High-order fractional linear differential equations do not describe stochastic processes with finite state space

P. Van Mieghem*

Delft University of Technology

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Abstract

The replacement of the classical derivative in the governing equation of a continuous-time Markov process with finite state space by a fractional derivative of arbitrary order $\alpha > 1$ produces solutions that are mathematically incompatible with the theory of stochastic processes.

1 Introduction

We focus on the extension of continuous-time Markov processes with finite state space to cover a more general dependence of a stochastic process over time. The essence of a continuous-time Markov process $X(t)$ with time $t \geq 0$ lies in the semi-group property $P(t+u) = P(t)P(u) = P(u)P(t)$ of the transition probability matrix $P(t)$, with elements $P_{jk}(t) = \Pr[X(t) = j | X(0) = k]$. The semi-group property results at any state in exponential transition times, that feature the memoryless property, which induces the one-step dependence of the event $\{X(t) = k\}$ on $\{X(u) = j\}$ at the previous time $u < t$ at which the Markov process changes. Real processes do not always possess the Markov properties and motivate the search for a more realistic modeling.

Fractional calculus (see e.g. [1], [2], [3], [4], [5]) may provide a particular extension of continuous-time Markov processes with the main attractive property of an exact, analytic solution [6]. The fractional derivative defined in Section 2 is, however, far more complicated than the classical derivative. While the properties of a fractional derivative are mathematically studied and understood, its repercussions and its meaning to physical phenomena, in particular stochastic processes, is far less understood.

Here, we highlight a serious limitation for stochastic processes: the extension of the governing equation (6) of a continuous-time Markovian process with N states to a fractional setting is shown to be mathematically consistent only for fractional order $0 < \alpha \leq 1$. We demonstrate, generally, that the exact solution (12) of the general fractional order, linear differential equation (7) for $m = \lceil \alpha \rceil > 1$ cannot describe a stochastic process with a finite state space. Previously in the literature (see e.g. [7],

*Faculty of Electrical Engineering, Mathematics and Computer Science, P.O Box 5031, 2600 GA Delft, The Netherlands; *email*: P.F.A.VanMieghem@tudelft.nl

[8], [9], [10], [11]), the confinement of the fractional order α to the interval $(0,1]$ was made based upon the complete monotonicity [6, Sec. 5.1] of the Mittag-Leffler function $E_{a,b}(z)$ for $0 < a \leq 1$ and $b = 1$. However, also for values of a outside the interval $(0,1]$, some combinations of Mittag-Leffler functions are completely monotonous [6, art. 91]. Moreover, we deduce in [6, art. 31] that the set $\{E_{a,b-j}\}_{j\geq 0}$ is independent, implying that a function $f(z)$ can be written as a linear combination $\sum_{j=0}^{\infty} c_j E_{a,b-j}(z)$ with the coefficients c_j, depending on the function $f(z)$. Hence, real non-negative functions such as probability density functions can be expressed as a linear combination of Mittag-Leffler functions, while some of those individual Mittag-Leffler functions might violate complete monotonicity. In summary, complete monotonicity alone is insufficient to dismiss the possibility that fractional linear differential equations of order $\alpha > 1$ cannot describe stochastic processes.

Merely for readability and to motivate of the aim of this paper, we review earlier established theory, such as the complex fractional derivative in Section 2 and the fractional extension (7) of the Markov governing equation (6) in Section 3. Theorem 1 in Section 4.2 gives the general solution of (7) and Section 5.2 formalizes the equation structure for stochastic processes on a finite state space. Finally, our main result in Section 6 illustrates a matrix structure incompatibility between a general-order fractional, linear process and a continuous-time stochastic process on a finite state space. We conclude in Section 7 that the mere replacement of the classical derivative by a fractional derivative needs a reconsideration.

2 Fractional derivative of complex order

The k-th order derivative of a function $f(z)$ can be expressed by Cauchy's integral [12]

$$\left.\frac{d^k f(z)}{dz^k}\right|_{z=z_0} = \frac{\Gamma(1+k)}{2\pi i}\int_{C(z_0)} \frac{f(\omega)\,d\omega}{(\omega - z_0)^{k+1}} \tag{1}$$

where $C(z_0)$ is a contour around the point z_0 in a region of the complex plane where the function $f(z)$ is analytic. In [13], we have proposed and studied the *fractional derivative of complex order* α, defined by the *formal* extension of the scope of the integer number k in (1) to the complex number α, as

$$D^{\alpha} f(z_0) \equiv \left.\frac{d^{\alpha} f(z)}{dz^{\alpha}}\right|_{z=z_0} = \frac{\Gamma(1+\alpha)}{2\pi i}\int_{C(z_0)} \frac{f(\omega)\,d\omega}{(\omega - z_0)^{\alpha+1}} \tag{2}$$

The fundamental problem in (2) with the extension of the integer k to the complex α is generated by the branch cut of the function $z^{\alpha} = e^{\alpha \log z}$ along the negative real z-axis, causing that the function $\frac{f(\omega)}{(\omega - z_0)^{\alpha+1}}$ is not analytic around the point z_0 and that the contour $C(z_0)$ in (2) cannot be closed over the branch cut. Consequently, the formal extension (2) is ill defined and its reparations, as shown in [13], lead to infinitely many possible definitions of the fractional derivative (see also [4, Appendix E]). A general form for real t, that already appeared in the early work of Riemann (see [6, p. 6]), is

$$D^{\alpha}_{m,p} f(t) = \left.\frac{d^{\alpha} f(z)}{dz^{\alpha}}\right|_{z=t} = \frac{1}{\Gamma(m-\alpha)}\int_p^t \frac{f^{(m)}(x)}{(t-x)^{\alpha+1-m}} dx \quad \text{for } \operatorname{Re}(\alpha) < m \tag{3}$$

where the real number $p < t$ and the m-th classical derivative $f^{(m)}(x)$ is assumed to be finite on the interval $x \in [0,t]$. The substitution $u = t - x$ in (3),

$$\left.\frac{d^{\alpha} f(z)}{dz^{\alpha}}\right|_{z=t} = \frac{1}{\Gamma(m-\alpha)}\int_0^{t-p} u^{m-\alpha-1} f^{(m)}(t-u) du \quad \text{for } \operatorname{Re}(\alpha) < m \tag{4}$$

further motivates to choose the point $p = 0$. The simplest definition of the fractional derivative is the case of $m = 1$, real α and $p = 0$ in (3),

$$D_0^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^{\alpha}} dx \quad \text{for } 0 < \alpha < 1 \tag{5}$$

which is known as the case $m = 1$ of the Caputo fractional derivative [4, Chapter E.2].

3 Fractional extension of continuous-time Markov processes

We revisit the generalization of continuous-time Markov processes [14, Chapter 10] to a "fractional setting". The governing equation of a continuous-time Markovian process with N states is

$$\frac{ds(t)}{dt} = -Qs(t) \tag{6}$$

where $-Q$ is the infinitesimal generator in [14, (10.13) after transpose], equal to minus the $N \times N$ weighted Laplacian matrix Q of the underlying Markov graph, in which a node defines a state in the Markov process. The $N \times 1$ probability state vector[1] is

$$s(t) = (\Pr[X(t) = 1], \Pr[X(t) = 2], \ldots, \Pr[X(t) = N])$$

and $s_j(t) = \Pr[X(t) = j]$ denotes the probability that the Markov process $X(t)$ is in state j at time t. The probability operator $\Pr[.]$ on event X measures the event X by a non-negative, real number in the interval $[0, 1]$ so that $0 \leq s_j(t) \leq 1$. Since the Markov process $X(t)$ at time t must be in one of the N possible states, axiom 1 in probability theory [14, p. 8] states that $u^T s(t) = \sum_{j=1}^{N} \Pr[X(t) = j] = 1$, where the unity vector $u = (1, 1, \ldots, 1)$ is the all-one vector. Hence, it follows from the governing equation (6) after left-multiplication by u^T that $u^T Q = 0$, which is the characteristic property of the weighted, generally asymmetric Laplacian Q. Indeed, the unity or all-one vector u is the left-eigenvector of any Laplacian matrix Q belonging to its zero eigenvalue.

We confine the analysis to an infinitesimal generator Q that is fixed and does not depend upon time t. In other words, the Markov "master" equation (6) is equivalent to an N-th order linear differential equation with constant, time-independent coefficients, whose general solution

$$s(t) = e^{-Qt} s(0)$$

possesses exponential functions at different eigenvalues of Q as its eigenfunctions (see [6, p. 7-8]).

Instead of replacing the classical derivative in (6) by the Caputo fractional derivative (5) with limited order $0 < \alpha < 1$ as in [13], we formally extend the governing equation (6) with fixed Laplacian Q to the general fractional derivative (3), valid for $\mathrm{Re}(\alpha) < m$,

$$D_{m,0}^{\alpha} s_{\alpha}(t) = -Q s_{\alpha}(t) \tag{7}$$

[1] In contrast to Markov theory [14], where vectors are usually written as $1 \times N$ row vectors, we follow here linear algebra as in [15] and write $N \times 1$ column vectors.

4 Solution of the general fractional linear differential equation (7)

This section is devoted to the analytic solution of the α-fractional stochastic process $s_\alpha(t)$. We first derive in Section 4.1 a solution of (7) under restricted conditions by Laplace transformation. Inspired by that solution, we prove in Section 4.2 the general solution of the general fractional linear differential equation (7).

4.1 Laplace transformation of (7)

Taking the Laplace transform of both sides in (7) and using (28) yields, for $\mathrm{Re}(\alpha) < m$,

$$Q\mathcal{L}[s_\alpha(t)] = z^{\alpha-1}\sum_{l=0}^{m-1} z^{-l} s_\alpha^{(l)}(0) - z^\alpha \mathcal{L}[s_\alpha(t)] \tag{8}$$

which is reorganized as

$$(Q + z^\alpha I)\,\mathcal{L}[s_\alpha(t)] = \sum_{l=0}^{m-1} z^{\alpha-1-l} s_\alpha^{(l)}(0)$$

Multiplying both sides by the matrix $(Q + z^\alpha I)^{-1}$, which we assume to exist, then results into the Laplace transform of the probability state vector $s_\alpha(t)$,

$$\mathcal{L}[s_\alpha(t)] = \sum_{l=0}^{m-1} z^{\alpha-1-l}\,(Q + z^\alpha I)^{-1}\, s_\alpha^{(l)}(0) \tag{9}$$

For simplicity[2], we first proceed by assuming that the weighted Laplacian Q is diagonalizable and the eigenvalue decomposition [17] is

$$Q = \sum_{k=1}^{N} \mu_k x_k y_k^T = XMY^T$$

where μ_k, with $\mathrm{Re}(\mu_k) \geq 0$, is the eigenvalue belonging to the left-eigenvector x_k and the right-eigenvector y_k of the weighted Laplacian Q and we assume the ordering $|\mu_1| \geq |\mu_2| \geq \cdots \geq \mu_N = 0$. The $N \times N$ matrix X (and Y) contains the left-(right-) eigenvectors in its columns and $M = \mathrm{diag}(\mu)$ is the $N \times N$ diagonal matrix of the eigenvalue vector $\mu = (\mu_1, \mu_2, \ldots, 0)$ with k-th diagonal element $M_{kk} = \mu_k$. Since $\sum_{k=1}^N x_k y_k^T = I$, because the right- and left-eigenvectors form a bi-orthogonal set of vectors [15, art. 237-239] and the corresponding right- and left-eigenvector matrices obey $Y^T X = I$, we have

$$Q + z^\alpha I = \sum_{k=1}^{N} (\mu_k + z^\alpha)\, x_k y_k^T = X\,(M + z^\alpha I)\, Y^T$$

Provided that $\mu_k + z^\alpha \neq 0$, we find

$$(Q + z^\alpha I)^{-1} = \sum_{k=1}^{N} \frac{x_k y_k^T}{\mu_k + z^\alpha} = X\mathrm{diag}\left(\frac{1}{\mu + z^\alpha}\right) Y^T$$

[2] A general way invokes Jordan forms [16], that lead to a considerably more involved derivation without adding new insights. We have chosen to combine a Laplace approach – a most common approach in fractional calculus –, with a Taylor series approach. The latter provides more clearly the restriction $m - 1 < \mathrm{Re}(\alpha) < m$ in Theorem 1.

and the Laplace transform (9) becomes

$$\mathcal{L}\left[s_\alpha(t)\right] = \sum_{k=1}^{N} x_k y_k^T \sum_{l=0}^{m-1} \frac{z^{\alpha-1-l}}{\mu_k + z^\alpha} s_\alpha^{(l)}(0) = X \sum_{l=0}^{m-1} \mathrm{diag}\left(\frac{z^{\alpha-1-l}}{\mu + z^\alpha}\right) Y^T s_\alpha^{(l)}(0) \tag{10}$$

The Laplace transform (see e.g. [6, art. 43],[4]) of the Mittag-Leffler function $E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)}$, which restricts z by $|z^a| > |x|$ for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$,

$$\int_0^\infty e^{-zt} t^{b-1} E_{a,b}(xt^a)\, dt = \frac{z^{a-b}}{z^a - x} \tag{11}$$

indicates that the inverse Laplace transform is

$$\mathcal{L}^{-1}\left[\frac{z^{\alpha-1-l}}{z^\alpha + \mu_k}\right] = t^l E_{\alpha,l+1}(-\mu_k t^\alpha)$$

Inverse Laplace transformation of (10) thus leads to

$$s_\alpha(t) = \sum_{l=0}^{m-1} t^l \left(\sum_{k=1}^{N} E_{\alpha,l+1}(-\mu_k t^\alpha)\, x_k y_k^T\right) s_\alpha^{(l)}(0)$$

Finally, after using $f(Q) = \sum_{k=1}^{N} f(\mu_k)\, x_k y_k^T$ in [15, (A.88)] for any function f defined on the eigenvalues of the matrix Q, we arrive at the solution (12) below of the fractional matrix differential equation (7) for $0 < \mathrm{Re}(\alpha) < m$,which was already given without derivation in [13, Appendix C].

4.2 General solution of (7) for any $N \times N$ matrix Q

The derivation in Section 4.1 has assumed that the matrix Q is diagonalizable. Here, we prove the general solution:

Theorem 1 *The general solution of the fractional, linear differential equation $D_{m,0}^\alpha s_\alpha(t) = -Q s_\alpha(t)$ in (7), where the matrix Q can be any $N \times N$ matrix with constant, time-independent elements and where m is a non-negative integer, is*

$$s_\alpha(t) = \sum_{l=0}^{m-1} t^l E_{\alpha,l+1}(-Qt^\alpha)\, s_\alpha^{(l)}(0) \tag{12}$$

with the restriction that the complex fractional order α must obey $m - 1 < \mathrm{Re}(\alpha) < m$.

Proof: Substituting the suggestion $s_\alpha(t) = \sum_{l=0}^{m-1} t^l E_{\alpha,l+1}(-Qt^\alpha)\, s_\alpha^{(l)}(0)$ in (12) into the matrix differential equation $D_{m,0}^\alpha s_\alpha(t) = -Q s_\alpha(t)$ in (7) with the explicit form of the general fractional derivative (3) with $p = 0$ first needs the normal derivative $s_\alpha^{(m)}(x) = \frac{d^m}{dx^m} s_\alpha(x)$. The Taylor series $E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}$ of the Mittag-Leffler function [6] converges for any complex number z for $\mathrm{Re}(a) > 0$. A function of a matrix is defined by a converging Taylor series [15, Sec. 9.4; art. 232]. The Mittag-Leffler matrix function $E_{a,b}(Q) = \sum_{k=0}^{\infty} \frac{Q^k}{\Gamma(ak+b)}$ for $\mathrm{Re}(a) > 0$ and any complex b converges for any matrix Q.

After introducing the Taylor series in $s_\alpha(t) = \sum_{l=0}^{m-1} t^l E_{\alpha,l+1}(-Qt^\alpha)\, s_\alpha^{(l)}(0)$,

$$s_\alpha(t) = \sum_{l=0}^{m-1} \sum_{k=0}^{\infty} \frac{(-Q)^k\, t^{\alpha k+l}}{\Gamma(\alpha k + l + 1)} s_\alpha^{(l)}(0) \tag{13}$$

the derivative $s_\alpha^{(m)}(x) = \frac{d^m}{dx^m} s_\alpha(x)$ is, with $\frac{d^m x^{\alpha k+l}}{dx^m} = \frac{\Gamma(\alpha k+l)}{\Gamma(\alpha k+l-m+1)} x^{\alpha k+l-m}$, equal to

$$s_\alpha^{(m)}(x) = \sum_{l=0}^{m-1} \sum_{k=0}^{\infty} \frac{(-Q)^k x^{\alpha k+l-m}}{\Gamma(\alpha k+l-m+1)} s_\alpha^{(l)}(0)$$

While $s_\alpha(x)$ is regular for all real $x \geq 0$, we observe that the m-th derivative $s_\alpha^{(m)}(x)$ has a singularity at $x = 0$ if $\alpha k + l - m < 0$ and $\alpha \notin \mathbb{N}$, because for non-integer αk, it holds that $\frac{1}{\Gamma(\alpha k+l-m+1)} \neq 0$. Thus, for $l \leq m-1$, the $k = 0$ term disappears, resulting in

$$s_\alpha^{(m)}(x) = \sum_{l=0}^{m-1} \sum_{k=1}^{\infty} \frac{(-Q)^k x^{\alpha k+l-m}}{\Gamma(\alpha k+l-m+1)} s_\alpha^{(l)}(0) \tag{14}$$

The fractional derivative (3) thus becomes with (14)

$$\begin{aligned} D_{m,0}^\alpha s_\alpha(t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{s_\alpha^{(m)}(x)}{(t-x)^{\alpha+1-m}} dx \\ &= \frac{1}{\Gamma(m-\alpha)} \sum_{l=0}^{m-1} \sum_{k=1}^{\infty} \frac{(-Q)^k}{\Gamma(\alpha k+l-m+1)} \left(\int_0^t \frac{x^{\alpha k+l-m} dx}{(t-x)^{\alpha+1-m}} \right) s_\alpha^{(l)}(0) \end{aligned}$$

The integral

$$\int_0^t \frac{x^{\alpha k+l-m} dx}{(t-x)^{\alpha+1-m}} = t^{-\alpha-1+m} \int_0^t \left(1 - \frac{x}{t}\right)^{-\alpha+m-1} x^{k\alpha+l-m} dx = t^{(k-1)\alpha+l} \int_0^1 (1-u)^{m-\alpha-1} u^{k\alpha+l-m} du$$

equals, provided that $k \operatorname{Re}(\alpha) + l - m + 1 > 0$,

$$\int_0^t \frac{x^{k\alpha-1}}{(t-x)^\alpha} dx = t^{(k-1)\alpha+l} \frac{\Gamma(m-\alpha)\Gamma(k\alpha+l-m+1)}{\Gamma((k-1)\alpha+l+1)}$$

where we have used the Beta-function integral [6, Chapter 17], valid for $\operatorname{Re}(z) > 0$ and $\operatorname{Re}(q) > 0$,

$$B(z,q) = \frac{\Gamma(z)\Gamma(q)}{\Gamma(z+q)} = \int_0^1 u^{z-1} (1-u)^{q-1} du$$

Hence, the first validity condition of the Beta-integral requires that $\operatorname{Re}(\alpha) > \frac{m-1-l}{k}$ for $k \geq 1$, which amounts to the requirement that $m - 1 < \operatorname{Re}(\alpha)$ for all $k \geq 1$, while the second condition of the Beta-integral requires that $\operatorname{Re}(\alpha) < m$. Thus, for $m - 1 < \operatorname{Re}(\alpha) < m$, it holds that

$$\begin{aligned} D_{m,0}^\alpha s_\alpha(t) &= \frac{1}{\Gamma(m-\alpha)} \sum_{l=0}^{m-1} \sum_{k=1}^{\infty} \frac{(-Q)^k}{\Gamma(\alpha k+l-m+1)} \left(t^{(k-1)\alpha+l} \frac{\Gamma(m-\alpha)\Gamma(k\alpha+l-m+1)}{\Gamma((k-1)\alpha+l+1)} \right) s_\alpha^{(l)}(0) \\ &= \sum_{l=0}^{m-1} \sum_{k=1}^{\infty} \frac{(-Q)^k t^{(k-1)\alpha+l}}{\Gamma((k-1)\alpha+l+1)} s_\alpha^{(l)}(0) = -Q \sum_{l=0}^{m-1} \sum_{k=0}^{\infty} \frac{(-Q)^k t^{k\alpha+l}}{\Gamma(k\alpha+l+1)} s_\alpha^{(l)}(0) \end{aligned}$$

Finally with (13), we arrive at $D_{m,0}^\alpha s(t) = -Q s_\alpha(t)$, which proves Theorem 1. □

To some extent, the restriction $m - 1 < \operatorname{Re}(\alpha) < m$ in Theorem 1 coincides with the fact that a higher-order derivative in a physical governing equation describes a different process. For example, the differential equations $\frac{ds}{dt} = -Qs$ and $\frac{d^2 s}{dt^2} = -Qs$ have rather different solutions and appear in different physical processes. A second fact is that the Mittag-Leffler function $E_{a,b}(z)$, an entire function of order $\frac{1}{a}$, behaves seriously different in a and different expressions (see [6, Chapter 4, e.g. art. 69]) also appear in the regimes for $m - 1 < \operatorname{Re}(\alpha) < m$ for integer $m = \lceil \operatorname{Re}(\alpha) \rceil$, where $\lceil x \rceil$ is the smallest integer larger than or equal to the real x.

5 Stochastic processes

5.1 The law of total probability

The law of total probability [18], [14, p. 23], applied to any continuous-time stochastic process $Y(t)$ with a finite number of discrete states $n \in \{1, 2, \ldots, N\}$ and time $t \geq 0$, states that, for any real $v \geq 0$,

$$\Pr[Y(t+v) = j] = \sum_{n=1}^{N} \Pr[Y(t+v) = j | Y(v) = n] \Pr[Y(v) = n] \tag{15}$$

Appendix B adds a proof of the law (15) of total probability.

5.2 Derivatives of the law of total probability

The definition $\left(\mathfrak{s}^{(n)}(t)\right)_j = \frac{d^n}{dt^n} \Pr[Y(t) = j]$ and axiom 1 in probability theory [14, p. 8], $u^T \mathfrak{s}(t) = 1 = \sum_{n=1}^{N} \Pr[Y(v) = n]$ show, after differentiation with respect to t, that any stochastic process obeys

$$u^T \mathfrak{s}^{(n)}(t) = 1_{\{n=0\}} \tag{16}$$

In other words, the vectors $\mathfrak{s}^{(n)}(t)$ for $n > 0$ are orthogonal to the unity vector or all-one vector $u = (1, 1, \ldots 1)$.

Axiom 1 in probability theory "$\sum_{j=1}^{N} \Pr[X_\alpha(t) = j] = u^T s_\alpha(t) = 1$ for all times t", which equals (16) for $n = 0$, means that the process at time t should be in one state in the probability state space. Its application to $s_\alpha(t) = \sum_{l=0}^{m-1} t^l E_{\alpha,l+1}(-Qt^\alpha) s_\alpha^{(l)}(0)$ in (12) is investigated in [19], where we show that the solution (12) satisfies the necessary conditions $u^T s_\alpha^{(l)}(0) = 0$ for $0 < l \leq m-1$. However, we also show, by a counter example, that the necessary conditions are not sufficient for $m = \lceil \alpha \rceil > 1$ to guarantee that $s_\alpha(t)$ is a probability state vector, i.e. that each component $(s_\alpha(t))_n$ for $1 \leq n \leq N$ is a probability that lies in the interval $[0, 1]$. In the sequel, we formalize the observation in [19].

6 Formal comparison between stochastic and fractional process

This Section 6 compares the characteristic properties of both a stochastic process on a finite state space with a general-order fractional, linear process by recasting the generalized state vector, defined in (19), in the corresponding matrix representation (20) and (23). Section 6.3 illustrates that the corresponding matrices $\Upsilon(t, 0)$ and $F(t, 0)$ exhibit a different structure, which causes an incompatibility if a general-order fractional, linear process would describe a stochastic process on a finite state space.

6.1 Matrix equation of stochastic processes

Leibniz' rule applied to the law (15) of total probability, expresses the m-th derivative as

$$\begin{aligned}
\frac{d^m}{dv^m} \Pr[Y(t+v) = j] &= \sum_{n=1}^{N} \frac{d^m}{dv^m} \left(\Pr[Y(t+v) = j | Y(v) = n] \Pr[Y(v) = n] \right) \\
&= \sum_{n=1}^{N} \sum_{l=0}^{m} \binom{m}{l} \frac{d^{m-l}}{dv^{m-l}} \Pr[Y(t+v) = j | Y(v) = n] \frac{d^l}{du^l} \Pr[Y(v) = n]
\end{aligned}$$

With the notation $\left(\mathfrak{s}^{(n)}(t)\right)_j = \frac{d^n}{dt^n}\Pr\left[Y(t)=j\right]$, we obtain the m-th derivative of the law of total probability

$$\left(\mathfrak{s}^{(m)}(t+v)\right)_j = \sum_{n=1}^{N}\sum_{l=0}^{m}\binom{m}{l}\frac{d^{m-l}}{dv^{m-l}}\Pr\left[Y(t+v)=j\,|\,Y(v)=n\right]\left(\mathfrak{s}^{(l)}(v)\right)_n \tag{17}$$

If we define the $N\times N$ matrix $\Phi(t,v;k)$ with elements $\Phi_{jn}(t,v;k)=\frac{d^k}{du^k}\Pr\left[Y(t+v)=j\,|\,Y(v)=n\right]$, then (17) is written in matrix form as

$$\mathfrak{s}^{(m)}(t+v)=\sum_{l=0}^{m}\binom{m}{l}\Phi(t,v;m-l)\,\mathfrak{s}^{(l)}(v) \tag{18}$$

Thus, the m-th derivative $\mathfrak{s}^{(m)}(t+v)$ of the probability state vector $\mathfrak{s}(t+v)$ at time $t+v$ is a linear combination of the vectors $\Phi(t,v;m-l)\,\mathfrak{s}^{(l)}(v)$, illustrating that all the derivatives of the probability state vector $\mathfrak{s}(v)$ at time v are involved. In particular, since $\Phi(t,v;0)=P(t,v)$, the transition probability matrix with elements $\Pr\left[Y(t+v)=j\,|\,Y(v)=n\right]$ and $P(t,0)=P(t)$ corresponds to the transition probability matrix of a stationary continuous-time Markov process (i.e. independent of the time v). Left-multiplying (18) with the vector u^T and invoking (16) shows that

$$u^T\Phi(t,v;k)=\begin{cases}u^T & \text{if } k=0\\ 0 & \text{if } k>0\end{cases}$$

Let us define the $mN\times 1$ vector $\sigma(t)$ consisting of the m-concatenation of the $N\times 1$ vectors $\mathfrak{s}^{(l)}(t)$ for $0\le l<m$, which is the column block vector

$$\sigma(t)=\begin{bmatrix}\mathfrak{s}(t)\\ \mathfrak{s}^{(1)}(t)\\ \vdots\\ \mathfrak{s}^{(m-1)}(t)\end{bmatrix} \tag{19}$$

then the vector relation (18) can be written in matrix form as

$$\sigma(t+v)=\boldsymbol{\Phi}(t,v)\sigma(v) \tag{20}$$

where the $mN\times mN$ square matrix

$$\boldsymbol{\Phi}(t,v)=\begin{bmatrix}\Phi(t,v;0) & O & O & \cdots & O\\ \Phi(t,v;1) & \Phi(t,v;0) & O & \cdots & O\\ \Phi(t,v;2) & \binom{2}{1}\Phi(t,v;1) & \Phi(t,v;0) & \cdots & O\\ \vdots & \vdots & \vdots & \ddots & \\ \Phi(t,v;m-1) & \binom{m-1}{1}\Phi(t,v;m-2) & \binom{m-1}{2}\Phi(t,v;m-3) & \cdots & \Phi(t,v;0)\end{bmatrix}$$

is an lower triangular block matrix with determinant $\det\boldsymbol{\Phi}(t,v)=(\det\Phi(t,v;0))^m$ and whose inverse is also a lower triangular block matrix as shown in [20]. Thus,

$$\sigma(v)=\boldsymbol{\Phi}^{-1}(t,v)\sigma(t+v) \tag{21}$$

If $t=0$ in (20), then $\boldsymbol{\Phi}(0,v)=I$ for any time t. Thus, $\Phi(0,v;0)=I$ and $\Phi(0,v;k)=O$ for $k>0$.

6.2 Matrix equation of fractional processes

Returning to the general-order fractional, linear differential equation (7) with general solution $s_\alpha(t) = \sum_{l=0}^{m-1} t^l E_{\alpha,l+1}\left(-Qt^\alpha\right) s_\alpha^{(l)}(0)$ in (12), the n-th derivative is

$$s_\alpha^{(n)}(t) = \sum_{l=0}^{m-1} \frac{d^n}{dt^n}\left(t^l E_{\alpha,l+1}\left(-Qt^\alpha\right)\right) s_\alpha^{(l)}(0)$$

Invoking the key differential recursion [6, art. 16] of the Mittag-Leffler function, $\frac{d^n}{dz^n}\left(z^{b-1}E_{a,b}\left(xz^a\right)\right) = z^{b-1-n}E_{a,b-n}\left(xz^a\right)$, yields

$$s_\alpha^{(n)}(t) = \sum_{l=0}^{m-1} t^{l-n} E_{\alpha,l-n+1}\left(-Qt^\alpha\right) s_\alpha^{(l)}(0) \tag{22}$$

Employing (22) for $n = m$ leads to the matrix equation

$$\sigma_\alpha(t) = \Upsilon(t,0)\sigma_\alpha(0) \tag{23}$$

where the block matrix

$$\Upsilon(t,0) = \begin{bmatrix} E_{\alpha,1}\left(-Qt^\alpha\right) & tE_{\alpha,2}\left(-Qt^\alpha\right) & t^2E_{\alpha,3}\left(-Qt^\alpha\right) & \cdots & t^{m-1}E_{\alpha,m}\left(-Qt^\alpha\right) \\ t^{-1}E_{\alpha,0}\left(-Qt^\alpha\right) & E_{\alpha,1}\left(-Qt^\alpha\right) & tE_{\alpha,2}\left(-Qt^\alpha\right) & \cdots & t^{m-2}E_{\alpha,m-1}\left(-Qt^\alpha\right) \\ t^{-2}E_{\alpha,-1}\left(-Qt^\alpha\right) & t^{-1}E_{\alpha,0}\left(-Qt^\alpha\right) & E_{\alpha,1}\left(-Qt^\alpha\right) & \cdots & t^{m-3}E_{\alpha,m-2}\left(-Qt^\alpha\right) \\ \vdots & \vdots & \vdots & \ddots & \\ t^{1-m}E_{\alpha,-m}\left(-Qt^\alpha\right) & t^{2-m}E_{\alpha,1-m}\left(-Qt^\alpha\right) & t^{3-m}E_{\alpha,2-m}\left(-Qt^\alpha\right) & \cdots & E_{\alpha,1}\left(-Qt^\alpha\right) \end{bmatrix}$$

is not a lower triangular block matrix as $F(t,0)$ in (20) for $v = 0$.

6.3 Matrix structure comparison between stochastic and fractional process

The general-order fractional, linear differential equation (7) possesses a general matrix solution $\sigma_\alpha(t) = \Upsilon(t,0)\sigma_\alpha(0)$ in (23) in the generalized state vector $\sigma_\alpha(t)$ in (19). If a general-order fractional, linear differential equation (7) describes a stochastic process on a finite state space, then its solution must also obey the general matrix solution $\sigma(t) = F(t,0)\sigma(0)$ in (20). Equating the generalized state vectors $\sigma_\alpha(t) = \sigma(t)$ requires, for all times $t > 0$, that the matrices $\Upsilon(t,0)$ and $F(t,0)$ have the same lower triangular block matrix structure. However, the matrices $\Upsilon(t,0)$ and $F(t,0)$ possess a different structure.

In summary, we must conclude that the general solution (12) does not describe a stochastic process if $m = \lceil \alpha \rceil > 1$.

7 Conclusions

Our analysis indicates that the generalization of the Markovian governing equation $\frac{ds(t)}{dt} = -Qs(t)$ in (6) by only replacing the left-hand side $\frac{ds(t)}{dt}$ by a fractional derivative $D_{m,0}^\alpha s(t)$, culminating in the generalized governing equation (7), leads to a mathematical incompatibility for $\alpha > 1$. Since there is no obvious reason why the fractional order α should be limited to the interval $(0,1]$ and

since measurements indicate that $\alpha > 1$ for an overwhelming amount of respiratory infectious diseases [21], we believe that the formal process of replacing only the left-hand side of the Markovian governing equation (6) with a fractional derivative without modifying the right-hand side, leading to $D^{\alpha}_{m,0} s_{\alpha}(t) = -Q s_{\alpha}(t)$, needs a reconsideration. A mathematical equation acts as a balance and changing only one-side logically destroys the balance. A consistent generalization of the Markovian governing equation (6) may need a modification of both sides of the equation.

Since both equations (6) and (7) possess the same right-hand side $-Qs(t)$, their steady-state vector $s_{\infty} = \lim_{t\to\infty} s(t) = \lim_{t\to\infty} s_{\alpha}(t)$ is the same, independent of the fractional order α and satisfying $Qs_{\infty} = 0$. Thus, if both processes start (at $t = 0$) with the same probability state vector $s(0) = s_{\alpha}(0)$, then they also end (at $t \to \infty$) with the same probability state vector s_{∞}, which poses a physical constraint on the generalization of Markov processes.

Even the case for fractional order $\alpha \in (0, 1]$, where *mathematically* no inconsistencies appear, is *physically* questionable, because the solution $s_{\alpha}(t)$ only seems to be consistent with a semi-Markov process [22]. Thus, the case $\alpha \in (0, 1]$ induces an inconsistency: the fractional derivative (4) with $p = 0$ is a convolution of the process over all times since its start at $t = 0$ and causes the current state at time t to depend on its entire past history. The semi-Markov process, on the other hand, only depends on the previous states with Mittag-Leffler transition times. For epidemics on a contact graph, a semi-Markov process causes two major problems (I) and (II), which contradict physical observations. We quote from [23]:

"(I) all competing infection and curing processes are not independent and (II) when a state transition occurs, all residual times of competing infection and curing processes are neglected, due to the semi-Markov "restart" property. Dependence in (I) implies that the curing process of some node i in the contact graph directly influences the infection and curing processes of every other node in the network [19], [22]. Restart (II) implies that, if a state transition event occurs before node i can cure, then the curing process of node i is reset as if that curing of node i so far has never happened. The beautiful memoryless property of the exponential and fundamental property in Markov processes allows a reset, because the remaining time to cure is the same as the time to cure after reset [14, pp. 42-43, pp. 399-401]. Both effects (I) and (II) are nonphysical in epidemic spread."

8 Statements and Declarations

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A Laplace transform of $\left.\frac{d^\alpha f(z)}{dz^\alpha}\right|_{z=t}$ in (3)

We summarize results from [13]. The single-sided[3] Laplace transform for complex z is defined (see e.g. [24], [25, Chapter VII], [26]) as

$$\varphi(z) = \int_0^\infty e^{-zt} f(t) dt \tag{24}$$

and in operator form $\varphi(z) = \mathcal{L}_z [f(t)]$, simplified to $\varphi(z) = \mathcal{L} [f(t)]$. The inverse Laplace transform, in operator form $f(t) = \mathcal{L}_t^{-1} [\varphi(z)]$, simplified to $f(t) = \mathcal{L}^{-1} [\varphi(z)]$, is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi(z) e^{zt} dz \tag{25}$$

where c is the smallest real value of $\mathrm{Re}(z)$ for which the integral in (24) converges.

The Laplace transform of the derivative[4] $\mathcal{L} [f'(t)] = \int_0^\infty e^{-zt} f'(t) dt = z\mathcal{L} [f(t)] - f(0)$. If we replace f by $f^{(k-1)}$, then we find with $y_k = \mathcal{L} \left[f^{(k)}(t)\right]$ the recursion $y_k = z y_{k-1} - f^{(k-1)}(0)$, which becomes after l iterations

$$y_k = z^l y_{k-l} - \sum_{j=1}^{l} z^{j-1} f^{(l-j)}(0)$$

Since $y_0 = \mathcal{L} [f(t)]$, choosing $l = k$ yields the general form

$$\mathcal{L} \left[f^{(k)}(t)\right] = z^k \mathcal{L} [f(t)] - \sum_{j=1}^{k} z^{j-1} f^{(k-j)}(0)$$

The k-fold integral

$$F_k(t, a) = \int_a^t du_1 \int_a^{u_1} du_2 \ldots \int_a^{u_{k-1}} du_k f(u_k) \tag{26}$$

can be reduced to a single integral, by repeated partial integration, for $k \geq 1$,

$$F_k(t, a) = \frac{1}{(k-1)!} \int_a^t (t-u)^{k-1} f(u) du \tag{27}$$

The expression (27), which possesses the form of (3), was known to Cauchy, but is also contributed to Riemann and Liouville [27, p. 43],[4, Appendix E]. Using the Riemann-Liouville integral in (27) for k-fold integration with $a = 0$ and the important convolution property $\mathcal{L} [(h * g)(t)] = \mathcal{L} \left[\int_0^t h(t-x) g(x) dx\right] = \mathcal{L} [h(t)] \mathcal{L} [g(t)]$ results in

$$\mathcal{L} [F_k(t, 0)] = \mathcal{L} \left[\frac{1}{(k-1)!} \int_0^t (t-u)^{k-1} f(u) du\right] = \frac{\mathcal{L} [f(t)]}{z^k}$$

Invoking the convolution property and $\mathcal{L} \left[t^{-\beta}\right] = \int_0^\infty t^{-\beta} e^{-zt} dt = \frac{\Gamma(1-\beta)}{z^{1-\beta}}$ valid for $\mathrm{Re}(\beta) < 1$, the Laplace transform (24) of the generalization (3) with $p = 0$ is

$$\mathcal{L} \left[\left.\frac{d^\alpha f(z)}{dz^\alpha}\right|_{z=t}\right] = \frac{1}{\Gamma(m-\alpha)} \mathcal{L} \left[t^{-\alpha-1+m}\right] \mathcal{L} \left[f^{(m)}(t)\right]$$

[3] The double-sided Laplace transform extends the lower integration bound of the single-sided Laplace transform over all negative real values of t and is $\varphi(z) = \int_{-\infty}^\infty e^{-zt} f(t) dt$.

[4] We assume that $f(t)$ is analytic at $t = 0$, else $f(0)$ should be replaced by $\lim_{t \downarrow 0} f(t) = f\left(0^+\right)$, because then the direction of the path towards the point $t = 0$ matters and the left-limit $\lim_{t \uparrow 0} f(t) = f\left(0^-\right)$ and right-limit may be different and the function $f(t)$ may not be continuous at $t = 0$.

and

$$\mathcal{L}\left[\left.\frac{d^{\alpha}f(z)}{dz^{\alpha}}\right|_{z=t}\right] = z^{\alpha}\left\{\mathcal{L}\left[f(t)\right] - \sum_{l=0}^{m-1} z^{-l-1} f^{(l)}(0)\right\} \quad \text{for } \operatorname{Re}(\alpha) < m \tag{28}$$

B Proof of the law (15) of total probability for stochastic processes

The definition of conditional probability, $\Pr[A \cap B] = \Pr[A|B]\Pr[B]$, illustrates that (15) can be written as

$$\Pr\left[Y(t+v) = j\right] = \sum_{n=1}^{N} \Pr\left[\{Y(t+v) = j\} \cap \{Y(v) = n\}\right] \tag{29}$$

The case for $t = 0$ is immediate, because $\{Y(v) = j\} \cap \{Y(v) = n\} = \emptyset$ for $j \neq n$ as the process can only be in one state at time v. Now,

$$\Pr\left[\cup_{n=1}^{N}\left\{\{Y(t+v) = j\} \cap \{Y(v) = n\}\right\}\right] = \Pr\left[\{Y(t+v) = j\} \cap \left\{\cup_{n=1}^{N} Y(v) = n\right\}\right]$$

by the distributive property $\{A \cap B_1\} \cup \{A \cap B_2\} = A \cap \{B_1 \cup B_2\}$. Since $\cup_{n=1}^{N}\{Y(v) = n\} = \Omega$ comprises the entire sample space Ω and $A \cap \Omega = A$, we find that

$$\Pr\left[Y(t+v) = j\right] = \Pr\left[\cup_{n=1}^{N}\left\{\{Y(t+v) = j\} \cap \{Y(v) = n\}\right\}\right]$$

Since all events $W_n = \{\{Y(t+v) = j\} \cap \{Y(v) = n\}\}$ are mutually exclusive[5] in that $W_n \cap W_k = \emptyset$ for $n \neq k$ and with relation

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \tag{30}$$

in [14, eq. (2.4)], we arrive at (29) and thus prove (15). □

[5] Associativity shows that

$$\begin{aligned} W_n \cap W_k &= \{\{Y(t+v) = j\} \cap \{Y(v) = n\}\} \cap \{\{Y(t+v) = j\} \cap \{Y(v) = k\}\} \\ &= \{Y(t+v) = j\} \cap \{\{Y(v) = n\} \cap \{Y(v) = k\}\} = \{Y(t+v) = j\} \cap \emptyset = \emptyset \end{aligned}$$