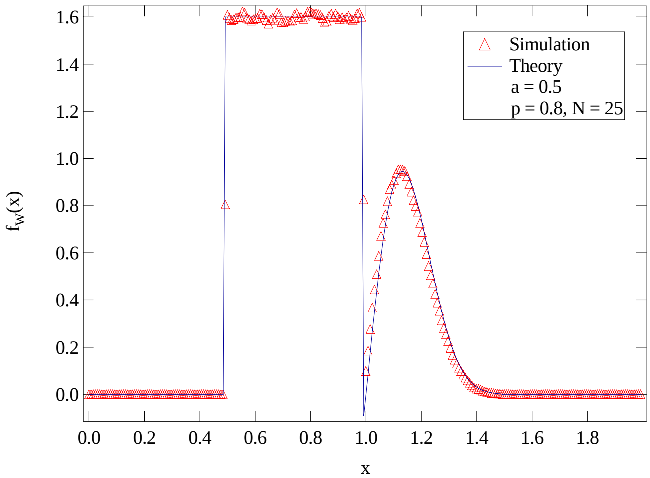

Fig. 5. The pdf of the weight of the shortest path with $a = 0.5$, $p = 0.8$ and $N = 25$ computed by (7) and by simulations.

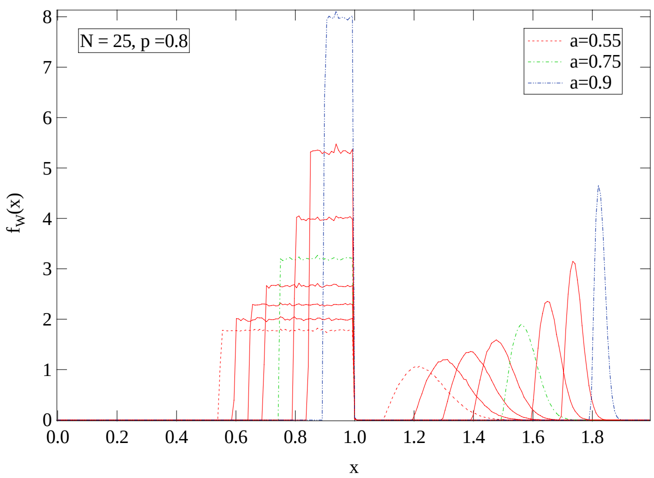

Fig. 6. The pdf of the weight of the shortest path with $a > 0.5$ (in creased in steps of $\Delta a = 0.05$)

in the previous case $a = 0.5$. The *pdf* of the weight of the shortest path in this case can be calculated analogously as for the case $a = 0.5$ with the result

$$\begin{aligned} f_W(x) &= \frac{p}{1-a} \cdot 1_{a<x\le 1} + p^2(1-p)(N-2) \\ &\quad \times \frac{x-2a}{(1-a)^2}\left(1 - 0.5p^2\left(\frac{x-2a}{1-a}\right)^2\right)^{N-3} \\ &\quad \times 1_{2a<x\le 1+a} \\ &\quad + p^2(1-p)(N-2)\frac{2-x}{(1-a)^2} \cdot 1_{1+a<x\le 2} \\ &\quad \times \left(0.5p^2\left(\frac{2-x}{1-a}\right)^2 + 1 - p^2\right)^{N-3} \end{aligned}$$

The third part is very small when p and N are large enough and can be approximated by 0.

3) The case $\frac{1}{3} \le a < \frac{1}{2}$*:* When the direct link exists and its weight is uniformly distributed within $(a, 2a]$, the weight is

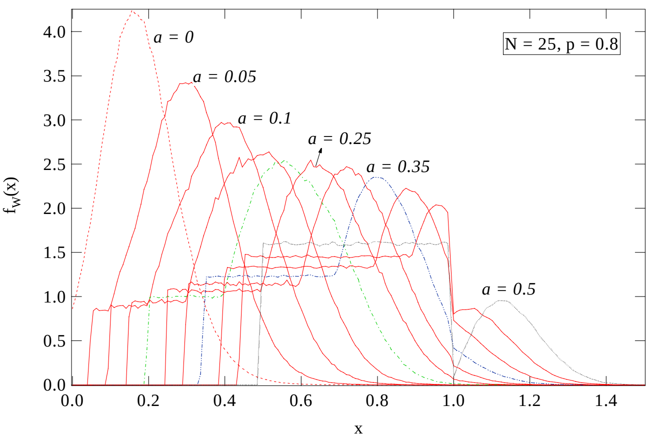

Fig. 7. The pdf of the weight of the shortest path in $G_{0.8}(25)$ with $a \leq 0.5$ (in steps of $\Delta a = 0.05$).

uniformly distributed with the density $\frac{p}{1-a}$. When the weight is distributed within $(2a, 1]$, the shortest path must have one or two hops because the weight of any 3 hops path is larger than 1. The shortest path with weight distributed within $(1, 3a]$ is caused by the competition among two hops paths to be the shortest path. When the weight of the shortest path is larger than $3a$, the computation of the *pdf* is complicated due to overlapping links. For example, the weight of the shortest path distributed within $(3a, 4a]$ is the result of the competition among paths with 2 hops and 3 hops, where complex correlations exist among these paths.

4) The case $a < \frac{1}{3}$: Similar to the corresponding case for $p = 1$, no simple analysis is expected for this case due to the dependence of paths that compete to be the shortest. Simulation results are shown in Figure 7.

C. Summary

We have shown in this section that the case $a \geq \frac{1}{3}$ is analytically tractable. Earlier [12], the case for $a = 0$ has been computed analytically, which leaves the case $a \in \left(0, \frac{1}{3}\right)$ open as a problem that still requires an analytic solution.

The random graph of the class $G_p(N)$ are reasonable models for Ad-Hoc networks [11]. If a is not too small, almost all shortest paths are shown to consist of a few hops which seems to agree with practice in multi-hop wireless networks. In these networks where the link weight represents the delay, the value of a is indeed bounded from below by (a) the propagation delay and (b) the minimum processor time to transmit an IP packet. On the other hand, interpretations of simulations that target e.g. to compare routing algorithms or protocols should take the quite small hopcount into account when a shifted uniform link weight distribution as (2) is used in small world networks.

V. THE SHORTEST PATH IN A SQUARE LATTICE WITH SHIFTED UNIFORMLY DISTRIBUTED LINK WEIGHTS

The Erdös-Rényi random graphs $G_p(N)$ belongs to the class of "small-world" graphs [25], where the average hopcount of the shortest path is usually small, with average on the order $O(\log N)$. In a lattice with N nodes, the hopcount of the shortest path is much larger than that in a random graph, on average on the order $O(\sqrt{N})$. In this Section, we investigate the weight and the hopcount of the shortest path in a two-dimensional square lattice with shifted uniformly distributed link weight specified by (3). The lattice has size x and contains $N = (x+1)^2$ nodes. Two cases are studied: (a) the source and destination are positioned at the diagonal points and (b) they are randomly chosen among the N nodes in the lattice.

A. The source and destination nodes are positioned at diagonal points

Here, the diagonal points are always chosen as the source and destination nodes. For the class of square lattices with N nodes, the minimum hopcount between the diagonal points is $h_{\min} = 2\sqrt{N} - 2$ and the number of paths with such minimum hopcount is $\binom{2x}{x}$, where $2x = h_{\min}$.

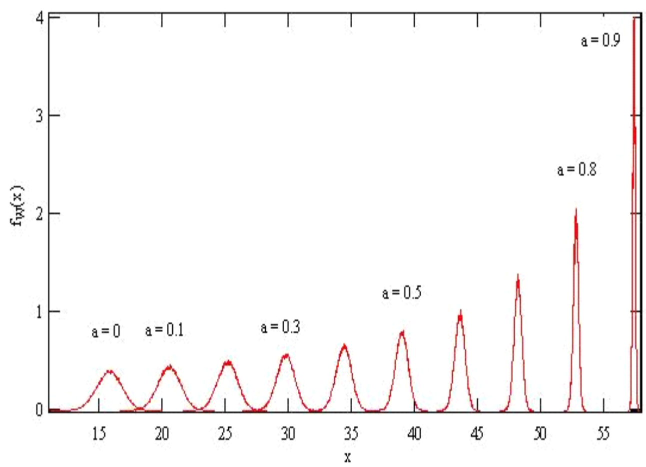

Fig. 8. The pdf of the weight of the shortest path in square lattice with $0 \leq a < 1$ (in steps of $\Delta a = 0.1$) and $N = 1024$.

Figure 8 shows the *pdf* $f_W(x)$ of the weight of the shortest path for different values of $0 \leq a < 1$ in a square lattice with $N = 1024$ nodes. Each *pdf* with a specified a resembles a Gaussian which is characterized by its mean and standard deviation. This is in contrast to the random graph, where the *pdf* change dramatically as a increases as shown previously in Section IV.

We further examine the mean and standard deviation $\sigma(W) = \sqrt{Var(W)}$ of the weight of the shortest path as a function of a, which are shown in Figure 9 and Figure 10. Both the average and standard deviation seem to be linear with a. When a is large, and exactly $a = 1$, the shortest path must have $h_{\min} = 2\sqrt{N} - 2 = 62$ hops. In this case, the average weight of the shortest path must be linear with a.

Assume that we have three graphs G_1, G_2 and G_3, which have the same topology, a square lattice with 1024 nodes. The links in G_1 are uniformly distributed within $(0, 1]$. The graph G_2 with uniform links distributed within $(b, 1+b]$ is obtained by adding a constant b to all links of G_1. Scaling all links

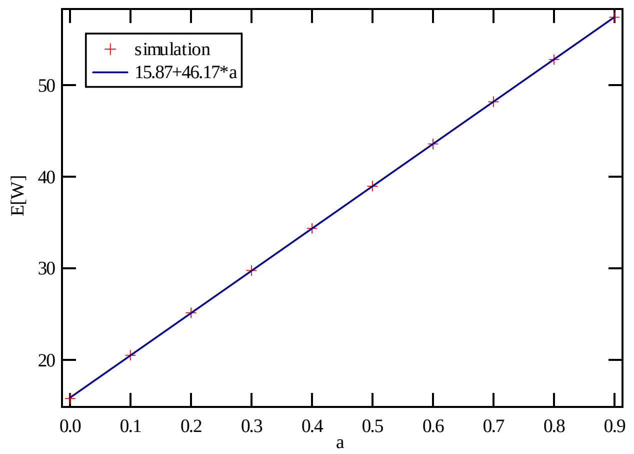

Fig. 9. The average weight of the shortest path in a square lattice with 1024 nodes and $0 \leq a < 1$ (in steps of $\Delta a = 0.1$).

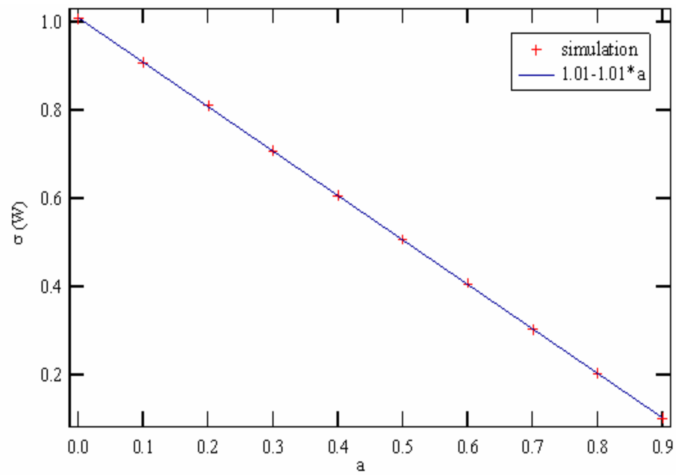

Fig. 10. The standard deviation of the weight of the shortest path in a square lattice with 1024 nodes and $0 \leq a < 1$ (in steps of $\Delta a = 0.1$).

in G_2 by $\frac{1}{1+b}$, we get G_3 which has the shifted uniformly distributed link weights specified by (3) with $a = \frac{b}{1+b}$. The shortest paths in G_2 and G_3 are the same, because the shortest path will not change when all the links are scaled. When a is large, the shortest path in G_2 and G_3 has hopcount $h_{\min}$. Moreover, it is equal to the shortest path among paths with $h_{\min}$ hops in G_1. Hence, in G_3, which corresponds to the graph we simulated, the average weight of the shortest path obeys

$$\begin{aligned} E[W_3(P^*)] &= \frac{E[W_1(P^*_{h_{\min}})] + b * h_{\min}}{1+b} \\ &= \frac{E[W_1(P^*_{h_{\min}})] + \frac{a}{1-a} * h_{\min}}{1+\frac{a}{1-a}} \\ &= (h_{\min} - E[W_1(P^*_{h_{\min}})]) * a + E[W_1(P^*_{h_{\min}})] \end{aligned} \quad (8)$$

where $P^*_{h_{\min}}$ denotes the shortest path among paths with $h_{\min}$ hops and $E[W_1(P^*_{h_{\min}})]$ is the average weight of $P^*_{h_{\min}}$ in G_1. Our simulation results show that when $a \geq 0.5$, the shortest path always has $h_{\min}$ hops. These simulations indicate that (8) only holds for $a \geq 0.5$. For any $0 \leq a < 1$,

$$E[W_3(P^*)] \geq \frac{E[W_1(P^*)] + b * h_{\min}}{1+b}$$

where the shortest path in G_3 with weight $W_3(P^*)$ may be different from the shortest path in the corresponding G_1 with weight $W_1(P^*)$. The reasons, why in Figure 9, the average weight seems always linear with a, are:

- In G_1 where $a = 0$, the weight of the shortest path $E[W(P^*)]$ is very close to $E[W(P^*_{h_{\min}})]$. We further fit the points with $a \geq 0.5$ and obtain the fitting curve $15.93 + 46.07 * a$, which indicates that $E[W(P^*_{h_{\min}})] = 15.93$. The simulation results show that $E[W(P^*)] = 15.77$ which is close to $E[W(P^*_{h_{\min}})]$.
- The hopcount of the shortest path in G_1 is very close to $h_{\min}$. In simulations, the average hopcount of the shortest path in G_1 is 64.2, while $h_{\min} = 62$.

Similarly, when a is large, the variance of $W_1(P^*_{h_{\min}})$ in G_1 is equal to the variance of $W_2(P^*)$ in G_2. However, the variance in G_3 is

$$\begin{aligned} Var[W_3(P^*)] &= (\frac{1}{1+b})^2 \cdot Var[W_1(P^*_{h_{\min}})] \\ &= (1-a)^2 \cdot Var[W_1(P^*_{h_{\min}})] \end{aligned}$$

where $Var[W_1(P^*_{h_{\min}})]$ is the variance of $P^*_{h_{\min}}$ in G_1. Hence, the standard deviation is

$$\sigma[W_3(P^*)] = -a * \sigma[W_1(P^*_{h_{\min}})] + \sigma[W_1(P^*_{h_{\min}})]$$

Since the $W_1(P^*_{h_{\min}})$ is close to $W_1(P^*)$ in G_1, the standard deviation $\sigma[W_3(P^*)]$ of the weight of the shortest path in G_3 is almost linear with a.

B. The source and destination nodes are chosen randomly

The analysis can be extended to a more general case, where the source and destination nodes are randomly chosen within a graph with N nodes. We show by simulation again the two points: in G_1 or when $a = 0$, the hopcount of the shortest path is very close to $h_{\min}$ and the weight of the shortest path $W(P^*)$ is very close to $W(P^*_{h_{\min}})$, which are responsible for the linear behavior of the average weight of the shortest path with a as shown in previous section V-A.

We carried out 10^6 iterations for each simulation. Within each iteration, uniformly distributed link weights within $[0, 1)$ are assigned independently to all the links in the square lattice with $N = 1024$ nodes. The minimum hopcount and the hopcount of the shortest path between the randomly chosen source and destination are calculated. The average hopcount of the shortest path with a given minimum hopcount is shown in Figure 11. The hopcount of the shortest path appears to be close to the minimum hopcount. The largest difference 3.8 occurs when the minimum hopcount is 30. According to the definition of G_1 and G_2 in Section V-A, by adding a constant link weight b to all the links in G_1 which are uniformly distributed within $[0, 1)$, we arrive at G_2. The constant link weight added may be caused by e.g. reserving certain resources of the network or by the delay due to traffic jam. The fact that $H(P^*) \approx H(P^*_{h\min})$ in G_1 indicate that,

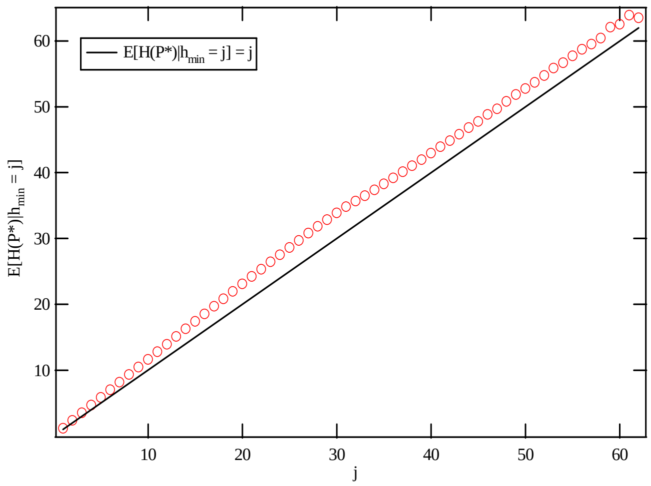

Fig. 11. The average hopcount of the shortest path in a square lattice with $N = 1024$ and $a = 0$, given the minimum hopcount between the source and destination is j.

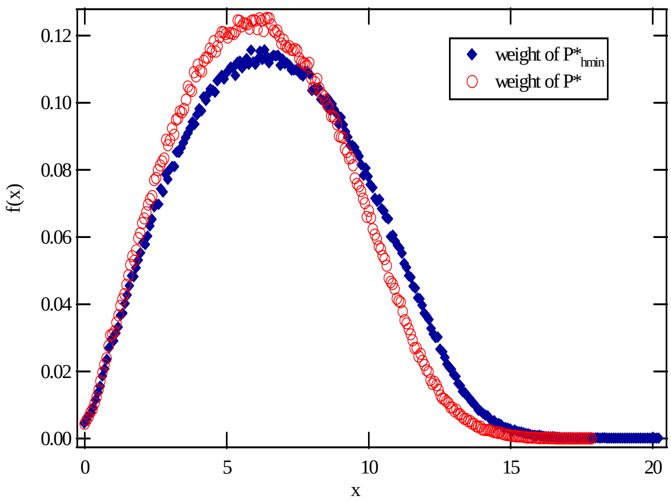

Fig. 12. The weight of the shortest path $W(P^*)$ and the weight of the shortest path among paths with the minimum hopcount $W(P^*_{h\,\min})$ in a square lattice with $N = 1024$ and $a = 0$.

when adding a constant to all links, the path is stable and rerouting is not needed. The difference in path weight is upper bounded by $(H(P^*) - H(P^*_{h\,\min})) * b$, which is small if b is not so large.

The shortest path subject to a given hopcount is more complex to calculate than the unconstrained shortest path problem. In fact, that problem is NP-complete. By shifting link weights, we can calculate the shortest minimum hopcount path much simpler. We observe that, when $a = 0.9$, all shortest paths follow the shortest minimum hopcount path. Hence, we first find the shortest path in G_3 with $N = 1024$ and $a = 0.9$ and corresponding $b = 9$. The weight of the shortest path is denoted by $W_3(P^*)$. Then, the weight of the shortest $h_{\min}$ hop path in the corresponding graph G_1 is

$$W_1(P^*_{h_{\min}}) = W_3(P^*) \cdot (1 + b) - h_{\min} \cdot b$$

The *pdf* of weight of the shortest minimum hopcount path $W_1(P^*_{h_{\min}})$ and the weight of the shortest path $W_1(P^*)$ are shown in Figure 12. We compare the *pdf* instead of presenting the difference in weight due to the large variance of $W_1(P^*)$ when the source and destination are chosen randomly. Figure 12 indicates that, when $a = 0$, $W(P^*_{h_{\min}}) \approx W(P^*)$ with average $E[W(P^*_{h_{\min}})] = 6.77$ and $E[W(P^*_h)] = 6.31$. Hence, even when we add a large constant to the links in G_1, the path can be stable if we choose the shortest minimum hopcount path $P^*_{h_{\min}}$ in both G_1 and G_2.

In summary, after adding a small constant to all links in a square lattice, the weight of the shortest path by rerouting is close to the weight of the original shortest path. The weight of the shortest minimum hopcount path $W(P^*_{h_{\min}})$ is close to the weight of the shortest path $W(P^*)$. Hence, the route $P^*_{h_{\min}}$ can always be chosen, no matter how large the constant added is. The shortest path in the lattice is more stable than that in the random graph. By shifting the link weight, the shortest path is more likely to follow the shortest minimum hopcount path, which leads to a simpler method to find the shortest path with hopcount $h_{\min}$ in the lattice.

VI. Conclusion

We have shown that the properties of the shortest path crucially depend on the extreme value index of the probability distribution of the link weights. Further, we have analyzed the effect of shifting the uniform distribution for the link weights from $[0, 1]$ to $(a, 1]$ where $a > 0$. By tuning the link weight parameter a to a larger value, the shortest path is more probable to have a less hopcount. The network resources are used more efficiently with balanced traffic traversing the network. In the Erdös-Rényi random graph, the case that $a > 0$ causes the properties of the shortest path (hopcount and weight) to be dramatically different than for a small $(a \to 0)$. However, the shortest paths in the square lattice are more stable in contrast to the small-world graphs. After shifting the link weights, the weight of the shortest path does not differ much from the weight of the original shortest path. The intuition is that, inrespective of the link weights, if $h_{\min}$ is large, the $i.i.d.$ link weights only seem a small perturbation of the $w = 1$ case. As a final remark, the scale-free networks are tree-like sparse graphs. There are few paths between the source and destination nodes [26]. Hence, the scale-free networks are expected to be stable when link weights are shifted.

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Appendix

A. Proof of Theorem 1

When the direct link weight lies in $(a, 2a]$, the weight of shortest path is uniformly distributed as explained above.

When the direct link is uniformly distributed within $(2a, 1]$, we only consider the direct link and $N - 2$ paths with 2 hops, because only one and two hop paths compete to be the shortest path and they are independent. The weight of the shortest path $w(P^*)$ is the minimum of the weights of these $N - 1$

paths. The weight of any path with 2 hops is the sum of two independent uniform link weights. The *pdf* of the weight of such a two hops path is (see e.g. [15, Chapt. 3])

$$\begin{aligned} f_{w(P_{h=2})}(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{e^{-az}-e^{-z}}{(1-a)z}\right)^2 \cdot e^{zx}dz \\ &= \frac{(x-2a)}{(1-a)^2}\cdot 1_{x>2a} - \frac{2(x-a-1)}{(1-a)^2}\cdot 1_{x>(1+a)} \\ &\quad + \frac{(x-2)}{(1-a)^2}\cdot 1_{x>2} \end{aligned}$$

After integration, we obtain

$$\Pr[w(P_{h=2})>x] = \begin{cases} 1, & x\le 2a \\ 1-\frac{1}{2}(\frac{x-2a}{1-a})^2, & 2a<x\le 1+a \\ \frac{1}{2}(\frac{2-x}{1-a})^2, & 1+a<x\le 2 \\ 0, & x>2 \end{cases} \tag{9}$$

If $w(P_{h=1}) > 2a$, then $w(P^*_{h\le 2})$ is the minimum weight of $N-1$ independent paths. For independent random variables $\{X_k\}_{1\le k\le m}$ holds that

$$\Pr\left[\min_{1\le k\le m} X_k \le x\right] = 1-\prod_{k=1}^{m}\Pr[X_k > x] \tag{10}$$

Thus, we have that

$$\begin{aligned} y &= \Pr[w(P^*_{h\le 2})\le x | w(P_{h=1})>2a] \\ &= 1-\Pr[w(P_{h=1})>x]\cdot(\Pr[w(P_{h=2})>x])^{N-2} \\ &= \begin{cases} 0, & x\le 2a \\ 1-\frac{1-x}{1-2a}\left(1-\frac{1}{2}(\frac{x-2a}{1-a})^2\right)^{N-2}, & 2a<x<1 \\ 1, & 1<x \end{cases} \end{aligned}$$

By derivation with respect to x, we obtain

$$\begin{aligned} f_{W|w(P_{h=1})>2a}(x) &= \frac{1_{2a<x\le 1}}{1-2a}\left(1-\frac{1}{2}\left(\frac{x-2a}{1-a}\right)^2\right)^{N-2} \\ &\quad + \frac{(N-2)(1-x)(x-2a)}{(1-2a)(1-a)^2} \\ &\quad \times\left(1-\frac{1}{2}\left(\frac{x-2a}{1-a}\right)^2\right)^{N-3} 1_{2a<x\le 1} \end{aligned}$$

As mentioned above, if and only if the weight of the direct link is distributed within $(a, 2a]$, the shortest path must be the direct link with path length uniformly distributed within $(a, 2a]$. By the law of total probability (see e.g. [15, Chapt. 1]), the *pdf* of the link weight of the shortest path then is

$$\begin{aligned} f_W(x) &= f_{W|w(P_{h=1})\le 2a}(x)\Pr[w(P_{h=1})\le 2a] \\ &\quad + f_{W|w(P_{h=1})>2a}(x)\Pr[w(P_{h=1})>2a] \\ &= \frac{1}{1-a}\cdot 1_{a<x\le 2a} \\ &\quad + \frac{1-2a}{1-a}\cdot f_{W|w(P_{h=1})>2a}(x)\cdot 1_{2a<x\le 1} \end{aligned}$$

where the first part follows from the definition of the conditional probability,

$$\begin{aligned} &\Pr[W\le x | w(P_{h=1})\le 2a]\Pr[w(P_{h=1})\le 2a] \\ =\ &\Pr[\{W\le x\}\cap\{w(P_{h=1})\le 2a\}] \\ =\ &\Pr[W\le x]\cdot 1_{a<x\le 2a} \end{aligned}$$

Finally, for $\frac{1}{3}\le a<\frac{1}{2}$, we arrive at the exact result 5. □

B. Proof of Theorem 2

We will first compute the weight of the shortest path in case that path consists of merely 2 hops. Applying (10), we have

$$\Pr[w(P^*_{h=2})>x] = (\Pr[w(P_{h=2})>x])^{N-2} \tag{11}$$

For an arbitrary path with two hops,

$$\begin{aligned} \Pr[w(P_{h=2})<x] &= \Pr[w(P_{h=2})<x | P_{h=2} \text{ exists }] \\ &\quad \times\Pr[P_{h=2} \text{ exists}] \end{aligned}$$

Using (9), we find that

$$\Pr[w(P_{h=2})<x | P_{h=2} \text{ exists }] = \begin{cases} 0, & x\le 1 \\ 2(x-1)^2, & 1<x\le 1.5 \\ 1-2(2-x)^2, & 1.5<x\le 2 \\ 1, & 2<x \end{cases}$$

such that

$$\Pr[w(P_{h=2})>x] = \begin{cases} 1, & x\le 1 \\ 1-2p^2(x-1)^2, & 1<x\le 1.5 \\ 1-p^2+2p^2(2-x)^2, & 1.5<x\le 2 \\ 1-p^2, & 2<x \end{cases}$$

Substituted into (11) yields

$$\Pr[w(P^*_{h=2})>x] = \begin{cases} 1, & x\le 1 \\ \left(1-2p^2(x-1)^2\right)^{N-2}, & 1<x\le 1.5 \\ \left(1-p^2+2p^2(2-x)^2\right)^{N-2}, & 1.5<x\le 2 \\ \left(1-p^2\right)^{N-2}, & 2<x \end{cases} \tag{12}$$

If $w(P^*_{h=2}) \le 1.5$ and the direct link does not exist, the shortest path must have 2 hops, because the weight of any 3 hops path is larger than 1.5. From (12), we observe that

$$\Pr[w(P^*_{h=2})\le 1.5] = 1-\left(1-\frac{1}{2}p^2\right)^{N-2}$$

For N sufficiently large and $p\le 1$, $\Pr[w(P^*_{h=2})\le 1.5]$ tends to 1 exponentially fast. This justifies the approximation

$$f_{W|P_{h>1}}(x) \approx f_{w(P^*_{h=2})}(x)$$

where

$$\begin{aligned} f_{W|P_{h>1}}(x) &\approx \frac{d\Pr[w(P^*_{h=2})\le x]}{dx} \\ &\approx 4p^2(N-2)(x-1) \\ &\quad \times(1-2p^2(x-1)^2)^{N-3}\cdot 1_{1<x\le 1.5} \end{aligned}$$

because, as explained before, $\Pr[w(P^*_{h=2})\le 1.5]\approx 1$, so the *pdf* for $1.5<x\le 2$ is approximated by 0. With (6), we finally arrive at (7). □

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