A new type of lower bound for the largest eigenvalue of a symmetric matrix

Piet Van Mieghem

Delft University of Technology, P.O. Box 356, 2600 AJ Delft, The Netherlands

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Abstract

Let $A_{m \times m}$ denote a symmetric matrix. We present an order expansion (4) based on Lagrange series that allows us to improve the classical bound $\frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \leq \lambda_{\text{max}}(A)$. © 2007 Elsevier Inc. All rights reserved.

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1. Introduction

Let $\lambda_{\text{min}}(A) \leq \lambda_{m-1} \leq \cdots \leq \lambda_2 \leq \lambda_{\text{max}}(A)$ denote the ordered, real eigenvalues of a symmetric $m \times m$ matrix $A$. The largest eigenvalue can be expressed [4, p. 549] as

$$\lambda_{\text{max}}(A) = \max_{x \neq 0} \frac{x^T A x}{x^T x},$$

(1)

where $\frac{x^T A x}{x^T x}$ is called the Rayleigh quotient. The maximum in (1) is only attained if $x$ is the eigenvector belonging to $\lambda_{\text{max}}(A)$. Hence, for any other vector $y$ that is not the corresponding eigenvector, it holds that

$$\frac{y^T A y}{y^T y} \leq \lambda_{\text{max}}(A)$$

E-mail address: P.VanMieghem@ewi.tudelft.nl

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from which the commonly used lower bound,
\[
\frac{u^T A u}{m} = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \leq \lambda_{\text{max}}(A)
\] (2)
follows for the choice \( y = u \), where \( u = [1 1 \cdots 1]^T \) is the \( m \times 1 \) all-one vector.

The main result of this article is

**Theorem 1.1.** Consider a symmetric matrix \( A_{m \times m} \) and define \( T = \frac{1}{\sqrt{m}} \max_{1 \leq j \leq m}(a_{jj} + \sum_{i=1; i \neq j}^{m} |a_{ij}|) \). For any real number \( t \geq T \) and \( \lambda_0 = t \sqrt{m} \), the largest eigenvalue of \( A \) can be bounded from below by
\[
\frac{N_1}{m} + 2 \left( \frac{N_3}{2m} - \frac{N_1 N_2}{m^2} + \frac{N_1^3}{2m^3} \right) \lambda_0^{-2} + O(t^{-4}) \leq \lambda_{\text{max}}(A),
\] (3)
where \( N_k = u^T A^k u = \sum_{i=1}^{m} \sum_{j=1}^{m} (A^k)_{ij} \) and \( N_0 = m \).

Although the theorem is stated with order term \( O(t^{-4}) \), the method of Appendix B allows us to sharpen the classical bound to any desired order \( O(t^{-2j}) \), where \( j \) is a positive integer. If the \( \lambda_0^{-2} \) term and the order term are ignored, we find the classical bound (2).

In the theory on the spectra of graphs, numerous lower and upper bounds for the largest eigenvalue \( \lambda_{\text{max}}(A) \) of the adjacency matrix \( A \) of a graph \( G \) exist (see e.g. [6, Appendix B]). In a graph, \( N_k \) is the total number of walks of length \( k \) [6, Appendix B] and the degree of node \( k \) is \( d_k(A) = \sum_{j=1}^{m} (A)_{kj} \). Then, the first few \( N_k \) are \( N_1 = 2L_A \), where \( L_A \) is the number of links in \( G \) and \( N_2 = D_A = \sum_{k=1}^{m} d_k^2(A) \). For graphs, we show in Section 3 that (3) can be rephrased as
\[
\frac{2L_A}{m} + 2 \left( \frac{N_3}{2m} - \frac{2L_A D_A}{m^2} + \frac{4L_A^3}{m^3} \right) \lambda_0^{-2} + O(N^{-2}) \leq \lambda_{\text{max}}(A)
\] (4)
for any \( N \geq 2m \) and where \( \lambda_0 = \sqrt{(N - m)m} \). Equality in the classical bound (2) is attained in regular graphs where each node has the same degree \( r \) and where \( \lambda_{\text{max}}(A) = \frac{2L_A}{m} = r \). Thus, for regular graphs, the coefficient of the \( \lambda_0^{-2} \) term in (4) is precisely zero.

In Section 2, Theorem 1.1 is proved. Section 3 discusses the applications to graphs, while Section 4 revisits Theorem 1.1 from the viewpoint of perturbation theory.

### 2. Proof of Theorem 1.1

The ingredients of the proof of Theorem 1.1 rely on the possibility to compute the largest eigenvalue \( \lambda_{\text{max}}(A_t) \) and the smallest eigenvalue \( \lambda_{\text{min}}(A_t) \) of the symmetric matrix
\[
A_t = \begin{bmatrix}
A_{m \times m} & t \cdot u_{m \times 1} \\
(t \cdot u^T)_{1 \times m} & 0
\end{bmatrix},
\]
where \( t \in \mathbb{R} \) and on Lemma A.1 that yields
\[
\lambda_{\text{max}}(A_t) + \lambda_{\text{min}}(A_t) \leq \lambda_{\text{max}}(A).
\] (5)
The sequel is devoted to the computation of \( \lambda_{\text{max}}(A_t) \) and \( \lambda_{\text{min}}(A_t) \). Lemma A.1 restricts the validity of the analysis to symmetric matrices.
The characteristic polynomial of $A_t$ is

$$\det(A_t - \lambda I)_{(m+1) \times (m+1)} = \begin{vmatrix} (A - \lambda I)_{m \times m} & t.m_{m \times 1} \\ t.(u^T)_{1 \times m} & -\lambda \end{vmatrix}. $$

The general relation (15) gives

$$\det(A_t - \lambda I) = \det(A - \lambda I)_{m \times m} \det(-\lambda - t^2(u^T)_{1 \times m} ((A - \lambda I)_{m \times m})^{-1}u_{m \times 1}).$$

For any matrix $X$, the sum of all its elements is $s_X = u^T X u = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$. Let us denote by $s_\lambda$ the sum of all elements of the resolvent $(A - \lambda I)^{-1}$, then

$$\det(A_t - \lambda I) = -(\lambda + t^2 s_\lambda) \det(A - \lambda I)_{m \times m}. \tag{6}$$

An explicit expression for $s_\lambda$ is given in (24).

Consider the expansion of the resolvent of $A$,

$$(A - \lambda I)^{-1} = \frac{1}{-\lambda} \left( I - \frac{A}{\lambda} \right)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{A^k}{\lambda^k} = -\frac{1}{\lambda} \left( I + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots \right) \tag{7}$$

such that

$$s_\lambda = -\frac{1}{\lambda} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=0}^{\infty} \frac{(A^k)_{ij}}{\lambda^k} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} \sum_{i=1}^{m} \sum_{j=1}^{m} (A^k)_{ij} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{N_k}{\lambda^k}. $$

Introduced into the equation $\lambda + s_\lambda t^2 = 0$, gives

$$\lambda + s_\lambda t^2 = \lambda - \frac{t^2}{\lambda} \left( m + \sum_{k=1}^{\infty} \frac{N_k}{\lambda^k} \right) = 0 \tag{8}$$

or,

$$\lambda = \pm t \sqrt{m \left( 1 + \sum_{k=1}^{\infty} \frac{N_k}{m \lambda^k} \right)}. \tag{9}$$

If $t$ is large and $m$ fixed, (9) reveals that the first order expression, $\lambda_0 = \pm t \sqrt{m}$ is accurate up to $O(1)$. Gerschgorin’s Theorem [7, p. 71] shows that each eigenvalue of the (symmetric) matrix $A$ lies in at least one of the $1 \leq j \leq m$ intervals $(a_{jj} - \sum_{i=1;i \neq j}^{m} |a_{ij}|, a_{jj} + \sum_{i=1;i \neq j}^{m} |a_{ij}|)$.

Hence, if $t > T = \frac{1}{\sqrt{m}} \max_{1 \leq j \leq m} \left( a_{jj} + \sum_{i=1; i \neq j}^{m} |a_{ij}| \right)$, then relation (9) further shows that the largest eigenvalues of $A_t$ in absolute value are the roots of $\lambda + s_\lambda t^2 = 0$. This observation leads to a lower bound for $t \geq T$.

2.1. Lagrange series for the zero of (8)

We solve the equation $f(\lambda) = \lambda + s_\lambda t^2 \approx \lambda - t^2 \left( \frac{m}{\lambda} + \frac{N_1}{\lambda^2} + \frac{N_2}{\lambda^3} \right) = 0$ by Langrange expansion first\(^1\) up to $O(t^{-2})$ while the general method is outlined in Appendix B. The zero $\zeta(\lambda_0)$ of $f(\lambda)$ around $\lambda_0 = t \sqrt{m}$ can be written as a Lagrange series [3, II, pp. 88]. Since $f(\lambda)$ is known up to order $O(t^{-2})$, only three terms in the general Lagrange series are needed.

\[^1\] The same result can be obtained by two iterations in Newton-Raphson’s method.
\[ \zeta(\lambda_0) \approx \lambda_0 - \frac{f(\lambda_0)}{f'(\lambda_0)} - \frac{f''(\lambda_0)}{2f'(\lambda_0)^2} \left( \frac{f(\lambda_0)}{f'(\lambda_0)} \right)^2 \]  

(10)

to guarantee that the zero is also accurate up to order \( O(t^{-2}) \). Indeed,

\[ f(\lambda_0) = -\frac{N_1}{m} - \frac{N_2}{mt\sqrt{m}} + O(t^{-2}), \]

\[ f'(\lambda_0) = 2 + \frac{2N_1}{m t \sqrt{m}} + O(t^{-2}), \]

\[ f''(\lambda_0) = -\frac{2}{t \sqrt{m}} + O(t^{-2}) \]

and \( f''(\lambda_0) = O(t^{-2}) \). After substitution in (10) and some reorganization, the extreme eigenvalues of \( A_t \) are, accurate up to order \( O(t^{-2}) \) for large \( t \),

\[ \lambda_{\text{max}}(A_t) = t \sqrt{m} + \frac{N_1}{2m} + \left( \frac{N_2}{2m} - \frac{3N_1^2}{8m^2} \right) \frac{1}{t \sqrt{m}} + O(t^{-2}) \]

(11)

and

\[ \lambda_{\text{min}}(A_t) = -t \sqrt{m} + \frac{N_1}{2m} - \left( \frac{N_2}{2m} - \frac{3N_1^2}{8m^2} \right) \frac{1}{t \sqrt{m}} + O(t^{-2}), \]

(12)

where the last expression is obtained analogously from (10) for \( \lambda_0 = -t \sqrt{m} \). The method presented in Appendix B allows us to compute \( \zeta(\lambda_0) \) to any desired order in \( t \), although the amount of computations rapidly becomes impressive. Computed up to \( O(t^{-3}) \), we find

\[ \zeta(\lambda_0) = \lambda_0 + \frac{N_1}{2m} + \left( \frac{N_2}{2m} - \frac{3N_1^2}{8m^2} \right) \frac{1}{\lambda_0} + \left( \frac{N_3}{2m} - \frac{N_1 N_2}{m^2} + \frac{N_1^3}{2m^3} \right) \frac{1}{\lambda_0^2} + O(t^{-3}), \]

(13)

where \( \lambda_{\text{max}}(A_t) \) and \( \lambda_{\text{min}}(A_t) \) follow for \( \lambda_0 = t \sqrt{m} \) and \( \lambda_0 = -t \sqrt{m} \), respectively.

Finally, our Theorem 1.1, in particular the bound (3), is proved by combining the bound (5) and equation (13).

3. Application to the spectra of graphs

Since \( A_t \) is not an adjacency matrix for \( t \neq 1 \), we consider the adjacency matrix of the \( G \)-connected \( m \) star topology with \( N \) nodes,

\[ A_{\text{mstarG}} = \begin{bmatrix} A_{m \times m} & J_{m \times (N-m)} \\ J_{(N-m) \times m} & O_{(N-m) \times (N-m)} \end{bmatrix}, \]

where \( J \) is the all-one matrix and \( A_{m \times m} \) is the adjacency matrix of an arbitrary graph \( G \) that connects \( m \) nodes. Each of those \( m \) nodes is connected to each of the \( N-m \) other nodes in the topology called \( \text{mstarG} \). We note that the bi-partite structure of \( A_{\text{mstarG}} \) is crucial. Similarly as above, we find

\[ \det(A_{\text{mstarG}} - \lambda I) = \det(A - \lambda I)_{m \times m} (-1)^{N-m} \lambda^{N-m-1} (\lambda + s_{\lambda}(N-m)), \]

where \( t^2 = N - m \). Hence, by modifying the size of the matrix \( A_{\text{mstarG}} \), the zeros of the same function \( f(\lambda) = \lambda + s_{\lambda}(N-m) \) are the maximum and minimum eigenvalue of \( A_{\text{mstarG}} \). Moreover, since the largest eigenvalue of \( A \) (for any graph) is smaller than the maximum degree \( d_{\text{max}}(A) \leq m - 1 \), a tighter bound for \( N \) compared to \( t \) is found, \( N \geq 2m \). The general result is then given in (4).
Example. The spectrum of a $m$-fully meshed star topology where $A = J - I$ can be computed exactly as

$$(\lambda_{\text{max}})_{\text{star}} = \sqrt{m(N - m) + \left(\frac{m - 1}{2}\right)^2 + \frac{m - 1}{2}},$$

$$(\lambda_{\text{min}})_{\text{star}} = -\sqrt{m(N - m) + \left(\frac{m - 1}{2}\right)^2 + \frac{m - 1}{2}},$$

and with an eigenvalue at $-1$ with multiplicity $m - 1$ and at $0$ with multiplicity $N - m - 1$. Comparing (13) with the exact result of a $m$-fully meshed star topology,

$$(\lambda_{\text{max}})_{\text{star}} = \sqrt{(N - m)m + \frac{m - 1}{2} \left(\frac{m - 1}{2}\right)^2 \frac{1}{\sqrt{(N - m)m}}} + O(N^{-3/2})$$

shows, indeed, that (13) is correct, since $N_1 = \frac{m(m-1)}{2}$, $N_2 = m(m - 1)^2$ and $N_3 = m(m - 1)^3$ for the complete graph $K_m$.

4. Perturbation theory

Consider the symmetrix matrix

$$B = \begin{bmatrix} O_{m \times m} & u_{m \times 1} \\ (u^T)_{1 \times m} & 0 \end{bmatrix}$$

that represents the adjacency matrix of the bi-partite graph $K_{m,1}$ or the star topology, a central node that connects $m$ other, not interconnected nodes. The eigenvalues of $B$ are well-known: $-\sqrt{m}$, $\sqrt{m}$ and $0$ with multiplicity $m - 1$. The corresponding eigenvectors to the eigenvalue $\sqrt{m}$ and $-\sqrt{m}$ are $v = [u_{1 \times m} \sqrt{m}]^T$ and $w = [u_{1 \times m} -\sqrt{m}]^T$, respectively. Hence, apart from the zero eigenvalues, the eigenvalues of $tB$ are precisely $\lambda_0 = \pm t\sqrt{m}$.

Further, we can write

$$A_t = tB + A_0 = t \left( B + \frac{1}{t} A_0 \right)$$

which implies that the eigenvalues of $A_t$ are equal to those of $B + \frac{1}{t} A_0$ multiplied by $t$. Since we know the eigenvalues of $B$ exactly, and if $t$ is sufficiently large, perturbation theory [2,7] can be applied. Since $B + zA_0$ is analytic in $z$, real symmetric on the real axis, all eigenvalues of $B + zA_0$ are analytic functions of $z$ in the neighborhood of the real axis ($\text{Im} z = 0$). Hence, there exists a real number $R > 0$, for which $B + zA_0$ has two, simple eigenvalues $\lambda_+(z)$ and $\lambda_-(z)$ with Taylor expansion around $\sqrt{m}$ and $-\sqrt{m}$,

$$\lambda_+(z) = \sqrt{m} + \sum_{k=1}^{\infty} \alpha_k z^k \quad |z| < R,$$

$$\lambda_-(z) = -\sqrt{m} + \sum_{k=1}^{\infty} \beta_k z^k \quad |z| < R,$$
where all coefficients $\alpha_k$ and $\beta_k$ are real. Perturbation theory [7, p. 69] gives explicitly the first order coefficients as

$$\alpha_1 = \frac{v^T Av}{v^Tv} = \frac{u^T Au}{2m} \quad \text{and} \quad \beta_1 = \frac{w^T Aw}{w^Tw} = \frac{u^T Au}{2m}.$$ 

Thus,

$$\alpha_1 = \beta_1 = \frac{N_1}{2m}.$$ 

For sufficiently small $z$, $\lambda_+(z)$ and $\lambda_-(z)$ are the maximum and minimum eigenvalue of $B + zA_0$. Hence, with (5), we obtain

$$\lambda_{\text{max}}(A) \geq t \left( \lambda_+ \left( \frac{1}{t} \right) + \lambda_- \left( \frac{1}{t} \right) \right) = \sum_{k=1}^{\infty} (\alpha_k + \beta_k) t^{1-k}$$

$$= \frac{N_1}{m} + \frac{(\alpha_2 + \beta_2)}{t} + \frac{(\alpha_3 + \beta_3)}{t^2} + \sum_{k=4}^{\infty} (\alpha_k + \beta_k) t^{1-k}.$$ 

The specific (bi-partite) structure of $A_t$ enables us to write the characteristic polynomial (6) explicitly, from which we deduce, for sufficiently large $t$, that both $t \lambda_+(t^{-1})$ and $t \lambda_-(t^{-1})$ are zeros of the function $f(\lambda) = \lambda + s\lambda^2$. If $y(t)$ is a zero of $\lambda_+(t)$, then also $y(-t)$ is a zero, which shows that $t \lambda_+(t^{-1}) = -t \lambda_-(t^{-1})$. The zeros of $f(\lambda)$ can be expanded in a Lagrange series around $\lambda_0 = \pm t \sqrt{m}$. By also expanding the coefficients of this Lagrange series into a power series expansion in $\lambda_0$ as shown in Appendix B, all coefficients $\alpha_k$ can be computed and we indeed find that $\lambda_+(t^{-1}) = -\lambda_-(t^{-1})$. Hence,

$$t \left( \lambda_+ \left( \frac{1}{t} \right) + \lambda_- \left( \frac{1}{t} \right) \right) = 2 \sum_{k=0}^{\infty} \alpha_{2k+1} t^{-2k}$$

$$= \frac{N_1}{m} + \frac{2\alpha_3}{t^2} + 2 \sum_{k=2}^{\infty} \alpha_{2k+1} t^{-2k}.$$ 

If $\alpha_3 = \frac{N_3}{2m} - \frac{N_1 N_2}{m^2} + \frac{N_3}{2m^2} > 0$, a tighter lower bound for $\lambda_{\text{max}}(A)$ than the classical $\lambda_{\text{max}}(A) \geq \frac{N_1}{m}$ is obtained.

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**Appendix A. Results from linear algebra**

From the Schur identity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ O & D - CA^{-1}B \end{bmatrix}$$

we find that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B)$$

and $D - CA^{-1}B$ is called the Schur complement of $A$. 

Lemma A.1. If

\[ X = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \]

is a real symmetric matrix, where A and B are square, and consequently symmetric, matrices, then

\[ \lambda_{\max}(X) + \lambda_{\min}(X) \leq \lambda_{\max}(A) + \lambda_{\max}(B) \]  \hspace{1cm} (16)

Proof. See e.g. [1, p. 56]. \( \square \)

Appendix B. Characteristic coefficients of a complex function

If \( f(z) \) has a Taylor series around \( z_0 \),

\[ f(z) = \sum_{k=0}^{\infty} f_k(z_0)(z - z_0)^k \quad \text{with} \quad f_k(z_0) = \frac{1}{k!} \frac{d^k f(z)}{dz^k} \bigg|_{z=z_0} \]

then the general relation where \( G(z) \) is analytic around \( f(z_0) \) is

\[ G(f(z)) = G(f(z_0)) + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \frac{1}{k!} \frac{d^k G(p)}{dp^k} \bigg|_{p=f(z_0)} s[k, m]_{f(z)}(z_0) \right) (z - z_0)^m, \]  \hspace{1cm} (17)

where the characteristic coefficient \([5]\) of a complex function \( f(z) \) is

\[ s[k, m]_{f(z)}(z_0) = \sum_{\sum_{i=1}^{k} j_i = m; j_i > 0} \prod_{i=1}^{k} f_{j_i}(z_0) \]

which obeys the recursion relation

\[ s[1, m]_{f(z)}(z_0) = f_m(z_0), \]

\[ s[k, m]_{f(z)}(z_0) = \sum_{j=1}^{m-k+1} f_{j}(z_0)s[k-1, m-j]_{f(z)}(z_0) \quad (k > 1). \]  \hspace{1cm} (18)

The zero \( \zeta(z_0) \) of \( f(z) \) closest to \( z_0 \) is given \([5]\) in terms of the coefficients \( f_k(z_0) \) of the series expansion of \( f(z) \) around \( z_0 \) as

\[ \zeta(z_0) = f^{-1}(0) = z_0 - \frac{f_0(z_0)}{f_1(z_0)} + \sum_{n=2}^{\infty} \left[ \sum_{k=1}^{n-1} \frac{(-1)^k \binom{n+k-1}{k-1}}{k(f_1(z_0))^k} s^*[k, n-1](z_0) \right] \left( -\frac{f_0(z_0)}{f_1(z_0)} \right)^n, \]  \hspace{1cm} (19)

where \( s^*[k, m] = s[k, m]_{f_m \to f_{m+1}} \) denotes that the index of all Taylor coefficients appearing in (18) is augmented by 1. Explicitly summing the first five terms \( (n \leq 5) \),
\[
\zeta(z_0) \approx z_0 - \frac{f_0(z_0)}{f_1(z_0)} - \frac{f_2(z_0)}{f_1(z_0)} \left( \frac{f_0(z_0)}{f_1(z_0)} \right)^2 + \left[ -2 \left( \frac{f_2(z_0)}{f_1(z_0)} \right) + \frac{f_3(z_0)}{f_1(z_0)} \right] \left( \frac{f_0(z_0)}{f_1(z_0)} \right)^3 \\
+ \left[ -5 \left( \frac{f_2(z_0)}{f_1(z_0)} \right)^3 + 5 \frac{f_3(z_0)}{f_1(z_0)} \frac{f_2(z_0)}{f_1(z_0)} - \frac{f_4(z_0)}{f_1(z_0)} \right] \left( \frac{f_0(z_0)}{f_1(z_0)} \right)^4 \\
+ \left[ -14 \left( \frac{f_2(z_0)}{f_1(z_0)} \right)^4 + 21 \frac{f_3(z_0)}{f_1(z_0)} \left( \frac{f_2(z_0)}{f_1(z_0)} \right)^2 - 3 \left( \frac{f_3(z_0)}{f_1(z_0)} \right)^2 \\
- 6 \frac{f_4(z_0)}{f_1(z_0)} \frac{f_2(z_0)}{f_1(z_0)} + \frac{f_5(z_0)}{f_1(z_0)} \right] \left( \frac{f_0(z_0)}{f_1(z_0)} \right)^5.
\]

(20)

**Appendix B.1. Lagrange expansion**

The zero of \( f(\lambda) = \lambda + s_1 t^2 = 0 \) will be computed using the Lagrange series which can be efficiently computed to any order with characteristic coefficients \([5]\). The Lagrange expansion (19) in terms of characteristic coefficients needs the Taylor coefficients of \( f(\lambda) \) around \( \lambda_0 = \pm t \sqrt{m} \),

\[
f_n(\lambda_0) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left( \lambda - t^2 \sum_{k=0}^{\infty} \frac{N_k}{\lambda_0^{k+1}} \right) \bigg|_{\lambda = \lambda_0}
\]

For \( n = 0 \),

\[
f_0(\lambda_0) = \lambda_0 - t^2 \sum_{k=0}^{\infty} \frac{N_k}{\lambda_0^{k+1}} = -t^2 \sum_{k=1}^{\infty} \frac{N_k}{\lambda_0^{k+1}}
\]

because \( \lambda_0 - \frac{t^2 m}{\lambda_0} = 0 \) and,

\[
f_0(\lambda_0) = -\frac{1}{m} \sum_{k=0}^{\infty} \frac{N_{k+1}}{\lambda_0^k}.
\]

(21)

Similarly, for \( n = 1 \),

\[
f_1(\lambda_0) = 1 + t^2 \sum_{k=0}^{\infty} \frac{(k + 1)N_k}{\lambda_0^{k+2}} = 2 + \frac{1}{m} \sum_{k=1}^{\infty} \frac{(k + 1)N_k}{\lambda_0^k}
\]

(22)

and, for all \( n > 1 \),

\[
f_n(\lambda_0) = \frac{(-1)^n + 1}{\lambda_0^{n+1}} \sum_{k=0}^{\infty} \binom{n + k}{k} \frac{N_k}{\lambda_0^k}
\]

(23)

We now confine ourselves to computing the zero \( \zeta(\lambda_0) \) of \( f(\lambda) \) accurate up to order \( O(t^{-2q}) \), where \( q \) is fixed, but specified later. Since \( \lambda_0 = O(t) \), for \( f_n(\lambda_0) \) to be accurate up to order \( O(t^{-2q}) \), we need to take in the computation \( k = 2(q - 1) - n \) terms in the \( k \)-sums. Also, it follows that \( 2(q - 1) > n \) derivatives or Taylor coefficients in the Lagrange series are needed. The Lagrange expansion (19) indicates that we need order expansions for \( (f_1(\lambda_0))^{-k} \) and \( \left( \frac{f_0(z_0)}{f_1(z_0)} \right)^n \). Both of these expansions can be given in terms of characteristic coefficients. Applying

\[
g^{-k}(z) = g_0^{-k} + \sum_{m=1}^{\infty} \left( \sum_{n=1}^{m} (-1)^n \binom{n + k - 1}{n} g_0^{-k-n} s[n, m] \right) z^m
\]
to (22), we have, with \( z = \lambda_0^{-1} \) and \( g_0 = 2 \) and \( g_n = \frac{(n+1)}{m} N_n \) for \( n > 0 \), that

\[
(f_1(\lambda_0))^{-k} = 2^{-k} + \sum_{m=1}^{\infty} \left( \sum_{n=1}^{m} \frac{(-1)^n}{n} \left( \frac{(n+k-1)}{n} \right) s[n, m] \right) \frac{g(z)}{2^{k+n}} \lambda_0^{-m}.
\]

Similarly, applying

\[
h^n(z) = h^n_0 + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \left( \frac{n}{k} \right) h^{n-k} \frac{s[k, m]}{1} \right) z^m
\]
to (21), we obtain, with \( z = \lambda_0^{-1} \) and with \( h_n = -\frac{N_{n+1}}{m} \) for \( n \geq 0 \), that

\[
(f_0(\lambda_0))^{n} = \frac{(-1)^n N_1^n}{m^n} \left[ 1 + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \left( \frac{n}{k} \right) \left( -m \right)^k \frac{s[k, m]}{1} \right) \frac{h(z)}{N_1^k} \right] \lambda_0^{-m}.
\]

The quotient \( \left( \frac{f_0(z_0)}{f_1(z_0)} \right)^n \) follows by Cauchy’s product rule for series,

\[
\left( \frac{f_0(z_0)}{f_1(z_0)} \right)^n = \frac{(-1)^n N_1^n}{m^n 2^n} \left[ 1 + \frac{n}{m} \left( -m \frac{N_2}{N_1} + N_1 \right) \right] \lambda_0^{-1}
\]

\[
+ \frac{(-1)^n N_1^n}{m^n 2^n} \sum_{r=2}^{\infty} \left( \sum_{j=1}^{r} \left( \frac{n}{j} \right) \left( -m \right)^j \frac{s[j, r]}{1} \right) \frac{h(z)}{N_1^j}
\]

\[
+ \frac{r}{r} \sum_{q=1}^{r} \left( \sum_{j=1}^{r-q} \left( \frac{n}{j} \right) \left( -m \right)^j \frac{s[j, r-q]}{1} \right) \frac{h(z)}{N_1^j}
\]

\[
\times \left( \sum_{k=1}^{q} \left( \frac{n}{k} \right) \left( -m \right)^k \frac{s[k, q]}{1} \right) \lambda_0^{-r}.
\]

Explicitly up to order \( O(t^{-3}) \),

\[
\left( \frac{f_0(z_0)}{f_1(z_0)} \right)^n = \frac{(-1)^n N_1^n}{m^n 2^n} \left[ 1 + \frac{n}{m} \left( -m \frac{N_2}{N_1} + N_1 \right) \right] \lambda_0^{-1}
\]

\[
+ \frac{(-1)^n N_1^n}{m^n 2^n} \left( \frac{n N_3}{N_1} + \frac{n(n-1)N_2^2}{2N_1^2} \right) - \left( \frac{3n + 2n^2}{2m} \right) N_2 + \frac{n(n+1)N_2^2}{2m^2} \lambda_0^{-2}.
\]

Let us compute the zero of \( f(\lambda) = \lambda + s_\lambda t^2 = 0 \) up to \( O(t^{-3}) \). For \( q = 3/2 \), we need for each derivative \( n < 4 \), only \( k = 4 - n \) terms. The corresponding Lagrange series is

\[
\zeta(\lambda_0) = \lambda_0 - \frac{f_0(\lambda_0)}{f_1(\lambda_0)} - \frac{f_2(\lambda_0)}{f_1(\lambda_0)} \left( \frac{f_0(\lambda_0)}{f_1(\lambda_0)} \right)^2
\]

\[
+ \left[ -2 \left( \frac{f_2(\lambda_0)}{f_1(\lambda_0)} \right)^2 + \frac{f_3(\lambda_0)}{f_1(\lambda_0)} \right] \left( \frac{f_0(\lambda_0)}{f_1(\lambda_0)} \right) \quad \text{O}(t^{-3}).
\]
We list the separate terms,

\[ \frac{f_0(z_0)}{f_1(z_0)} = \frac{-N_1}{2m} \left( 1 + \frac{1}{m} \left( \frac{mN_2}{N_1} - N_1 \right) \lambda_0^{-1} + \left( \frac{N_3}{N_1} - \frac{5N_2}{2m} + \frac{N_1^2}{m^2} \right) \lambda_0^{-2} \right), \]

\[ \left( \frac{f_0(\lambda_0)}{f_1(\lambda_0)} \right)^2 = \frac{N_1^2}{4m^2} \left( 1 + \frac{2}{m} \left( \frac{mN_2}{N_1} - N_1 \right) \lambda_0^{-1} + O(\lambda_0^{-2}) \right), \]

\[ \left( -\frac{N_1}{f_1(z_0)} \right)^3 = \frac{-N_1^3}{8m^3} (1 + O(\lambda_0^{-1})). \]

and

\[ \frac{f_2(\lambda_0)}{f_1(\lambda_0)} = -\left( \frac{1}{2} - \frac{N_1}{2m} \lambda_0^{-1} + \left( \frac{N_1^2}{2m^2} - \frac{3N_2}{2m} \right) \lambda_0^{-2} \right) \left( \frac{1}{\lambda_0} + \frac{3N_1}{m^2} \lambda_0^{-1} \right) \]

\[ = -\left( \frac{1}{2} \lambda_0^{-1} + \frac{N_1}{m} \lambda_0^{-2} \right), \]

\[ \left( \frac{f_2(\lambda_0)}{f_1(\lambda_0)} \right)^2 = \frac{1}{4} \lambda_0^{-2} \]

and

\[ \frac{f_3(\lambda_0)}{f_1(\lambda_0)} = 2^{-1} - \frac{N_1}{2m} \lambda_0^{-1} + \left( \frac{N_1^2}{2m^2} - \frac{3N_2}{2m} \right) \lambda_0^{-2} \lambda_0^{-2} = \frac{1}{2} \lambda_0^{-2}. \]

Combined yields the final result (13).

**Appendix C. A finite sum expression for \( s_\lambda \)**

Since \( A \) is symmetrix, \( A^k = X \text{diag}(\lambda_j^k)X^T \) where the columns of the orthogonal matrix \( X \) consists of eigenvectors \( x_j \) of \( A \),

\[ (A - \lambda I)^{-1} = -\frac{1}{\lambda} X \left( \text{diag} \left( \sum_{k=0}^\infty \frac{\lambda_j^k}{\lambda^k} \right) \right) X^T = -X \text{diag} \left( \frac{1}{\lambda - \lambda_j} \right) X^T. \]

Then,

\[ s_\lambda = u(A - \lambda I)^{-1}u^T = -\sum_{j=1}^m \frac{1}{\lambda - \lambda_j} \left( \sum_{k=1}^m x_{j;k} \right)^2. \]  

(24)

Unless the all-one vector \( u \) is an eigenvector of \( A \), \( \left( \sum_{k=1}^m x_{j;k} \right)^2 \neq 0 \). Hence, in that case

\[ \det(A_t - \lambda I) = -\det(A - \lambda I) m \times m \left( \lambda - t^2 \sum_{j=1}^m \frac{\left( \sum_{k=1}^m x_{j;k} \right)^2}{\lambda - \lambda_j} \right) \]

and no eigenvalue of \( A \) is a zero of \( f(\lambda) = \lambda - t^2 \sum_{j=1}^m \frac{\left( \sum_{k=1}^m x_{j;k} \right)^2}{\lambda - \lambda_j} \).
References