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# A new type of lower bound for the largest eigenvalue of a symmetric matrix

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**Abstract**

Let $A_{m\times m}$ denote a symmetric matrix. We present an order expansion (4) based on Lagrange series that allows us to improve the classical bound $\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}\leqslant\lambda_{\max}(A)$.


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## 1. Introduction

Let $\lambda_{\min}(A)\leqslant\lambda_{m-1}\leqslant\cdots\leqslant\lambda_2\leqslant\lambda_{\max}(A)$ denote the ordered, real eigenvalues of a symmetric $m\times m$ matrix $A$. The largest eigenvalue can be expressed [4, p. 549] as

$$\lambda_{\max}(A)=\max_{x\neq 0}\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x}, \tag{1}$$

where $\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x}$ is called the Rayleigh quotient. The maximum in (1) is only attained if $x$ is the eigenvector belonging to $\lambda_{\max}(A)$. Hence, for any other vector $y$ that is not the corresponding eigenvector, it holds that

$$\frac{y^{\mathrm{T}}Ay}{y^{\mathrm{T}}y}\leqslant\lambda_{\max}(A)$$

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from which the commonly used lower bound,

$$\frac{u^{\mathrm{T}}Au}{m} = \frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \leqslant \lambda_{\max}(A) \tag{2}$$

follows for the choice $y = u$, where $u = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$ is the $m \times 1$ all-one vector.

The main result of this article is

**Theorem 1.1.** *Consider a symmetric matrix $A_{m\times m}$ and define $T = \frac{1}{\sqrt{m}} \max_{1\leqslant j\leqslant m}(a_{jj} + \sum_{i=1;i\neq j}^{m} |a_{ij}|)$. For any real number $t \geqslant T$ and $\lambda_0 = t\sqrt{m}$, the largest eigenvalue of $A$ can be bounded from below by*

$$\frac{N_1}{m} + 2\left(\frac{N_3}{2m} - \frac{N_1 N_2}{m^2} + \frac{N_1^3}{2m^3}\right)\lambda_0^{-2} + \mathrm{O}(t^{-4}) \leqslant \lambda_{\max}(A), \tag{3}$$

*where $N_k = u^{\mathrm{T}}A^k u = \sum_{i=1}^{m}\sum_{j=1}^{m}(A^k)_{ij}$ and $N_0 = m$.*

Although the theorem is stated with order term $\mathrm{O}(t^{-4})$, the method of Appendix B allows us to sharpen the classical bound to any desired order $\mathrm{O}(t^{-2j})$, where $j$ is a positive integer. If the $\lambda_0^{-2}$ term and the order term are ignored, we find the classical bound (2).

In the theory on the spectra of graphs, numerous lower and upper bounds for the largest eigenvalue $\lambda_{\max}(A)$ of the adjacency matrix $A$ of a graph $G$ exist (see e.g. [6, Appendix B]). In a graph, $N_k$ is the total number of walks of length $k$ [6, Appendix B] and the degree of node $k$ is $d_k(A) = \sum_{j=1}^{m}(A)_{kj}$. Then, the first few $N_k$ are $N_1 = 2L_A$, where $L_A$ is the number of links in $G$ and $N_2 = D_A = \sum_{k=1}^{m} d_k^2(A)$. For graphs, we show in Section 3 that (3) can be rephrased as

$$\frac{2L_A}{m} + 2\left(\frac{N_3}{2m} - \frac{2L_A D_A}{m^2} + \frac{4L_A^3}{m^3}\right)\lambda_0^{-2} + \mathrm{O}(N^{-2}) \leqslant \lambda_{\max}(A) \tag{4}$$

for any $N \geqslant 2m$ and where $\lambda_0 = \sqrt{(N-m)m}$. Equality in the classical bound (2) is attained in regular graphs where each node has the same degree $r$ and where $\lambda_{\max}(A) = \frac{2L_A}{m} = r$. Thus, for regular graphs, the coefficient of the $\lambda_0^{-2}$ term in (4) is precisely zero.

In Section 2, Theorem 1.1 is proved. Section 3 discusses the applications to graphs, while Section 4 revisits Theorem 1.1 from the viewpoint of perturbation theory.

## 2. Proof of Theorem 1.1

The ingredients of the proof of Theorem 1.1 rely on the possibility to compute the largest eigenvalue $\lambda_{\max}(A_t)$ and the smallest eigenvalue $\lambda_{\min}(A_t)$ of the symmetric matrix

$$A_t = \begin{bmatrix} A_{m\times m} & t.u_{m\times 1} \\ t.(u^{\mathrm{T}})_{1\times m} & 0 \end{bmatrix},$$

where $t \in \mathbb{R}$ and on Lemma A.1 that yields

$$\lambda_{\max}(A_t) + \lambda_{\min}(A_t) \leqslant \lambda_{\max}(A). \tag{5}$$

The sequel is devoted to the computation of $\lambda_{\max}(A_t)$ and $\lambda_{\min}(A_t)$. Lemma A.1 restricts the validity of the analysis to symmetric matrices.

The characteristic polynomial of $A_t$ is

$$\det(A_t - \lambda I)_{(m+1)\times(m+1)} = \begin{bmatrix} (A-\lambda I)_{m\times m} & t.u_{m\times 1} \\ t.(u^{\mathrm{T}})_{1\times m} & -\lambda \end{bmatrix}.$$

The general relation (15) gives

$$\det(A_t - \lambda I) = \det(A-\lambda I)_{m\times m} \det(-\lambda - t^2 (u^{\mathrm{T}})_{1\times m}((A-\lambda I)_{m\times m})^{-1} u_{m\times 1}).$$

For any matrix $X$, the sum of all its elements is $s_X = u^{\mathrm{T}} X u = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}$. Let us denote by $s_\lambda$ the sum of all elements of the resolvent $(A-\lambda I)^{-1}$, then

$$\det(A_t - \lambda I) = -(\lambda + t^2 s_\lambda)\det(A-\lambda I)_{m\times m}. \tag{6}$$

An explicit expression for $s_\lambda$ is given in (24).

Consider the expansion of the resolvent of $A$,

$$(A-\lambda I)^{-1} = \frac{1}{-\lambda}\left(I - \frac{A}{\lambda}\right)^{-1} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{A^k}{\lambda^k} = -\frac{1}{\lambda}\left(I + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\right) \tag{7}$$

such that

$$s_\lambda = -\frac{1}{\lambda}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=0}^{\infty}\frac{(A^k)_{ij}}{\lambda^k} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\sum_{i=1}^{m}\sum_{j=1}^{m}(A^k)_{ij} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{N_k}{\lambda^k}.$$

Introduced into the equation $\lambda + s_\lambda t^2 = 0$, gives

$$\lambda + s_\lambda t^2 = \lambda - \frac{t^2}{\lambda}\left(m + \sum_{k=1}^{\infty}\frac{N_k}{\lambda^k}\right) = 0 \tag{8}$$

or,

$$\lambda = \pm t\sqrt{m\left(1 + \sum_{k=1}^{\infty}\frac{N_k}{m\lambda^k}\right)}. \tag{9}$$

If $t$ is large and $m$ fixed, (9) reveals that the first order expression, $\lambda_0 = \pm t\sqrt{m}$ is accurate up to O(1). Gerschgorin's Theorem [7, p. 71] shows that each eigenvalue of the (symmetric) matrix $A$ lies in at least one of the $1 \leqslant j \leqslant m$ intervals $\left(a_{jj} - \sum_{i=1;i\neq j}^{m}|a_{ij}|, a_{jj} + \sum_{i=1;i\neq j}^{m}|a_{ij}|\right)$. Hence, if $t > T = \frac{1}{\sqrt{m}}\max_{1\leqslant j\leqslant m}\left(a_{jj} + \sum_{i=1;i\neq j}^{m}|a_{ij}|\right)$, then relation (9) further shows that the largest eigenvalues of $A_t$ in absolute value are the roots of $\lambda + s_\lambda t^2 = 0$. This observation leads to a lower bound for $t \geqslant T$.

### 2.1. *Lagrange series for the zero of (8)*

We solve the equation $f(\lambda) = \lambda + s_\lambda t^2 \simeq \lambda - t^2\left(\frac{m}{\lambda} + \frac{N_1}{\lambda^2} + \frac{N_2}{\lambda^3}\right) = 0$ by Langrange expansion first[1] up to $O(t^{-2})$ while the general method is outlined in Appendix B. The zero $\zeta(\lambda_0)$ of $f(\lambda)$ around $\lambda_0 = t\sqrt{m}$ can be written as a Lagrange series [3, II, pp. 88]. Since $f(\lambda)$ is known up to order $O(t^{-2})$, only three terms in the general Lagrange series are needed,

[1] The same result can be obtained by two iterations in Newton-Raphson's method.

$$\zeta(\lambda_0) \approx \lambda_0 - \frac{f(\lambda_0)}{f'(\lambda_0)} - \frac{f''(\lambda_0)}{2f'(\lambda_0)}\left(\frac{f(\lambda_0)}{f'(\lambda_0)}\right)^2 \tag{10}$$

to guarantee that the zero is also accurate up to order $O(t^{-2})$. Indeed,

$$f(\lambda_0) = -\frac{N_1}{m} - \frac{N_2}{mt\sqrt{m}} + O(t^{-2}),$$
$$f'(\lambda_0) = 2 + \frac{2N_1}{mt\sqrt{m}} + O(t^{-2}),$$
$$f''(\lambda_0) = -\frac{2}{t\sqrt{m}} + O(t^{-2})$$

and $f'''(\lambda_0) = O(t^{-2})$. After substitution in (10) and some reorganization, the extreme eigenvalues of $A_t$ are, accurate up to order $O(t^{-2})$ for large $t$,

$$\lambda_{\max}(A_t) = t\sqrt{m} + \frac{N_1}{2m} + \left(\frac{N_2}{2m} - \frac{3N_1^2}{8m^2}\right)\frac{1}{t\sqrt{m}} + O(t^{-2}) \tag{11}$$

and

$$\lambda_{\min}(A_t) = -t\sqrt{m} + \frac{N_1}{2m} - \left(\frac{N_2}{2m} - \frac{3N_1^2}{8m^2}\right)\frac{1}{t\sqrt{m}} + O(t^{-2}), \tag{12}$$

where the last expression is obtained analogously from (10) for $\lambda_0 = -t\sqrt{m}$. The method presented in Appendix B allows us to compute $\zeta(\lambda_0)$ to any desired order in $t$, although the amount of computations rapidly becomes impressive. Computed up to $O(t^{-3})$, we find

$$\zeta(\lambda_0) = \lambda_0 + \frac{N_1}{2m} + \left(\frac{N_2}{2m} - \frac{3N_1^2}{8m^2}\right)\frac{1}{\lambda_0} + \left(\frac{N_3}{2m} - \frac{N_1N_2}{m^2} + \frac{N_1^3}{2m^3}\right)\frac{1}{\lambda_0^2} + O(t^{-3}), \tag{13}$$

where $\lambda_{\max}(A_t)$ and $\lambda_{\min}(A_t)$ follow for $\lambda_0 = t\sqrt{m}$ and $\lambda_0 = -t\sqrt{m}$, respectively.

Finally, our Theorem 1.1, in particular the bound (3), is proved by combining the bound (5) and equation (13).

## 3. Application to the spectra of graphs

Since $A_t$ is not an adjacency matrix for $t \neq 1$, we consider the adjacency matrix of the $G$-connected $m$ star topology with $N$ nodes,

$$A_{m\text{star}G} = \begin{bmatrix} A_{m\times m} & J_{m\times(N-m)} \\ J_{(N-m)\times m} & O_{(N-m)\times(N-m)} \end{bmatrix},$$

where $J$ is the all-one matrix and $A_{m\times m}$ is the adjacency matrix of an arbitrary graph $G$ that connects $m$ nodes. Each of those $m$ nodes is connected to each of the $N - m$ other nodes in the topology called $m$star$G$. We note that the bi-partite structure of $A_{m\text{star}G}$ is crucial. Similarly as above, we find

$$\det(A_{m\text{star}G} - \lambda I) = \det(A - \lambda I)_{m\times m}(-1)^{N-m}\lambda^{N-m-1}(\lambda + s_\lambda(N-m)),$$

where $t^2 = N - m$. Hence, by modifying the size of the matrix $A_{m\text{star}G}$, the zeros of the same function $f(\lambda) = \lambda + s_\lambda(N - m)$ are the maximum and minimum eigenvalue of $A_{m\text{star}G}$. Moreover, since the largest eigenvalue of $A$ (for any graph) is smaller than the maximum degree $d_{\max}(A) \leqslant m - 1$, a tighter bound for $N$ compared to $t$ is found, $N \geqslant 2m$. The general result is then given in (4).

**Example.** The spectrum of a $m$-fully meshed star topology where $A = J - I$ can be computed exactly as

$$(\lambda_{\max})_{m\text{star}} = \sqrt{m(N-m) + \left(\frac{m-1}{2}\right)^2} + \frac{m-1}{2},$$

$$(\lambda_{\min})_{m\text{star}} = -\sqrt{m(N-m) + \left(\frac{m-1}{2}\right)^2} + \frac{m-1}{2}$$

and with an eigenvalue at $-1$ with multiplicity $m-1$ and at 0 with multiplicity $N-m-1$. Comparing (13) with the exact result of a $m$-fully meshed star topology,

$$(\lambda_{\max})_{m\text{star}} = \sqrt{m(N-m) + \left(\frac{m-1}{2}\right)^2} + \frac{m-1}{2}$$
$$= \sqrt{(N-m)m} + \frac{m-1}{2} + \frac{1}{2}\left(\frac{m-1}{2}\right)^2 \frac{1}{\sqrt{(N-m)m}} + \mathrm{O}(N^{-3/2})$$

shows, indeed, that (13) is correct, since $N_1 = \frac{m(m-1)}{2}$, $N_2 = m(m-1)^2$ and $N_3 = m(m-1)^3$ for the complete graph $K_m$.

## 4. Perturbation theory

Consider the symmetrix matrix

$$B = \begin{bmatrix} O_{m\times m} & u_{m\times 1} \\ (u^{\mathrm{T}})_{1\times m} & 0 \end{bmatrix}$$

that represents the adjacency matrix of the bi-partite graph $K_{m,1}$ or the star topology, a central node that connects $m$ other, not interconnected nodes. The eigenvalues of $B$ are well-known: $-\sqrt{m}$, $\sqrt{m}$ and 0 with multiplicity $m-1$. The corresponding eigenvectors to the eigenvalue $\sqrt{m}$ and $-\sqrt{m}$ are $v = [u_{1\times m} \quad \sqrt{m}]^{\mathrm{T}}$ and $w = [u_{1\times m} \quad -\sqrt{m}]^{\mathrm{T}}$, respectively. Hence, apart from the zero eigenvalues, the eigenvalues of $tB$ are precisely $\lambda_0 = \pm t\sqrt{m}$.

Further, we can write

$$A_t = tB + A_0 = t\left(B + \frac{1}{t}A_0\right)$$

which implies that the eigenvalues of $A_t$ are equal to those of $B + \frac{1}{t}A_0$ multiplied by $t$. Since we known the eigenvalues of $B$ exactly, and if $t$ is sufficiently large, perturbation theory [2,7] can be applied. Since $B + zA_0$ is analytic in $z$, real symmetric on the real axis, all eigenvalues of $B + zA_0$ are analytic functions of $z$ in the neighborhood of the real axis ($\mathrm{Im}z = 0$). Hence, there exists a real number $R > 0$, for which $B + zA_0$ has two, simple eigenvalues $\lambda_+(z)$ and $\lambda_-(z)$ with Taylor expansion around $\sqrt{m}$ and $-\sqrt{m}$,

$$\lambda_+(z) = \sqrt{m} + \sum_{k=1}^{\infty} \alpha_k z^k \quad |z| < R,$$

$$\lambda_-(z) = -\sqrt{m} + \sum_{k=1}^{\infty} \beta_k z^k \quad |z| < R,$$

where all coefficients $\alpha_k$ and $\beta_k$ are real. Perturbation theory [7, p. 69] gives explicitly the first order coefficients as

$$\alpha_1 = \frac{v^{\mathrm{T}} A v}{v^{\mathrm{T}} v} = \frac{u^{\mathrm{T}} A u}{2m} \quad \text{and} \quad \beta_1 = \frac{w^{\mathrm{T}} A w}{w^{\mathrm{T}} w} = \frac{u^{\mathrm{T}} A u}{2m}.$$

Thus,

$$\alpha_1 = \beta_1 = \frac{N_1}{2m}.$$

For sufficiently small $z$, $\lambda_+(z)$ and $\lambda_-(z)$ are the maximum and minimum eigenvalue of $B + zA_0$. Hence, with (5), we obtain

$$\begin{aligned} \lambda_{\max}(A) &\geqslant t\left(\lambda_+\left(\frac{1}{t}\right) + \lambda_-\left(\frac{1}{t}\right)\right) = \sum_{k=1}^{\infty} (\alpha_k + \beta_k)\, t^{1-k} \\ &= \frac{N_1}{m} + \frac{(\alpha_2 + \beta_2)}{t} + \frac{(\alpha_3 + \beta_3)}{t^2} + \sum_{k=4}^{\infty} (\alpha_k + \beta_k)\, t^{1-k}. \end{aligned}$$

The specific (bi-partite) structure of $A_t$ enables us to write the characteristic polynomial (6) explicitly, from which we deduce, for sufficiently large $t$, that both $t\lambda_+(t^{-1})$ and $t\lambda_-(t^{-1})$ are zeros of the function $f(\lambda) = \lambda + s_\lambda t^2$. If $y(t)$ is a zero of $\lambda + s_\lambda t^2 = 0$, which is even in $t$, then also $y(-t)$ is a zero, which shows that $t\lambda_+(t^{-1}) = -t\lambda_-(-t^{-1})$. The zeros of $f(\lambda)$ can be expanded in a Lagrange series around $\lambda_0 = \pm t\sqrt{m}$. By also expanding the coefficients of this Lagrange series into a power series expansion in $\lambda_0$ as shown in Appendix B, all coefficients $\alpha_k$ can be computed and we indeed find that $\lambda_+(t^{-1}) = -\lambda_-(-t^{-1})$. Hence,

$$\begin{aligned} t\left(\lambda_+\left(\frac{1}{t}\right) + \lambda_-\left(\frac{1}{t}\right)\right) &= 2\sum_{k=0}^{\infty} \alpha_{2k+1} t^{-2k} \\ &= \frac{N_1}{m} + \frac{2\alpha_3}{t^2} + 2\sum_{k=2}^{\infty} \alpha_{2k+1} t^{-2k}. \end{aligned}$$

If $\alpha_3 = \frac{N_3}{2m} - \frac{N_1 N_2}{m^2} + \frac{N_1^3}{2m^3} > 0$, a tighter lower bound for $\lambda_{\max}(A)$ than the classical $\lambda_{\max}(A) \geqslant \frac{N_1}{m}$ is obtained.

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## Appendix A. Results from linear algebra

From the Schur identity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ O & D - CA^{-1}B \end{bmatrix} \tag{14}$$

we find that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B) \tag{15}$$

and $D - CA^{-1}B$ is called the Schur complement of $A$.

**Lemma A.1.** *If*

$$X = \begin{bmatrix} A & C \\ C^{\mathrm{T}} & B \end{bmatrix}$$

*is a real symmetric matrix, where A and B are square, and consequently symmetric, matrices, then*

$$\lambda_{\max}(X) + \lambda_{\min}(X) \leqslant \lambda_{\max}(A) + \lambda_{\max}(B) \tag{16}$$

**Proof.** See e.g. [1, p. 56]. □

## Appendix B. Characteristic coefficients of a complex function

If $f(z)$ has a Taylor series around $z_0$,

$$f(z) = \sum_{k=0}^{\infty} f_k(z_0)(z - z_0)^k \quad \text{with} \quad f_k(z_0) = \frac{1}{k!} \left. \frac{d^k f(z)}{dz^k} \right|_{z=z_0}$$

then the general relation where $G(z)$ is analytic around $f(z_0)$ is

$$G(f(z)) = G(f(z_0)) + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \frac{1}{k!} \left. \frac{d^k G(p)}{dp^k} \right|_{p=f(z_0)} s[k,m]_{f(z)}(z_0) \right) (z - z_0)^m, \tag{17}$$

where the characteristic coefficient [5] of a complex function $f(z)$ is

$$s[k,m]_{f(z)}(z_0) = \sum_{\sum_{i=1}^{k} j_i = m;\, j_i > 0} \prod_{i=1}^{k} f_{j_i}(z_0)$$

which obeys the recursion relation

$$\begin{aligned} s[1,m]_{f(z)}(z_0) &= f_m(z_0), \\ s[k,m]_{f(z)}(z_0) &= \sum_{j=1}^{m-k+1} f_j(z_0) s[k-1, m-j]_{f(z)}(z_0) \quad (k > 1). \end{aligned} \tag{18}$$

The zero $\zeta(z_0)$ of $f(z)$ closest to $z_0$ is given [5] in terms of the coefficients $f_k(z_0)$ of the series expansion of $f(z)$ around $z_0$ as

$$\begin{aligned} \zeta(z_0) = f^{-1}(0) = z_0 &- \frac{f_0(z_0)}{f_1(z_0)} \\ &+ \sum_{n=2}^{\infty} \left[ \sum_{k=1}^{n-1} \frac{(-1)^k \binom{n+k-1}{k-1}}{k(f_1(z_0))^k} s^*[k, n-1](z_0) \right] \left( -\frac{f_0(z_0)}{f_1(z_0)} \right)^n, \end{aligned} \tag{19}$$

where $s^*[k,m] = s[k,m]|_{f_m \to f_{m+1}}$ denotes that the index of all Taylor coefficients appearing in (18) is augmented by 1. Explicitly summing the first five terms ($n \leqslant 5$),

$$\zeta(z_0) \approx z_0 - \frac{f_0(z_0)}{f_1(z_0)} - \frac{f_2(z_0)}{f_1(z_0)}\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^2 + \left[-2\left(\frac{f_2(z_0)}{f_1(z_0)}\right)^2 + \frac{f_3(z_0)}{f_1(z_0)}\right]\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^3$$
$$+ \left[-5\left(\frac{f_2(z_0)}{f_1(z_0)}\right)^3 + 5\frac{f_3(z_0)}{f_1(z_0)}\frac{f_2(z_0)}{f_1(z_0)} - \frac{f_4(z_0)}{f_1(z_0)}\right]\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^4$$
$$+ \left[-14\left(\frac{f_2(z_0)}{f_1(z_0)}\right)^4 + 21\frac{f_3(z_0)}{f_1(z_0)}\left(\frac{f_2(z_0)}{f_1(z_0)}\right)^2 - 3\left(\frac{f_3(z_0)}{f_1(z_0)}\right)^2\right.$$
$$\left. - 6\frac{f_4(z_0)}{f_1(z_0)}\frac{f_2(z_0)}{f_1(z_0)} + \frac{f_5(z_0)}{f_1(z_0)}\right]\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^5. \tag{20}$$

*Appendix B.1. Lagrange expansion*

The zero of $f(\lambda) = \lambda + s_\lambda t^2 = 0$ will be computed using the Lagrange series which can be efficiently computed to any order with characteristic coefficients [5]. The Lagrange expansion (19) in terms of characteristic coefficients needs the Taylor coefficients of $f(\lambda)$ around $\lambda_0 = \pm t\sqrt{m}$,

$$f_n(\lambda_0) = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\left(\lambda - t^2\sum_{k=0}^{\infty}\frac{N_k}{\lambda^{k+1}}\right)\Bigg|_{\lambda=\lambda_0}.$$

For $n = 0$,

$$f_0(\lambda_0) = \lambda_0 - t^2\sum_{k=0}^{\infty}\frac{N_k}{\lambda_0^{k+1}} = -t^2\sum_{k=1}^{\infty}\frac{N_k}{\lambda_0^{k+1}}$$

because $\lambda_0 - \frac{t^2 m}{\lambda_0} = 0$ and,

$$f_0(\lambda_0) = -\frac{1}{m}\sum_{k=0}^{\infty}\frac{N_{k+1}}{\lambda_0^k}. \tag{21}$$

Similarly, for $n = 1$,

$$f_1(\lambda_0) = 1 + t^2\sum_{k=0}^{\infty}\frac{(k+1)N_k}{\lambda_0^{k+2}} = 2 + \frac{1}{m}\sum_{k=1}^{\infty}\frac{(k+1)N_k}{\lambda_0^k} \tag{22}$$

and, for all $n > 1$,

$$f_n(\lambda_0) = \frac{(-1)^{n+1}t^2}{\lambda_0^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{N_k}{\lambda_0^k}, \tag{23}$$

We now confine ourselves to computing the zero $\zeta(\lambda_0)$ of $f(\lambda)$ accurate up to order $O(t^{-2q})$, where $q$ is fixed, but specified later. Since $\lambda_0 = O(t)$, for $f_n(\lambda_0)$ to be accurate up to order $O(t^{-2q})$, we need to take in the computation $k = 2(q-1) - n$ terms in the $k$-sums. Also, it follows that $2(q-1) > n$ derivatives or Taylor coefficients in the Lagrange series are needed. The Lagrange expansion (19) indicates that we need order expansions for $(f_1(\lambda_0))^{-k}$ and $\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^n$. Both of these expansions can be given in terms of characteristic coefficients. Applying

$$g^{-k}(z) = g_0^{-k} + \sum_{m=1}^{\infty}\left(\sum_{n=1}^{m}(-1)^n\binom{n+k-1}{n}g_0^{-k-n}s[n,m]\right)z^m$$

to (22), we have, with $z = \lambda_0^{-1}$ and $g_0 = 2$ and $g_n = \frac{(n+1)}{m} N_n$ for $n > 0$, that

$$(f_1(\lambda_0))^{-k} = 2^{-k} + \sum_{m=1}^{\infty} \left( \sum_{n=1}^{m} (-1)^n \binom{n+k-1}{n} \frac{s[n,m]|_{g(z)}}{2^{k+n}} \right) \lambda_0^{-m}.$$

Similarly, applying

$$h^n(z) = h_0^n + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \binom{n}{k} h_0^{n-k} s[k,m] \right) z^m$$

to (21), we obtain, with $z = \lambda_0^{-1}$ and with $h_n = -\frac{N_{n+1}}{m}$ for $n \geqslant 0$, that

$$(f_0(\lambda_0))^n = \frac{(-1)^n N_1^n}{m^n} \left( 1 + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \binom{n}{k} \frac{(-m)^k s[k,m]|_{h(z)}}{N_1^k} \right) \lambda_0^{-m} \right).$$

The quotient $\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^n$ follows by Cauchy's product rule for series,

$$\begin{aligned}\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^n = {} & \frac{(-1)^n N_1^n}{m^n 2^n} \left( 1 - \frac{n}{m} \left( \frac{-mN_2}{N_1} + N_1 \right) \lambda_0^{-1} \right. \\ & + \frac{(-1)^n N_1^n}{m^n 2^n} \sum_{r=2}^{\infty} \left( \sum_{k=1}^{r} \binom{n}{k} \frac{(-m)^k s[k,r]|_{h(z)}}{N_1^k} \right. \\ & + \sum_{j=1}^{r} (-1)^j \binom{j+n-1}{j} \frac{s[j,r]|_{g(z)}}{2^j} \\ & + \sum_{q=1}^{r-1} \left( \sum_{j=1}^{r-q} (-1)^j \binom{j+n-1}{j} \frac{s[j,r-q]|_{g(z)}}{2^j} \right) \\ & \left. \times \left( \sum_{k=1}^{q} \binom{n}{k} \frac{(-m)^k s[k,q]|_{h(z)}}{N_1^k} \right) \right) \lambda_0^{-r}.\end{aligned}$$

Explicitly up to order $O(t^{-3})$,

$$\begin{aligned}\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^n = {} & \frac{(-1)^n N_1^n}{m^n 2^n} \left( 1 + \frac{n}{m} \left( \frac{mN_2}{N_1} - N_1 \right) \right) \lambda_0^{-1} \\ & + \frac{(-1)^n N_1^n}{m^n 2^n} \left( \frac{nN_3}{N_1} + \frac{n(n-1)N_2^2}{2N_1^2} - \frac{(3n+2n^2)N_2}{2m} + \frac{n(n+1)N_1^2}{2m^2} \right) \lambda_0^{-2}.\end{aligned}$$

Let us compute the zero of $f(\lambda) = \lambda + s_\lambda t^2 = 0$ up to $O(t^{-3})$. For $q = 3/2$, we need for each derivative $n < 4$, only $k = 4 - n$ terms. The corresponding Lagrange series is

$$\begin{aligned}\zeta(\lambda_0) = {} & \lambda_0 - \frac{f_0(\lambda_0)}{f_1(\lambda_0)} - \frac{f_2(\lambda_0)}{f_1(\lambda_0)} \left( \frac{f_0(\lambda_0)}{f_1(\lambda_0)} \right)^2 \\ & + \left[ -2 \left( \frac{f_2(\lambda_0)}{f_1(\lambda_0)} \right)^2 + \frac{f_3(\lambda_0)}{f_1(\lambda_0)} \right] \left( \frac{f_0(\lambda_0)}{f_1(\lambda_0)} \right)^3 + O(t^{-3}).\end{aligned}$$

We list the separate terms,

$$\frac{f_0(z_0)}{f_1(z_0)} = \frac{-N_1}{2m}\left(1 + \frac{1}{m}\left(\frac{mN_2}{N_1} - N_1\right)\lambda_0^{-1} + \left(\frac{N_3}{N_1} - \frac{5N_2}{2m} + \frac{N_1^2}{m^2}\right)\lambda_0^{-2}\right),$$

$$\left(\frac{f_0(\lambda_0)}{f_1(\lambda_0)}\right)^2 = \frac{N_1^2}{4m^2}\left(1 + \frac{2}{m}\left(\frac{mN_2}{N_1} - N_1\right)\lambda_0^{-1} + \mathrm{O}(\lambda_0^{-2})\right),$$

$$\left(\frac{f_0(z_0)}{f_1(z_0)}\right)^3 = \frac{-N_1^3}{8m^3}(1 + \mathrm{O}(\lambda_0^{-1}))$$

and

$$\frac{f_2(\lambda_0)}{f_1(\lambda_0)} = -\left(\frac{1}{2} - \frac{N_1}{2m}\lambda_0^{-1} + \left(\frac{N_1^2}{2m^2} - \frac{3N_2}{2m}\right)\lambda_0^{-2}\right)\left(\frac{1}{\lambda_0} + \frac{3N_1}{m\lambda_0^2}\right)$$

$$= -\left(\frac{1}{2}\lambda_0^{-1} + \frac{N_1}{m}\lambda_0^{-2}\right),$$

$$\left(\frac{f_2(\lambda_0)}{f_1(\lambda_0)}\right)^2 = \frac{1}{4}\lambda_0^{-2}$$

and

$$\frac{f_3(\lambda_0)}{f_1(\lambda_0)} = \left(2^{-1} - \frac{N_1}{2m}\lambda_0^{-1} + \left(\frac{N_1^2}{2m^2} - \frac{3N_2}{2m}\right)\lambda_0^{-2}\right)\lambda_0^{-2} = \frac{1}{2}\lambda_0^{-2}.$$

Combined yields the final result (13).

## Appendix C. A finite sum expression for $s_\lambda$

Since $A$ is symmetrix, $A^k = X\mathrm{diag}(\lambda_j^k)X^{\mathrm{T}}$ where the colums of the orthogonal matrix $X$ consists of eigenvectors $x_j$ of $A$,

$$(A - \lambda I)^{-1} = -\frac{1}{\lambda}X\left(\mathrm{diag}\left(\sum_{k=0}^{\infty}\frac{\lambda_j^k}{\lambda^k}\right)\right)X^{\mathrm{T}} = -X\mathrm{diag}\left(\frac{1}{\lambda - \lambda_j}\right)X^{\mathrm{T}}.$$

Then,

$$s_\lambda = u(A - \lambda I)^{-1}u^{\mathrm{T}} = -\sum_{j=1}^{m}\frac{\left(\sum_{k=1}^{m} x_{j;k}\right)^2}{\lambda - \lambda_j}. \tag{24}$$

Unless the all-one vector $u$ is an eigenvector of $A$, $(\sum_{k=1}^{m} x_{j;k})^2 \neq 0$. Hence, in that case

$$\det(A_t - \lambda I) = -\det(A - \lambda I)_{m\times m}\left(\lambda - t^2\sum_{j=1}^{m}\frac{\left(\sum_{k=1}^{m} x_{j;k}\right)^2}{\lambda - \lambda_j}\right)$$

and no eigenvalue of $A$ is a zero of $f(\lambda) = \lambda - t^2\sum_{j=1}^{m}\frac{(\sum_{k=1}^{m} x_{j;k})^2}{\lambda - \lambda_j}$.

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