Graphs with given diameter maximizing the algebraic connectivity - numerical results

Huijuan Wang[1], R. E. Kooij[1,2] and P. Van Mieghem[1] *
[1]Faculty of Electrical Engineering, Mathematics, and Computer Science
Delft University of Technology, P.O. Box 5031, 2600 GA Delft
[2]TNO Information and Communication Technology,
P.O. Box 5050, 2600 GB Delft, The Netherlands

November 18, 2008

Abstract

In a submitted paper [1], we proposed a class of graphs $G_D^*(n_1, n_2, ..., n_{D+1})$, containing of a chain of $D+1$ cliques $K_{n_1}, K_{n_2}, ..., K_{n_{D+1}}$, where neighboring cliques are fully-interconnected. The class of graphs has diameter D and size $N = \sum_{1\leq i\leq D+1} n_i$. We proved that this class of graphs can achieve the maximal number of links, the minimum average hopcount, and more interestingly, the maximal of any Laplacian eigenvalue among all graphs with N nodes and diameter D. Numerically searching for the maximum of any eigenvalue is feasible, because (a) the searching within the class $G_D^*(n_1, n_2, ..., n_{D+1})$ is much smaller than within all graphs with N nodes and diameter D; (b) we reduce the calculation of the Laplacian spectrum from a $N \times N$ to a $(D+1)\times$ $(D+1)$ matrix. In this report, we present our numerical results: graphs in the class G_D^* that achieve the maximal algebraic connectivity, the second smallest Laplacian eigenvalue, among all graphs with N nodes and diameter D.

Let G be a graph and let $\mathcal{N}$ denote the set of nodes and $\mathcal{L}$ the set of links, with $N = |\mathcal{N}|$ nodes and $L = |\mathcal{L}|$ links, respectively. The Laplacian matrix of G with N nodes is a $N \times N$ matrix $Q = \Delta - A$, where $\Delta = \text{diag}(d_i)$ and d_i is the degree of node $i \in \mathcal{N}$ and A is the adjacency matrix of G. The Laplacian eigenvalues are all real and nonnegative [2]. The set of all N Laplacian eigenvalues $\mu_N = 0 \leq \mu_{N-1} \leq ... \leq \mu_1$ is called the Laplacian spectrum of G. The second smallest eigenvalue μ_{N-1}, also called after Fiedler's seminal paper [3], the algebraic connectivity, can be denoted as $\mu_{N-1} = a(G)$ for simplicity. The algebraic connectivity $a(G)$ is widely studied in the literature.

Definition 1 *The class of graphs $G_D^*(n_1, n_2, ..., n_{D+1})$ is composed of $D+1$ cliques $K_{n_1}, K_{n_2}, ..., K_{n_D}$ and $K_{n_{D+1}}$, where the variable $n_i \geq 1$ with $1 \leq i \leq D+1$ is the size or number of nodes of the $i-th$ clique. Each clique K_{n_i} is fully connected with its neighboring cliques $K_{n_{i-1}}$ and $K_{n_{i+1}}$ for $2 \leq i \leq D$. Two graphs G_1 and G_2 are fully connected if each node in G_1 is connected to all the nodes in G_2.*

We proved in [1] that the maximum algebraic connectivity of the class $G_D^*(n_1 = 1, n_2, ..., n_D, n_{D+1} = 1)$ is also the maximum $a_{\max}(N, D)$ over all graphs $G(N, D)$ with N nodes and diameter D. More generally, we prove that $G_D^*(n_1 = 1, n_2, ..., n_D, n_{D+1} = 1)$ can achieve the maximum of any Laplacian eigenvalue μ_i, $1 \leq i \leq N-1$, the maximum link density, the minimum average hopcount among all graphs $G(N, D)$.

The maximum algebraic connectivity can be searched numerically, which is feasible, because we search within the class $G_D^*(n_1 = 1, n_2, ..., n_D, n_{D+1} = 1)$ instead of all graphs with N nodes and diameter D. And,

*Email: h.wang@ewi.tudelft.nl, robert.kooij@tno.nl and p.vanmieghem@ewi.tudelft.nl

we reduce the computation of the Laplacian eigenvalue from a $N \times N$ to a $(D+1) \times (D+1)$ matrix. Here, we presents the numerical exhaustive searching results: graphs in G_D^* that obtain the maximal algebraic connectivity among graphs on N nodes and diameter D. Below, we list that for the given size N, given diameter D (column 1), the maximal algebraic connectivity (column 2) is achieved by a graph in the class G_D^*, whose sizes $(n_2, n_{3,} ... n_{D-1}, n_D)$ are presented since column 3. If the graph is not symetric, a star is marked at the beginning of the line.

$N = 10$

D=2 8 8

D=3 3.298438 4 4

D=4 1.438447 2 4 2

D=5 0.702929 2 2 2 2

D=6 0.438447 1 2 2 2 1

D=7 0.245717 1 1 2 2 1 1

D=8 0.152241 1 1 1 2 1 1 1

D=9 0.0979

$N = 15$

D=2 13 13

*D=3 5.582015 6 7

D=4 2.763932 4 5 4

*D=5 1.375193 2 4 4 3

D=6 0.837722 2 3 3 3 2

*D=7 0.490037 1 3 2 3 3 1

D=8 0.348558 1 2 2 3 2 2 1

*D=9 0.220619 1 1 2 2 2 2 2 1

*D=10 0.150965 1 1 2 1 2 2 2 1 1

*D=11 0.108933 1 1 1 2 1 2 2 1 1 1

D=12 0.079779 1 1 1 1 2 1 2 1 1 1 1

*D=13 0.057862 1 1 1 1 1 1 2 1 1 1 1 1

D=14 0.043705 1 1 1 1 1 1 1 1 1 1 1 1 1

$N = 20$

D=2 18 18

D=3 8.169048 9 9

D=4 4.000000 6 6 6

D=5 2.150032 4 5 5 4

D=6 1.245594 2 5 4 5 2

D=7 0.814923 2 3 4 4 3 2

D=8 0.513789 1 3 3 4 3 3 1

D=9 0.374821 1 2 3 3 3 3 2 1

D=10 0.265291 1 2 2 3 2 3 2 2 1

D=11 0.187141 1 1 2 2 3 3 2 2 1 1

D=12 0.145331 1 1 2 2 2 2 2 2 2 1 1

D=13 0.107786 1 1 1 2 2 2 2 2 2 1 1 1

D=14 0.079442 1 1 1 1 2 2 2 2 2 1 1 1 1

D=15 0.061792 1 1 1 1 2 1 2 2 1 2 1 1 1 1

D=16 0.048676 1 1 1 1 1 2 1 2 1 2 1 1 1 1 1

*D=17 0.038073 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 1

D=18 0.030384 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1

D=19 0.0246 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

$N = 25$

D=2 23 23

*D=3 10.512276 11 12

D=4 5.458619 7 9 7

*D=5 2.919904 4 7 7 5

D=6 1.760081 3 5 7 5 3

*D=7 1.100498 2 4 5 6 4 2

D=8 0.743292 2 3 4 5 4 3 2

*D=9 0.494043 1 3 3 4 4 4 3 1

D=10 0.369620 1 2 3 4 3 4 3 2 1

*D=11 0.277307 1 2 2 3 3 4 3 2 2 1

D=12 0.203396 1 2 2 2 3 3 3 2 2 2 1

*D=13 0.158140 1 1 2 2 3 2 3 3 2 2 1 1

D=14 0.123554 1 1 2 2 2 2 3 2 2 2 2 1 1

*D=15 0.096647 1 1 1 2 2 2 2 3 2 2 2 1 1 1

*D=16 0.075674 1 1 1 1 2 2 2 2 2 2 2 2 1 1 1

*D=17 0.060258 1 1 1 1 2 1 2 2 2 2 2 2 1 1 1 1

*D=18 0.048150 1 1 1 1 1 2 1 2 2 2 2 1 2 1 1 1 1

*D=19 0.039404 1 1 1 1 1 2 1 2 1 2 2 1 2 1 1 1 1 1

D=20 0.032674 1 1 1 1 1 1 2 1 2 1 2 1 2 1 1 1 1 1 1

*D=21 0.026858 1 1 1 1 1 1 1 1 2 1 2 1 2 1 1 1 1 1 1 1

D=22 0.022290 1 1 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 1 1 1 1

*D=23 0.018613 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1

D=24 0.015771 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

$N = 30$

D=2 28 28

D=3 13.118473 14 14

D=4 6.837722 9 10 9

*D=5 3.734477 6 8 9 5

D=6 2.251073 4 6 8 6 4

D=7 1.430406 3 5 6 6 5 3

D=8 0.967575 2 4 5 6 5 4 2

D=9 0.667334 2 3 4 5 5 4 3 2

D=10 0.467588 1 3 3 5 4 5 3 3 1

D=11 0.359826 1 2 3 4 4 4 4 3 2 1

D=12 0.273982 1 2 3 3 3 4 3 3 3 2 1

D=13 0.211939 1 2 2 3 3 3 3 3 3 2 2 1

*D=14 0.163294 1 1 2 2 3 3 3 3 3 3 2 1 1

D=15 0.132674 1 1 2 2 2 3 3 3 3 2 2 2 1 1

D=16 0.104274 1 1 2 2 2 2 3 2 3 2 2 2 2 1 1

*D=17 0.085787 1 1 1 2 2 2 2 3 2 3 2 2 2 1 1 1

D=18 0.069996 1 1 1 2 2 2 2 2 2 2 2 2 2 2 1 1 1

D=19 0.058281 1 1 1 1 2 2 2 2 2 2 2 2 2 2 1 1 1 1

D=20 0.047462 1 1 1 1 2 1 2 2 2 2 2 2 2 1 2 1 1 1 1

D=21 0.039315 1 1 1 1 1 2 1 2 2 2 2 2 2 1 2 1 1 1 1 1

D=22 0.032636 1 1 1 1 1 1 2 1 2 2 2 2 2 1 2 1 1 1 1 1 1

D=23 0.027377 1 1 1 1 1 1 2 1 2 1 2 2 1 2 1 2 1 1 1 1 1 1
D=24 0.023408 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1
*D=25 0.019914 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1
D=26 0.017011 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1
D=27 0.014550 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1
D=28 0.012576 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1
D=29 0.010956 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
$N = 35$
D=2 33 33
*D=3 15.478766 16 17
D=4 8.258343 10 13 10
*D=5 4.545953 6 10 10 7
D=6 2.731195 4 8 9 8 4
*D=7 1.758466 3 6 7 8 6 3
D=8 1.160770 2 5 6 7 6 5 2
*D=9 0.827059 2 4 5 5 6 5 4 2
D=10 0.592684 2 3 4 5 5 5 4 3 2
*D=11 0.436268 1 3 3 5 4 5 5 3 3 1
D=12 0.342576 1 2 3 4 4 5 4 4 3 2 1
*D=13 0.265883 1 2 2 3 4 4 4 4 3 3 2 1
D=14 0.210193 1 2 2 3 3 4 3 4 3 3 2 2 1
*D=15 0.163793 1 1 2 3 3 3 3 4 3 3 3 2 1 1
D=16 0.136953 1 1 2 2 3 3 3 3 3 3 3 2 2 1 1
*D=17 0.109695 1 1 2 2 2 3 2 3 3 3 3 2 2 2 1 1
D=18 0.089731 1 1 1 2 2 2 3 3 3 3 3 2 2 2 1 1 1
*D=19 0.075333 1 1 1 2 2 2 2 3 2 3 2 3 2 2 2 1 1 1
*D=20 0.061986 1 1 1 1 2 2 2 2 3 2 3 2 2 2 2 2 1 1 1
*D=21 0.053104 1 1 1 1 2 2 2 2 2 2 3 2 2 2 2 2 1 1 1 1
*D=22 0.044532 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1
*D=23 0.037812 1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1
*D=24 0.031963 1 1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 1 2 1 1 1 1 1
*D=25 0.027155 1 1 1 1 1 1 2 1 2 1 2 2 2 2 2 2 1 2 1 1 1 1 1 1
*D=26 0.023309 1 1 1 1 1 1 1 2 1 2 1 2 2 2 2 2 1 2 1 1 1 1 1 1 1
*D=27 0.020098 1 1 1 1 1 1 1 1 2 1 2 1 2 2 2 1 2 1 2 1 1 1 1 1 1 1
D=28 0.017577 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1
*D=29 0.015330 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1
D=30 0.013386 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1
*D=31 0.011672 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1
D=32 0.010255 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1
*D=33 0.009054 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
D=34 0.008051 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
$N = 40$
D=2 38 38
D=3 18.091288 19 19
D=4 9.725083 12 14 12
*D=5 5.385711 7 12 11 8
D=6 3.271170 5 9 10 9 5

D=7 2.079435 4 6 9 9 6 4
D=8 1.413674 3 5 7 8 7 5 3
D=9 0.988893 2 4 6 7 7 6 4 2
D=10 0.718208 2 3 5 6 6 6 5 3 2
D=11 0.521269 2 3 4 5 5 5 5 4 3 2
*D=12 0.404618 1 3 3 4 5 5 5 5 3 3 1
D=13 0.323122 1 2 3 4 4 5 5 4 4 3 2 1
D=14 0.255811 1 2 3 3 4 4 4 4 4 3 3 2 1
D=15 0.204993 1 2 2 3 3 4 4 4 4 3 3 2 2 1
D=16 0.161848 1 1 2 3 3 3 4 4 4 3 3 3 2 1 1
*D=17 0.136293 1 1 2 2 3 3 3 4 3 4 3 3 2 2 1 1
D=18 0.112624 1 1 2 2 2 3 3 3 4 3 3 3 2 2 2 1 1
D=19 0.092187 1 1 2 2 2 2 3 3 3 3 3 3 2 2 2 2 1 1
*D=20 0.078330 1 1 1 2 2 2 3 2 3 3 3 3 3 2 2 2 1 1 1
D=21 0.066003 1 1 1 2 2 2 2 3 2 3 3 2 3 2 2 2 2 1 1 1
D=22 0.055791 1 1 1 1 2 2 2 3 2 3 2 3 2 3 2 2 2 1 1 1 1
*D=23 0.048180 1 1 1 1 2 2 2 2 2 2 3 2 3 2 2 2 2 2 1 1 1 1
D=24 0.041052 1 1 1 1 1 2 2 2 2 2 3 2 3 2 2 2 2 2 1 1 1 1 1
D=25 0.035944 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1
D=26 0.031092 1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 2 2 2 1 2 1 1 1 1 1
D=27 0.026901 1 1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 2 2 2 1 2 1 1 1 1 1 1
D=28 0.023200 1 1 1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 2 2 1 2 1 1 1 1 1 1 1
D=29 0.020055 1 1 1 1 1 1 1 2 1 2 1 2 2 2 2 2 2 2 1 2 1 2 1 1 1 1 1 1 1
D=30 0.017568 1 1 1 1 1 1 1 1 2 1 2 1 2 2 2 2 2 2 1 2 1 2 1 1 1 1 1 1 1 1
D=31 0.015442 1 1 1 1 1 1 1 1 1 2 1 2 1 2 2 2 2 2 1 2 1 2 1 1 1 1 1 1 1 1 1
D=32 0.013676 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1
*D=33 0.012152 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1
D=34 0.010794 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1
*D=35 0.009561 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1
D=36 0.008514 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
*D=37 0.007604 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
D=38 0.006831 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
D=39 0.006165 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
$N = 50$
D=2 48 48
D=3 23.074278 24 24
D=4 12.641101 15 18 15
D=5 7.080889 9 15 15 9
D=6 4.290025 6 11 14 11 6
D=7 2.764758 5 8 11 11 8 5
D=8 1.859022 3 7 9 10 9 7 3
D=9 1.320825 3 5 7 9 9 7 5 3
D=10 0.959944 2 4 6 8 8 8 6 4 2
D=11 0.718518 2 4 5 6 7 7 6 5 4 2
D=12 0.543272 2 3 4 6 6 6 6 6 4 3 2
D=13 0.424975 1 3 4 5 5 6 6 5 5 4 3 1
D=14 0.341205 1 3 3 4 5 5 6 5 5 4 3 3 1

$N = 60$
D=3 28.062623 29 29
D=4 15.575571 18 22 18
$N = 100$
D=2 98 98
D=3 48.0385 49 49
D=4 27.6754 31 36 31
D=5 15.8799 19 30 30 19
D=6 9.7886 13 22 25 22 13
D=7 6.3833 9 17 23 23 17 9
D=8 4.358863 7 13 19 20 19 13 7
D=9 3.098801 5 10 16 18 18 16 10 5
$N = 122$
D=2 120 120
D=3 59.031762 60 60
D=4 34.442561 39 42 39
D=5 19.858188 24 36 36 24
D=6 12.266200 16 27 34 27 16
*D=7 8.021537 11 20 29 28 21 11
D=8 5.499296 8 16 23 26 23 16 8
*D=9 3.910465 6 13 18 23 22 19 13 6

The graph that maximizes the algebraic connectivity μ_{N-1}, has larger sizes for cliques in the middle. It is dense in the core and sparse at borders. A symmetric clique size $(n_1, n_2, ..., n_{D+1})$ or a symmetric structure seems to be necessary to maximize the algebraic connectivity $a_{\max}(N, D)$. We conjecture that the nonsymetric cases may come from the rounding error, where each element of $(n_1, n_2, ..., n_{D+1})$ has to be a positive integer.

References

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