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Effective graph resistance

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ABSTRACT

This paper studies an interesting graph measure that we call the effective graph resistance. The notion of effective graph resistance is derived from the field of electric circuit analysis where it is defined as the accumulated effective resistance between all pairs of vertices. The objective of the paper is twofold. First, we survey known formulae of the effective graph resistance and derive other representations as well. The derivation of new expressions is based on the analysis of the associated random walk on the graph and applies tools from Markov chain theory. This approach results in a new method to approximate the effective graph resistance.

A second objective of this paper concerns the optimisation of the effective graph resistance for graphs with given number of vertices and diameter, and for optimal edge addition. A set of analytical results is described, as well as results obtained by exhaustive search. One of the foremost applications of the effective graph resistance we have in mind, is the analysis of robustness-related problems. However, with our discussion of this informative graph measure we hope to open up a wealth of possibilities of applying the effective graph resistance to all kinds of networks problems.

1. Introduction

Over the past several years a variety of graph measures have been proposed with the aim of quantifying the relevant structural attributes of a graph. In this context of graph theory and network analysis

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we propose the effective graph resistance, a graph measure in our opinion highly valuable in the analysis of various network problems. The choice of this measure is inspired by the excellent paper of Klein and Randić [6]. The main contribution of their paper is the proof that the effective graph resistance can be written in terms of Laplacian eigenvalues. It is interesting to notice that the effective graph resistance is also called Kirchhoff index, named after Kirchhoff's circuit laws from which the notion of the effective resistance was initially derived.

The formal definition of the effective graph resistance is the sum of the effective resistances over all pairs of vertices. More informally, the effective resistance between two vertices of a network, assuming that a network is seen as an electrical circuit, can be calculated by the well-known series and parallel manipulations. Two edges, corresponding to resistors with resistance r_1 and r_2 Ohm, in series can be replaced by one edge with effective resistance $r_1 + r_2$. If the two edges are connected in parallel, then they can be replaced by an edge with effective resistance $\left(r_1^{-1} + r_2^{-1}\right)^{-1}$. From these series and parallel manipulations it follows that the effective graph resistance takes both the number of (not necessarily disjoint) paths between two vertices and their length into account, intuitively measuring the presence and quality of back-up possibilities.

The contribution of this paper is twofold. First, we survey known results of the effective graph resistance and give new representations based on the associated random walk on the graph, leading to a new method for approximating the effective graph resistance. Second, we optimise the effective graph resistance for graphs with a given number of vertices and diameter, and we consider optimal edge addition. In addition to these main contributions, we discuss a possible application related to network robustness. Overall, this paper establishes a path towards the identification of the set of graph measures that will serve in future analysis of various network problems.

The paper is organised as follows. Section 2 gives an overview of the preliminaries, the formal definition and some basic results on the effective graph resistance. Section 2 also gives a representation of the effective graph resistance based on the analysis of the associated random walk on the graph. In addition to the above, this section contains some examples giving an idea of the values the effective graph resistance can take and the reasoning behind the introduction of the effective graph resistance as a quantifier of robustness. Section 3 contains a set of results on the optimisation of the effective graph resistance: for a graph with given number of vertices and diameter in Sections 3.1 to 3.5, and for edge addition in Section 3.6. Section 4 summarises our main results on the effective graph resistance and states several interesting problems for further research.

2. Effective resistance

2.1. Preliminaries: Laplacian eigenvalues

Since the effective graph resistance is a function of the Laplacian eigenvalues of the graph, as shown in Section 2.2, this section provides a short introduction on the Laplacian and its eigenvalues. For a simple undirected graph $G = (V, E)$ the *Laplacian* Q is defined as the difference $\Delta - \mathsf{A}$ of the degree matrix Δ and the adjacency matrix A, i.e.

$$\mathsf{Q}_{ij} = \begin{cases} \delta_i & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases},$$

where δ_i is the degree of vertex i. For a graph with non-negative edge weights w_{ij}, the *weighted Laplacian* is $\mathsf{L}^W = \mathsf{S} - \mathsf{W}$, with W the matrix of weights $\mathsf{W} = (\mathrm{w_{ij}})$ and S the diagonal matrix of strengths ($\mathsf{S}_{ii} = \sum_{j=1}^{N} w_{ij}$).

Because the Laplacian is symmetric, positive semidefinite and the rows sum up to 0, its eigenvalues are real, non-negative and the smallest one is zero. Hence, we can order the eigenvalues and denote them as μ_i for $i = 1, \ldots, N = |V|$ such that $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_N$. The second smallest eigenvalue μ_2 of the Laplacian is called the *algebraic connectivity*.

For a complete graph K_N, we have $\mu_1 = 0$, $\mu_2 = \cdots = \mu_N = N$, because the vectors $\boldsymbol{u}_j$ with a one at the first position, a minus one at position $j \neq 1$, and zeroes everywhere else, are $N-1$ linearly independent eigenvectors corresponding to the eigenvalue N.

For more information we refer to [14] and to [7], in which Mohar gives a clear survey on the Laplacian, its properties, and its applications. In Appendix B.8 of [3] Cvetković gives an extensive review of publications on the Laplacian of graphs.

2.2. Definition of effective resistance and basic results

We will start by stating formal definitions of the pairwise effective resistance and the effective graph resistance together with some important theorems. The simple, undirected and connected graph is regarded as an electrical circuit, where an edge (i,j) corresponds to a resistor of $r_{ij} = 1$ Ohm. All definitions and results in this section carry over to weighted graphs when the edge resistance is defined as $r_{ij} = 1/w_{ij}$. For the proofs of the theorems in the weighted case see [4].

For each pair of vertices the *effective resistance* between these vertices – the resistance of the total system when a voltage is connected across them – can be calculated by Kirchhoff's circuit laws. Let a voltage be connected between vertices a and b and let $I > 0$ be the net current out of source a and into sink b, *Kirchhoff's current law* states that the current y_{ij} between vertices i and j (where $y_{ij} = -y_{ji}$) must satisfy

$$\sum_{j \in N(i)} y_{ij} = \begin{cases} I & \text{if } y = a \\ -I & \text{if } y = b \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

with $N(i)$ the neighbourhood of i, that is, the set of vertices adjacent to vertex i. This first law means that the total flow into a vertex equals the total flow out of it. The second of Kirchhoff's laws is equivalent to saying that a potential $\boldsymbol{v}$ may be associated with any vertex i, such that for all edges (i,j)

$$y_{ij} r_{ij} = \boldsymbol{v}_i - \boldsymbol{v}_j. \tag{2}$$

This is called Ohm's law.

Definition 2.1. The *effective resistance* R_{ab} between vertices a and b is defined as

$$R_{ab} = \frac{\boldsymbol{v}_a - \boldsymbol{v}_b}{I}.$$

The next theorem shows that R_{ab} exists and is uniquely defined. It is not known who first proved the theorem, but a continuous version was already known to Gauss.

Theorem 2.1. *The effective resistance R_{ab} between vertices a and b satisfies*

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^T \, \mathrm{Q}^{-1} (\boldsymbol{e}_a - \boldsymbol{e}_b),$$

where Q^{-1} is any matrix that on $(\mathrm{span}\{\mathbf{1}\})^{\perp}$ (the subspace perpendicular to the all-one vector) corresponds to an inverse of the Laplacian Q and on $\mathrm{span}\{\mathbf{1}\}$ to the zero map. The vector $\boldsymbol{e}_i$ has a one at position i and zeroes elsewhere.

This theorem will be used in Section 2.4 to derive an approximation formula for the computation of the effective graph resistance. We will now define the effective graph resistance.

Definition 2.2. The *effective graph resistance* R_G is the sum of the effective resistances over all pairs of vertices in the graph G:

$$R_G = \sum_{1 \leqslant i < j \leqslant N} R_{ij}.$$

In the literature the effective graph resistance is also called total effective resistance or Kirchhoff index. Klein and Randić [6] have proved that it can be written as a function of the non-zero Laplacian eigenvalues.

Theorem 2.2. *The effective graph resistance R_G satisfies*

$$R_G = N \sum_{i=2}^{N} \frac{1}{\mu_i}.$$

The next corollary specifies the relation between the algebraic connectivity and the effective graph resistance.

Corollary 2.3. *The effective graph resistance R_G can be bounded by functions of μ_2 in the following way*

$$\frac{N}{\mu_2} < R_G \leqslant \frac{N(N-1)}{\mu_2}.$$

Tighter bounds for R_G are presented in [14]. The effective graph resistance has been called resistance distance by Klein and Randić [6]. They have proved that it is indeed a distance function (metric). We will give some more (probably previously known) results from [6].

Theorem 2.4. *For the effective resistance and the ordinary distance d we have for any pair of vertices i, j:*

(1) $R_{ij} = d_{ij}$, if there is only one path between i and j;
(2) $R_{ij} < d_{ij}$, otherwise.

Corollary 2.5. *The effective resistance and the ordinary distance correspond on a tree, that is, for every pair of vertices i, j in a tree we have:*

$$R_{ij} = d_{ij}.$$

As a result of the Interlacing Theorem [14,10] the pairwise effective resistance is a non-increasing function of the edge weights. The result is generally referred to as Rayleigh's monotonicity law.

Theorem 2.6. *The pairwise effective resistance does not increase when edges are added or weights are increased.*

The effective graph resistance is even strictly decreasing in the edge weights.

Theorem 2.7. *The effective graph resistance strictly decreases when edges are added or weights are increased.*

Proof. Suppose edge weight w_{ij} is increased or edge (i, j) is added. It is enough to show that R_{ij} strictly decreases, since effective resistances between other pairs do not increase because of Theorem 2.6. The fact that R_{ij} strictly increases is a direct consequence of the well-known rule for resistors in parallel. □

2.3. An analogy with random walks

Let a random walk on the simple, undirected and connected graph $G = (V, E)$ be given by the transition probabilities $p_{ij} = a_{ij}/\delta_i$, where $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. We will consider the *expected commute time* between two vertices a and b in this random walk. This is the expected number of transitions needed to go from a to b and back. The following theorem from Chandra et al. [2] gives a relation between the average commute time and the effective resistance in the same graph.

Theorem 2.8. *Let a graph $G = (V, E)$ be given. First, define an electrical circuit as before. Secondly, define a random walk on G by the transition probabilities $p_{ij} = \frac{a_{ij}}{\delta_i}$. Let T_{ab} be the time (number of transitions) to reach vertex b starting in a. It holds that*

$$R_{ab} = \frac{1}{2L}(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba})),$$

with $L = |E|$.

The pairwise effective resistance is proportional to the expected commute time, which implies that the effective graph resistance is proportional to the expected commute time averaged over all pairs of vertices.

Corollary 2.9. *We have*

$$R_G = \frac{1}{2L}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{E}(T_{ij}).$$

The number of visits to vertex v in a random walk starting in a, going to b, and back to a, is also related to the expected commute time. This relation is given in the following lemma, which is easy to prove, but has not been found in the literature.

Theorem 2.10. *Let B_{avb} be the number of visits to vertex v strictly in between the start of the random walk in a and the stop in b. The expression*

$$\mathbf{E}(B_{avb}) + \mathbf{E}(B_{bva}) = \pi_v\left(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba})\right)$$

holds true. Here $\pi_v = \delta_v/2L$ denotes the stationary probability of vertex v, that is, the probability of being in vertex v in steady-state.

Proof. The theorem is clearly true for $a = b$. Suppose now that $a \neq b$. Lemma 9 in Chapter 2, Section 2 of [1] says

$$\mathbf{E}(B_{avb}) = \pi_v\left(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{bv}) - \mathbf{E}(T_{av})\right).$$

Adding $\mathbf{E}(B_{avb})$ and $\mathbf{E}(B_{bva})$ directly leads to the desired result. □

This theorem provides us an easy alternative way to prove that network criticality — proposed as a robustness measure by Tizghadam and Leon-Garcia [11] — is equal to two times the effective graph resistance. They define the random walk betweenness of vertex v as

$$B_v = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{E}(B_{ivj})$$

and the network criticality as

$$\tau = 2\frac{B_v}{\delta_v},$$

which turns out to be independent of the vertex v.

Theorem 2.11. *The network criticality τ satisfies*

$$\tau = 2R.$$

Proof. We use Corollary 2.9 and Theorem 2.10 to find

$$\frac{1}{2}\tau = \frac{B_v}{\delta_v} = \frac{1}{\delta_v}\sum_{1\leqslant i<j\leqslant N}(\mathbf{E}(B_{ivj}) + \mathbf{E}(B_{jvi})) = \frac{1}{\delta_v}\sum_{1\leqslant i<j\leqslant N}\pi_v\left(\mathbf{E}(T_{ij}) + \mathbf{E}(T_{ji})\right)$$

$$= \frac{1}{2L}\sum_{1\leqslant i<j\leqslant N}\left(\mathbf{E}(T_{ij}) + \mathbf{E}(T_{ji})\right) = R. \qquad \square$$

Most of the results of this section can be found in [1]. All proofs not given in this section are available in [4]. For the sake of presentation we have restricted ourselves to random walks on unweighted graphs, although all results are valid also for weighted graphs.

2.4. More alternative expressions and a computational method

In this section we will develop a method to approximate the effective graph resistance. In order to do so, we derive alternative expressions for our measure, using the analogy with random walks considered in Section 2.3.

The method that we will discuss here, proposes a way to approximate the pseudo inverse on $(\text{span}\{\mathbf{1}\})^{\perp}$. To this end, we consider the random walk on the simple, undirected and connected graph G with N vertices defined in Section 2.3. The associated transition matrix is equal to $\mathsf{P} = \Delta^{-1}\mathsf{A}$. The random walk is an irreducible Markov chain.

Let $y \perp \mathbf{1}$ be given. Clearly, $\mathsf{Q}x = y$ is equivalent to $(\mathsf{I} - \mathsf{P})x = \Delta^{-1}y$. In other words, x is a solution of $\mathsf{Q}x = y$ if and only if

$$x = \Delta^{-1}y + \mathsf{P}x. \tag{3}$$

This is precisely the so-called Poisson equation for Markov chains, which has a unique solution up to a constant vector. A typical method for solving this equation is to substract a specially chosen constant vector from both sides of (3), such that the resulting equation has a unique solution.

Let us be more precise. Fix any node k and choose $x_k\mathbf{1}$ to be the constant vector to substract:

$$x - x_k\mathbf{1} = \Delta^{-1}y + \mathsf{P}(x - x_k\mathbf{1}). \tag{4}$$

Since $x - x_k\mathbf{1}$ has k-th component equal to 0, we may replace the k-th column of P by zeroes. In other words, we delete the transitions to node k. The resulting matrix is the so-called taboo matrix with taboo set $\{k\}$, and is denoted by ${}_k\mathsf{P}$. The t-th iterate is denoted by ${}_k\mathsf{P}^{(t)}$, with ${}_k\mathsf{P}^{(0)} = \mathsf{I}$. The elements of ${}_k\mathsf{P}^{(t)}$ are denoted ${}_kp_{ij}^{(t)}$. Eq. (4) then becomes

$$x - x_k\mathbf{1} = \Delta^{-1}y + {}_k\mathsf{P}(x - x_k\mathbf{1}). \tag{5}$$

Since the taboo matrix ${}_k\mathsf{P}$ is the transition matrix of a transient Markov chain, $\mathsf{I} - {}_k\mathsf{P}$ is invertible with inverse $\sum_{t=0}^{\infty} {}_k\mathsf{P}^{(t)}$. This well-known fact follows e.g. from [9], using the fact that $\lambda = 1$ is the unique maximum eigenvalue in absolute value of an irreducible stochastic matrix.

As a consequence $x - x_k\mathbf{1} = \sum_{t=0}^{\infty} {}_k\mathsf{P}^{(t)}\Delta^{-1}y$. We can now describe the solution space of (3) in terms of the taboo matrix.

Lemma 2.12. *For any vector $y \perp \mathbf{1}$ the solution space of (3) is given by*

$$\left\{x \mid x = \sum_{t=0}^{\infty} {}_k\mathsf{P}^{(t)}\Delta^{-1}y + c\mathbf{1},\ c \in \mathbb{R}\right\}. \tag{6}$$

Proof. Let x be a solution of (3). From the above discussion it follows that

$$x = \sum_{t=0}^{\infty} {}_k\mathsf{P}^{(t)} \Delta^{-1} y + x_k \mathbf{1}.$$

Vice versa, let $x = \sum_{t=0}^{\infty} {}_k\mathsf{P}^{(t)} \Delta^{-1} y + c\mathbf{1}$. In other words $x - c\mathbf{1} = \Delta^{-1} y + {}_k\mathsf{P}(x - c\mathbf{1})$. By the Renewal Reward Theorem ([8] Theorem 3.16) $(x - c\mathbf{1})_k = \sum_j \pi_j y_j / \delta_j \pi_k = \sum_j y_i / \delta_k = 0$ and so $x - c\mathbf{1}$ is a solution of (3). But then x is a solution of (3) as well. □

Using Lemma 2.12 one may now compute the effective resistance by filling in $\mathsf{Q}^{-1} = \sum_{t=1}^{\infty} {}_k\mathsf{P}^{(t)} \Delta^{-1}$ in Theorem 2.1. For any k we find

$$R_{ij} = \sum_{t=0}^{\infty} {}_k p_{ii}^{(t)} \frac{1}{\delta_i} + \sum_{t=0}^{\infty} {}_k p_{jj}^{(t)} \frac{1}{\delta_j} - \sum_{t=0}^{\infty} {}_k p_{ij}^{(t)} \frac{1}{\delta_j} - \sum_{t=0}^{\infty} {}_k p_{ji}^{(t)} \frac{1}{\delta_i}. \tag{7}$$

The unknown constant of Lemma 2.12 cancels out because of the pre-multiplication by the vector $\boldsymbol{e_i} - \boldsymbol{e_j}$.

Taking taboo state i in (7) yields a simple characterisation of the effective resistance between nodes i and j. For its formulation we also need the taboo matrix ${}_{k,l}\mathsf{P}$ with taboo set $\{k, l\}$: it is obtained from P by deleting the transitions to nodes k and l, or, equivalently, by replacing the k-th and l-th columns by zeroes. The elements of the t-th iterate are denoted by ${}_{k,l}p_{ij}^{(t)}$.

Theorem 2.13. *It holds that*

$$R_{ij} = \sum_{t=0}^{\infty} {}_i p_{jj}^{(t)} \frac{1}{\delta_j} = \frac{1}{\delta_i g_{ij}}, \quad \textit{for } i \neq j, \tag{8}$$

where $g_{ij} = \sum_{t=0}^{\infty} \sum_{k=1}^{N} {}_{i,j}p_{ik}^{(t)} p_{kj}$ *is the probability that the random walk starting at i reaches j before returning to i.*

The expression $\sum_{t=0}^{\infty} {}_i p_{jj}^{(t)}$ has the interpretation of the expected number of visits of j, before returning to i, of the random walk starting at j.

Proof. Taking $k = i$ in (7) leads to

$$\begin{aligned} R_{ij} &= \frac{1}{\delta_i} \sum_{t=0}^{\infty} {}_i p_{ii}^{(t)} + \frac{1}{\delta_j} \sum_{t=0}^{\infty} {}_i p_{jj}^{(t)} - \frac{1}{\delta_j} \sum_{t=0}^{\infty} {}_i p_{ij}^{(t)} - \frac{1}{\delta_i} \sum_{t=0}^{\infty} {}_i p_{ji}^{(t)} \\ &= \frac{1}{\delta_i} + \sum_{t=0}^{\infty} {}_i p_{jj}^{(t)} \frac{1}{\delta_j} - \frac{1}{\delta_i} - 0 = \sum_{t=0}^{\infty} {}_i p_{jj}^{(t)} \frac{1}{\delta_j}. \end{aligned}$$

We have used the two following relations. The first is $\delta_j = \delta_i \sum_{t=0}^{\infty} {}_i p_{ij}^{(t)}$, which follows from [1, Chapter 2, Proposition 3], with stopping time S equal to the first return time to state i. The second relation is $\sum_{t=0}^{\infty} {}_i p_{ij}^{(t)} = g_{ij} \sum_{t=0}^{\infty} {}_i p_{jj}^{(t)}$. The validity of this expression can be argued as follows: $\sum_k {}_{i,j}p_{ik}^{(t)} p_{kj}$ is the probability that the random walk starting at node i visits node j for the first time at time $t + 1$, without passing node i in between. The event that the random walk starting at node i visits node j at all, without passing node i in between, is the disjoint union of events of the above type. □

A consequence of Theorem 2.13 is computable upper and lower bounds for R. Indeed, since the involved probabilities are always non-negative, for each pair of indices T_1, T_2

$$\frac{1}{\delta_j} \sum_{t=0}^{T_1} {}_i p_{jj}^{(t)} \leqslant R_{ij} \leqslant \frac{1}{\delta_i \sum_{t=0}^{T_2} \sum_{k=1}^{N} {}_{i,j}p_{ik}^{(t)} p_{kj}}, \quad \text{for } i \neq j. \tag{9}$$

For any $\epsilon > 0$, choose T_1 and T_2 such that the difference between upper and lower bound in (9) is less than 2ϵ. An approximation of the pairwise effective resistance R_{ij} up to ϵ precision is given by

$$R_{ij} \approx \frac{1}{2}\left(\frac{1}{\delta_j}\sum_{t=0}^{T_1} {}_ip_{jj}^{(t)} + \frac{1}{\delta_i \sum_{t=0}^{T_2}\sum_{k=1}^{N} {}_{i,j}p_{ik}^{(t)} p_{kj}}\right).$$

2.5. Some examples

As a consequence of Theorem 2.7, for graphs with a given number of vertices N the minimum effective graph resistance is reached by the complete graph K_N. By Theorem 2.2 and the eigenvalues of K_N given in Section 2.1 we have

$$R_{K_N} = N - 1.$$

The effective graph resistance cannot be calculated for unconnected graphs. For these graphs it is said to be infinity. Corollary 2.5 and Theorem 2.7 show that the connected graph with maximum effective graph resistance is the tree with maximum average distance. The path graph P_N has maximum average distance of all trees with N vertices and effective graph resistance

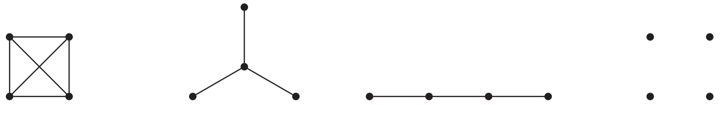
$$R_{P_N} = \sum_{1 \leqslant i < j \leqslant N} d_{ij} = \sum_{i=1}^{N-1}\sum_{j=1}^{N-i} j = \frac{1}{6}(N-1)N(N+1).$$

The tree with minimum effective graph resistance, that is with minimum average distance, is the star graph S_N. Its effective graph resistance is

$$R_{S_N} = \sum_{1 \leqslant i < j \leqslant N} d_{ij} = (N-1)\cdot 1 + \frac{1}{2}(N-1)(N-2)\cdot 2 = (N-1)^2.$$

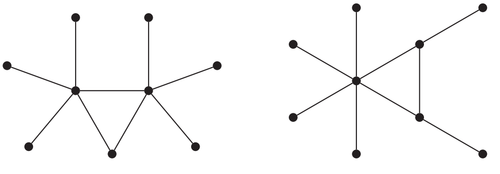
Fig. 1 gives examples of the graphs mentioned in this section.

Fig. 2 shows that different graphs with the same number of vertices and edges may have the same effective graph resistance [16].



(a) Complete graph K_4, $R_{K_4} = 3$ (b) Star graph S_4, $R_{S_4} = 9$ (c) Path graph P_4, $R_{P_4} = 10$ (d) Empty graph O_4, $R_{O_4} = \infty$

Fig. 1. Examples of graphs with four vertices.



(a) Graph G_1 with $R_{G_1} = 64$ (b) Graph G_2 with $R_{G_2} = 64$

Fig. 2. Two graphs with the same total effective resistance.

2.6. Effective graph resistance as a robustness measure

We believe that the effective graph resistance is a good measure for network robustness; the smaller the effective graph resistance the more robust the network. We have several arguments.

First, the effective graph resistance is the sum of pairwise effective resistances, which measure the robustness of the connection between two vertices, because pairwise effective resistance takes both the number of paths between two vertices and their length into account, therefore the number of back-up paths as well as their quality is considered.

A second indication is given by the fact that the effective graph resistance can be approximated by the algebraic connectivity (Corollary 2.3). Algebraic connectivity is used as a measure for network robustness [5].

Third, Theorem 2.7 states that effective graph resistance strictly decreases when edges are added or edge weights are increased. Algebraic connectivity for example does not show this strict monotonicity. Moreover, for the simple examples in Section 2.5 the effective graph resistance gives the same evaluation of robustness as does our intuition. Complete graphs are most robust, unconnected graphs least, trees are the least robust connected graphs, star graphs are the most robust trees, and path graphs the least robust trees.

The fourth reason is the analogy with random walks; the smaller the effective resistance between vertices a and b, the smaller the expected duration of a random walk from a to b and back (see Theorem 2.8). Short random walks suffer little from vertex or edge failures, and thus indicate a robust network.

In addition, the random walk analogy shows that the robustness measure defined in [11] is equal to two times the effective graph resistance (Theorem 2.11). Since both measures have been proposed independently and by different reasonings, it gives a strong indication that the effective graph resistance is indeed a useful robustness measure.

3. Optimising the effective graph resistance

3.1. Optimal graphs for fixed number of vertices and diameter: clique chains

Sections 3.1–3.5 treat the minimisation of the effective graph resistance for graphs with a given number of vertices and diameter. In this section (Section 3.1) we will first characterise the class of graphs, wherein the optimal graph must lie. In Section 3.2 the effective graph resistance of these graphs is calculated, in Section 3.3 we will compute analytically the optimal graphs for diameter $D \leqslant 3$, while in Section 3.4 we will find the optimal graphs for larger diameters by exhaustive search. Section 3.5 considers the question how many eigenvalues are needed in order to find the same optimal graph as for the effective graph resistance. The topic of the last Section 3.6 is the optimal addition of an edge.

Definition 3.1. The *graph* $G_D^*(n_1, n_2, \ldots, n_{D+1})$ is a graph obtained from the path graph P_{D+1} by replacing the i-th vertex by a clique (subset of vertices which are fully interconnected by edges) of size n_i, such that vertices in distinct cliques are adjacent if and only if the corresponding original vertices in the path graph are adjacent.

In [12] Van Dam has shown that, for fixed number of vertices N and a fixed diameter D, the class of graphs $G_D^*\,(n_1 = 1, n_2, \ldots, n_D, n_{D+1} = 1)$ with $N = \sum_{i=1}^{D+1} n_i$ contains a graph with maximum *spectral radius* (largest eigenvalue of the adjacency matrix). In [15] it has been shown that, for fixed N and D, also graphs with largest algebraic connectivity, maximum number of edges and smallest average distance are obtained within this class. For fixed N and D, the same class contains graphs with maximum vertex or edge connectivity and smallest average vertex or edge betweenness as well. We will show that the same holds for the effective graph resistance.

The following theorem [15] is the key to the proof of the statements above.

Theorem 3.1. *Any graph with N vertices and diameter D is a subgraph of at least one graph in the class $G_D^*\,(n_1 = 1, n_2, \ldots, n_D, n_{D+1} = 1)$ with $N = \sum_{i=1}^{D+1} n_i$.*

Using this theorem and Theorem 2.7 we find the next corollary.

Corollary 3.2. *The minimum effective graph resistance for fixed N and D is equal to the minimum effective graph resistance achieved in the class of the graphs* $G_D^*\left(n_1=1, n_2, \ldots, n_D, n_{D+1}=1\right)$ *with* $N=\sum_{i=1}^{D+1} n_i$.

3.2. The effective graph resistance of a clique chain

Theorem 3.3. *The characteristic polynomial of the Laplacian* $Q_{G_D^*}$ *of* $G_D^*(n_1, n_2, \ldots, n_{D+1})$ *equals*

$$\det\left(Q_{G_D^*}-\mu I\right)=p_D(\mu) \prod_{j=1}^{D+1}\left(d_j+1-\mu\right)^{n_j-1}, \tag{10}$$

where $d_j=n_{j-1}+n_j+n_{j+1}-1$ *denotes the degree of a vertex in clique j. The polynomial* $p_D(\mu)=\prod_{j=1}^{D+1} \theta_j$ *is of degree* $D+1$ *in* μ *and the function* $\theta_j=\theta_j(D;\mu)$ *obeys the recursion*

$$\theta_j=\left(d_j+1-\mu\right)-\left(\frac{n_{j-1}}{\theta_{j-1}}+1\right) n_j, \tag{11}$$

with initial condition $\theta_0=1$ *and with the convention that* $n_0=n_{D+2}=0$.

Applying Theorem 3.3, which is proven in [13], to the effective graph resistance (Theorem 2.2) of $G_D^*(n_1, n_2, \ldots, n_{D+1})$ yields

$$R_{G_D^*}=N \sum_{k=1}^{D} \frac{1}{z_k}+N \sum_{j=1}^{D+1} \frac{n_j-1}{d_j+1}, \tag{12}$$

where $\{z_k\}_{1 \leqslant k \leqslant D+1}$ with $z_{D+1}=0$ are the zeroes of the non-trivial polynomial

$$p_D(\mu)=\mu \sum_{k=0}^{D} c_{k+1}(D) \mu^k .$$

We invoke Newton's relation [14], valid for any polynomial $p(x)=\sum_{k=0}^n a_k x^k=a_n \prod_{k=1}^{n}\left(x-z_k\right)$,

$$\sum_{k=1}^n \frac{1}{z_k}=-\frac{a_1}{a_0}$$

to the polynomial $\frac{p_D(\mu)}{\mu}=\sum_{k=0}^D c_{k+1}(D) \mu^k$, because all the coefficients $c_k(D)$ of $\frac{p_D(\mu)}{\mu}$ are explicitly computed in [13]. From this we find that

$$\sum_{k=1}^D \frac{1}{z_k}=-\frac{c_2(D)}{c_1(D)}=\frac{1}{N} \sum_{q=2}^{D+1} \frac{\left(N-\sum_{k=1}^{q-1} n_k\right)}{n_{q-1} n_q} \sum_{k=1}^{q-1} n_k .$$

Substituted into (12) this leads to the explicit expression of the effective graph resistance of $G_D^*(n_1, n_2, \ldots, n_{D+1})$

$$R_{G_D^*}=\sum_{q=2}^{D+1} \frac{\left(N-\sum_{k=1}^{q-1} n_k\right)}{n_{q-1} n_q} \sum_{k=1}^{q-1} n_k+N \sum_{j=1}^{D+1} \frac{n_j-1}{n_{j-1}+n_j+n_{j+1}} \tag{13}$$

subject to $N=\sum_{m=1}^{D+1} n_m$. For example, the extreme case of the path graph P_n belongs to the class $G_D^*(n_1, n_2, \ldots, n_{D+1})$ with $N=D+1$ and all $n_k=1$. We can verify that (13) for the line topology reduces to $R_{P_N}=\frac{1}{6}(N-1)N(N+1)$, which was found earlier in Section 2.5.

3.3. The minimum effective graph resistance for diameter $D = 2, 3$

Before we start the $D = 2, 3$ cases, it should be mentioned that the graph $G^*_{D=1}(n_1, n_2)$ with N vertices and diameter 1 is unique: the complete graph. The effective graph resistance of a complete graph was already computed in Section 2.5 as $R_{K_N} = N - 1$.

Theorem 3.4. *For graphs with N vertices and diameter $D = 2$, the graph $G^*_{D=2}(1, N-2, 1)$ has the minimum effective graph resistance $R_{G^*_{D=2}(1,N-2,1)} = N - 1 + \frac{2}{N-2}$.*

Proof. The theorem follows directly from (13) for $n_1 = n_3 = 1, n_2 = N - 2$ and $D = 2$. $\square$

It is interesting to remark that $R_{G^*_{D=2}(1,N-2,1)} = R_{K_N} + \frac{2}{N-2}$.

Theorem 3.5. *For graphs with N vertices and diameter $D = 3$, the graph*

$$G^*_{D=3}\left(1, \left\lfloor \frac{N}{2} - 1 \right\rfloor, \left\lceil \frac{N}{2} - 1 \right\rceil, 1\right)$$

has the minimum effective graph resistance

$$R_{G^*_{D=3}\left(1,\left\lfloor \frac{N}{2}-1\right\rfloor,\left\lceil \frac{N}{2}-1\right\rceil,1\right)} = \frac{N-1}{\left\lfloor \frac{N}{2} - 1 \right\rfloor} + \frac{\left(\left\lceil \frac{N}{2} - 1 \right\rceil + 1\right)\left(\left\lfloor \frac{N}{2} - 1 \right\rfloor + 1\right)}{\left\lfloor \frac{N}{2} - 1 \right\rfloor \left\lceil \frac{N}{2} - 1 \right\rceil} + \frac{N-1}{\left\lceil \frac{N}{2} - 1 \right\rceil} + \frac{N(N-4)}{N-1}.$$

Proof. It follows from Corollary 3.2 that the graph with minimal effective graph resistance for $D = 3$ has the form $G^*_{D=3}(1, m, N - m - 2, 1)$. It can be assumed that $m \leqslant \left\lfloor \frac{N}{2} - 1 \right\rfloor$, because the case $m \geqslant \left\lfloor \frac{N}{2} - 1 \right\rfloor$ can be reduced to $m \leqslant \left\lfloor \frac{N}{2} - 1 \right\rfloor$, by swapping the order of the four cliques.

It follows from (13) that

$$R_{G^*_{D=3}(1,m,N-m-2,1)} = \frac{N-1}{m} + \frac{(N-1-m)(m+1)}{m(N-m-2)} + \frac{N-1}{N-m-2} + \frac{N(N-4)}{N-1} \equiv f(m, N). \tag{14}$$

A straightforward calculation reveals that

$$\frac{\partial f(m, N)}{\partial m} = -\frac{(N-1)^2(N-2-2m)}{m^2(N-m-2)^2}.$$

Hence, for m in the interval $[0, N - 2]$, $f(m, N)$ has a global optimum at $m = \frac{N}{2} - 1$. In addition, from $\lim_{m\downarrow 0} mf(m, N) = \frac{(N-1)^2}{N-2}$ it follows that $m = \frac{N}{2} - 1$ is a global minimum for m in the interval $[0, N - 2]$. Therefore, $G^*_{D=3}\left(1, \left\lfloor \frac{N}{2} - 1 \right\rfloor, \left\lceil \frac{N}{2} - 1 \right\rceil, 1\right)$ has the minimum effective graph resistance for all graphs with $D = 3$. The minimal value $R_{G^*_{D=3}\left(1,\left\lfloor \frac{N}{2}-1\right\rfloor,\left\lceil \frac{N}{2}-1\right\rceil,1\right)}$ of the effective graph resistance is obtained by substitution of $m = \left\lfloor \frac{N}{2} - 1 \right\rfloor$ in (14). $\square$

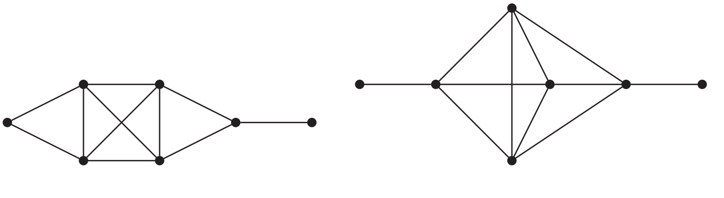
3.4. Exhaustive search

The clique sizes of the optimal graphs for some values of N and D can be found in Table 1. The same results for the algebraic connectivity are listed in [15].

For the algebraic connectivity and the effective graph resistance there exist different optimal graphs. In Fig. 3 an example is given. For $N = 7$ and $D = 4$ the graph with cliques of sizes $(1, 2, 2, 1, 1)$ minimises the effective graph resistance, while the graph with clique sizes $(1, 1, 3, 1, 1)$ maximises the algebraic connectivity.

Table 1
Graphs that minimise the effective graph resistance for given N, D.

	R	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$N = 26$									
$D = 2$	25.08	1	24	1					
$D = 3$	28.22	1	12	12	1				
$D = 4$	37.63	1	6	12	6	1			
$D = 5$	51.90	1	4	8	8	4	1		
$D = 6$	70.28	1	3	6	6	6	3	1	
$D = 7$	93.35	1	3	4	5	5	4	3	1
$N = 50$									
$D = 2$	49.04	1	48	1					
$D = 3$	52.11	1	24	24	1				
$D = 4$	64.03	1	9	29	10	1			
$D = 5$	84.31	1	6	18	18	6	1		
$D = 6$	110.01	1	5	11	15	12	5	1	
$D = 7$	139.36	1	4	9	11	11	9	4	1
$N = 100$									
$D = 2$	99.02	1	98	1					
$D = 3$	102.05	1	49	49	1				
$D = 4$	117.51	1	16	67	15	1			
$D = 5$	148.11	1	8	41	41	8	1		
$D = 6$	189.44	1	6	22	41	23	6	1	
$D = 7$	237.13	1	6	17	28	27	15	5	1
$N = 122$									
$D = 2$	121.01	1	120	1					
$D = 3$	124.04	1	60	60	1				
$D = 4$	140.68	1	18	84	18	1			
$D = 5$	175.11	1	9	51	51	9	1		
$D = 6$	222.84	1	7	27	51	28	7	1	
$D = 7$	278.35	1	6	19	35	35	19	6	1



(a) Graph with $\mu_2 \approx 0.68$, $R = 19.8$ (b) Graph with $\mu_2 \approx 0.70$, $R = 21$

Fig. 3. For $N = 7$ and $D = 4$, μ_2 and R give different optimal graphs.

The optimum for both the algebraic connectivity and the effective graph resistance is generally achieved for symmetric graphs. Surprisingly there are a few counterexamples. Regarding the effective graph resistance, for $N = 100$, $D = 7$ we found the optimal graph with clique sizes $(1, 6, 17, 28, 27, 15, 5, 1)$. While optimising the algebraic connectivity for $N = 122$ and $D = 7$ we found that the graph with clique sizes $(1, 11, 20, 29, 28, 21, 11, 1)$ is optimal. It would be interesting to find an explanation for this phenomenon.

The optimisation has shown that the clique sizes of the optimal graphs for both measures are larger for cliques closer to the middle. However for the algebraic connectivity there is an example that does not have this structure; the graph with clique sizes $(1, 2, 3, 5, 4, 5, 3, 2, 1)$ is optimal for $N = 26$ and $D = 8$.

3.5. Optimal graphs based on fewer Laplacian eigenvalues

In Corollary 2.3 we have shown that the algebraic connectivity provides bounds for effective graph resistance. In this section we will try to answer the question how many Laplacian eigenvalues are needed in order to find the same optimal graph as for the effective graph resistance.

The sum

$$N\sum_{i=2}^{k}\frac{1}{\mu_i}$$

with $k < N$ is a lower bound for the effective graph resistance that considers $k-1$ non-zero Laplacian eigenvalues instead of all $N-1$ of them. We have optimised this value within the class of clique graphs to find out how many eigenvalues are needed in order to find the same optimal graph as for the effective graph resistance. For the results see Table 2.

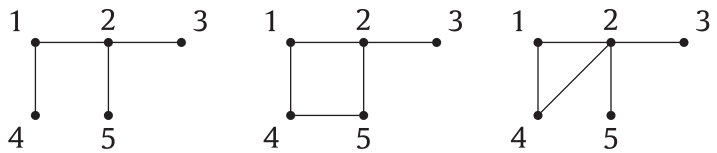
In general, for increasing k, the optimal graphs have an increasing number of vertices in the cliques in the middle, but a few surprising counterexamples have been found. For example, for $N = 26$, and $D = 4$ (Table 3) we have that for $k = 2$ (which corresponds to minimising N/μ_2) the graph with cliques sizes $(1, 7, 10, 7, 1)$ is optimal. For $k = 3, 4, 5, 6$ the graph with clique sizes $(1, 8, 8, 8, 1)$ is optimal. The graph with clique sizes $(1, 7, 10, 7, 1)$ is again optimal for $k = 7, \ldots, 13$.

Table 2
Minimal value k such that the graph that minimises $N\sum_{i=2}^{k}\frac{1}{\mu_i}$ also minimises R and $N\sum_{i=2}^{j}\frac{1}{\mu_i}$ for all $k < j < N$.

	$N = 26$	$N = 50$	$N = 100$	$N = 122$
$D = 2$	$k = 2$	$k = 2$	$k = 2$	$k = 2$
$D = 3$	$k = 2$	$k = 2$	$k = 2$	$k = 2$
$D = 4$	$k = 15$	$k = 23$	$k = 36$	$k = 42$
$D = 5$	$k = 14$	$k = 32$	$k = 93$	$k = 107$

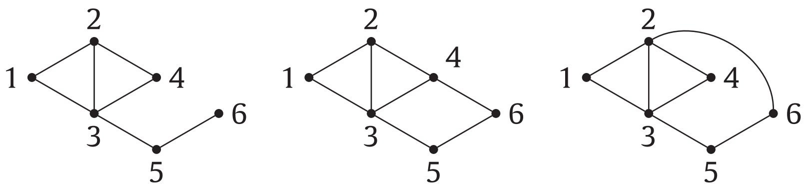
Table 3
Graphs that minimise $N\sum_{i=2}^{k}\frac{1}{\mu_i}$ for $N = 26, D = 4$.

$N = 26, D = 4$	$N\sum_{i=2}^{k}\frac{1}{\mu_i}$	n_1	n_2	n_3	n_4	n_5
$k = 2$	4.57	1	7	10	7	1
$k = 3$	7.68	1	8	8	8	1
$k = 4$	9.97	1	8	8	8	1
$k = 5$	11.50	1	8	8	8	1
$k = 6$	13.03	1	8	8	8	1
$k = 7$	14.53	1	7	10	7	1
$k = 8$	15.98	1	7	10	7	1
$k = 9$	17.42	1	7	10	7	1
$k = 10$	18.87	1	7	10	7	1
$k = 11$	20.31	1	7	10	7	1
$k = 12$	21.76	1	7	10	7	1
$k = 13$	23.20	1	7	10	7	1
$k = 14$	24.63	1	7	11	6	1
$k = 15$	25.74	1	6	12	6	1
$k = 16$	26.82	1	6	12	6	1
$k = 17$	27.91	1	6	12	6	1
$k = 18$	28.99	1	6	12	6	1
$k = 19$	30.08	1	6	12	6	1
$k = 20$	31.16	1	6	12	6	1
$k = 21$	32.24	1	6	12	6	1
$k = 22$	33.32	1	6	12	6	1
$k = 23$	34.41	1	6	12	6	1
$k = 24$	35.49	1	6	12	6	1
$k = 25$	36.57	1	6	12	6	1
$k = 26$	37.63	1	6	12	6	1



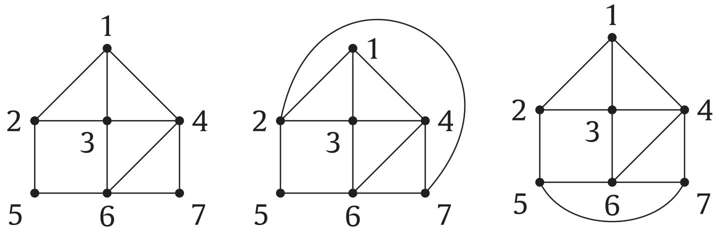
(a) $R = 18$ and $\mu_2 \approx 0.52$ (b) $R = 11\frac{1}{2}$ and $\mu_2 \approx 0.83$ (c) $R = 12\frac{2}{3}$ and $\mu_2 = 1$

Fig. 4. Adding the edge $(4, 5)$ is optimal for R, adding $(2, 4)$ is optimal for μ_2.



(a) Graph with $R = 20.5$ (b) Graph with $R \approx 12.6$ (c) Graph with $R \approx 12.4$

Fig. 5. Adding the edge $(4, 6)$ is not optimal, although R_{46} gives the maximum pairwise effective resistance.



(a) Graph with $R \approx 14.9$ (b) Graph with $R \approx 12.5$ (c) Graph with $R \approx 12.4$

Fig. 6. Adding the edge $(2, 7)$ is not optimal, although d_{27} gives the maximum distance.

3.6. Expanding the graph with an edge

For application purposes it is interesting to know which edge has to be added in order to optimise the effective graph resistance. The example of Fig. 4 demonstrates that the edge that decreases the effective graph resistance most, may not cause the largest increase in the algebraic connectivity.

A first hypothesis is that it is optimal to add the edge (i, j) for which R_{ij} is maximal. Unfortunately, the graph in Fig. 5a shows that this is not always the case. The corresponding matrix of effective resistances is:

$$(R_{ij}) = \begin{pmatrix} 0 & \frac{5}{8} & \frac{5}{8} & 1 & \frac{13}{8} & \frac{21}{8} \\ \frac{5}{8} & 0 & \frac{1}{2} & \frac{5}{8} & \frac{3}{2} & \frac{5}{2} \\ \frac{5}{8} & \frac{1}{2} & 0 & \frac{5}{8} & 1 & 2 \\ 1 & \frac{5}{8} & \frac{5}{8} & 0 & \frac{13}{8} & \frac{21}{8} \\ \frac{13}{8} & \frac{3}{2} & 1 & \frac{13}{8} & 0 & 1 \\ \frac{21}{8} & \frac{5}{2} & 2 & \frac{21}{8} & 1 & 0 \end{pmatrix}.$$

We see that the pairs $(1, 6)$ and $(4, 6)$ have the largest effective resistance. Nevertheless, edge $(2, 6)$ is the best edge to add.

In this counterexample the best edge to add is not the one with maximum pairwise effective resistance, but the one between vertices that lay furthest apart. However, it is not true that the edge (i, j) for which the distance d_{ij} is maximal always is the best edge to add, as the graph in Fig. 6a is again a counterexample. The distance matrix corresponding to the graph in Fig. 6a is:

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

Although the distance is maximal between vertices 2 and 7, it is optimal to add edge $(5, 7)$. The question which edge to add in order to minimise the effective graph resistance, is still open.

4. Discussion

In this paper, we have given a survey on the effective graph resistance, a graph measure in our opinion highly valuable in the analysis of various network problems. The results of this paper concern: (1) an overview of the known formulae of the effective graph resistance in Section 2.2, (2) a derivation of some new theorems based on the analysis of the associated random walk on the graph in Section 2.3 and Section 2.4, (3) the proposal of a possible application related to network robustness in Section 2.6, and (4) a set of results concerning the optimisation of the effective graph resistance for a given number of vertices and diameter, and for optimal edge addition in Section 3.

Specifically for (2), the analysis of the random walk analogy in Sections 2.3 and 2.4 has resulted in some new expressions (Eqs. (7) and (8)), and a new proof for the equivalence with network criticality (Theorem 2.11). Furthermore, in Section 2.4 we have a new approximation formula for the effective graph resistance. Then specifically for (3), we have argued that the effective graph resistance can be effectively used in the analysis of robustness-related problems. Finally concerning (4), we have found some interesting results by optimising the effective graph resistance for graphs with a given number of vertices and diameter, both analytically and by exhaustive search. These specific results include the identification of asymmetric optimal graphs, the characterisation of a class of graphs containing the optimal graph and the explicit calculation of the effective graph resistance for members of this class.

Further research on the effective graph resistance may include: (1) determination of the computation time of the approximation scheme presented in Section 2.4 and comparison with other algorithms used for computing the effective graph resistance, (2) further comparison of the effective graph resistance with other graph measures, in order to point out eventual dependencies, (3) analytical computation of the optimal graphs of Section 3.4 or at least finding an explanation for the presence of asymmetric optimal graphs, (4) search for real-world graphs that have the structure of the optimal graphs in Section 3.4, and (5) design of an algorithm for determining the edge that decreases the effective graphs resistance most, without having to try all possible edges.

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