Eigenvectors and Eigenvalues of the Effective Resistance Matrix of a Graph

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Abstract

We solve the problem "Given the spectrum (i.e. all eigenvalues and eigenvectors) of the Laplacian matrix of a graph, find the spectrum of the effective resistance matrix of that graph" and of the reversed problem. We find partial fraction expansions for the eigenvalues of either matrix in terms of the spectrum of the other matrix as well as the explicit characteristic polynomials and some sums of powers of eigenvalues.We also improve the bounds of the Interlacing Theorem for the eigenvalues of the Laplacian and effective resistance matrix of a same graph.

1 Introduction

We consider an undirected, possibly weighted and connected graph G, whose corresponding graph-related matrices are symmetric. The graph G contains a set $\mathcal{N}$ of N nodes and a set $\mathcal{L}$ of L links. As mentioned in my book [12], I believe that, after the adjacency matrix A and Laplacian matrix Q of a graph G, the effective resistance matrix Ω with elements ω_{ij} is the third important matrix associated with graph G. The effective resistance matrix Ω is closely related to the Laplacian matrix by

$$\Omega = \zeta u^T + u\zeta^T - 2Q^\dagger \tag{1}$$

where u is the all-one vector, the vector $\zeta = \left(Q^\dagger_{11}, Q^\dagger_{22}, \ldots, Q^\dagger_{NN}\right)$ and $Q^\dagger$ is the pseudoinverse of the Laplacian [13], [12, Secion 4.2]. In a graph G, two different types [8] of transport are possible that lead to either "path networks", such as telecommunication and transportation networks (for e.g. cars, trains, ships, airplanes), or "flow networks", such as power grids and utility (water, gas, etc.) networks. In a path network, the transport of items follows a single path $\mathcal{P}_{ij}$ between a node pair (i, j), whereas in a flow network, the transport from node i to node j propagates over all possible paths from node i to node j. The effective resistance matrix Ω plays a crucial role in "flow networks", such as electrical resistor networks [12, art. 14].

The effective resistance matrix Ω is a distance matrix [12, art. 8]. The spectrum of distance matrices was overviewed by Aouchiche and Hansen [1]. A fundamental relation in the theory of the

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effective resistance matrix Ω is Fiedler's block matrix [7],[12, (5.17)]

$$\begin{pmatrix} 0 & u^T \\ u & \Omega \end{pmatrix}^{-1} = \begin{pmatrix} -2\sigma^2 & p^T \\ p & -\frac{1}{2}Q \end{pmatrix} \qquad \text{with } \Omega p = 2\sigma^2 u \tag{2}$$

where the $N \times 1$ vector p is defined as

$$p = \frac{1}{2}Q\zeta + \frac{u}{N} \tag{3}$$

obeys $p^T u = 1$, and where[1]

$$\sigma^2 = \frac{R_G}{N^2} + \frac{1}{4}\zeta^T Q\zeta \tag{4}$$

and $R_G = N u^T \zeta = \frac{1}{2}u^T \Omega u = N \operatorname{trace}\left(Q^\dagger\right)$ is the effective graph resistance, an important graph metric [12, Section 5.2]. The effective resistance matrix Ω also possesses interesting geometric properties, recently explored by Devriendt and colleagues [2, 5, 4]. They demonstrate that the vector $p = \frac{1}{2}Q\zeta + \frac{u}{N}$ has several fundamental properties and that $\sigma^2 = \frac{\zeta^T \widetilde{Q}\zeta}{4} + \frac{R_G}{N^2}$ can be interpreted as a variance of a distribution on a graph. In particular [3], the vector p can be interpreted as a discrete scalar curvature on a graph, p is the solution of the optimization problem $\max_x x^T \Omega x$ for the vector x subject to $x^T u = 1$ and that maximum equals $2\sigma^2 = p^T \Omega p$, where

$$\sigma^2 = \frac{1}{4} \sum_{(i,j)\in\mathcal{L}} (\omega_{ik} - \omega_{jk})^2$$

for any node k in the graph G. We also add that the vector p is the barycentric coordinate of the circumcenter of the simplex S of the graph [6] and σ is the radius of the circumcenter of the simplex S. Each component ζ_i of the vector ζ has the property that $\frac{1}{\zeta_i} = \frac{1}{Q^\dagger_{ii}}$ equals the altitude of the vertex i in the simplex S towards the face containing all other vertices[2]. Following the terminology of Devriendt, the vector p is called the "resistance curvature" and σ is "resistance radius" of the circumcircle of the simplex S. In comparison with the Laplacian $Q = \Delta - A$, where Δ is the diagonal matrix with element $\Delta_{ii} = d_i$ equal to the degree of node i and being a component of the degree vector $d = (d_1, d_2, \ldots, d_N)$, we observe that the vector ζ, which we call the "simplex-altitude" vector, is equally important as the degree vector d, which is perhaps the most prominent characterizing vector of any graph. The simplex-altitude vector ζ, whose component ζ_i quantifies the spreading capacity in node i, is studied in [13], where the upper bound for the Euclidean norm $\zeta^T\zeta = \|\zeta\|_2^2$ is deduced

$$\zeta^T \zeta = \sum_{m=1}^{N} \left(Q^\dagger_{mm}\right)^2 \le \left(1 - \frac{1}{N}\right) \sum_{n=1}^{N-1} \frac{1}{\mu_n^2} \tag{5}$$

which is tight in the sense that equality is achieved if $Q^\dagger_{mm} = \frac{R_G}{N^2}$ for any node m in the graph. Such a graph, characterized by $\zeta = \frac{R_G}{N^2}u$, is called "resistance regular" [15].

The eigenvalue equation [12, Section 5.5] of the $N \times N$ non-negative, symmetric, effective resistance matrix Ω is

$$\Omega v_j = \rho_j v_j \tag{6}$$

[1] Unfortunately, there is an error in [12, p. 182], where $\sigma^2 = R_G + \frac{1}{4}\zeta^T Q\zeta$ is written, whereas (4) correct is.

[2] Any undirected graph G, with N nodes and L links, possesses a simplex S in the $(N-1)$-dimensional Euclidian space, consisting of N vertices. Each vertex in the simplex S corresponds with a node in the graph G, but a link in the graph G does *not* correspond to an edge in the simplex S. To avoid confusion, we have proposed in [12, p. 2] to use nodes and links in a graph G, while vertices and edges in the simplex S of that graph G.

where ρ_j is the j-th eigenvalue of Ω belonging to the normalized eigenvector v_j, i.e. $v_j^T v_j = 1$, for $1 \leq j \leq N$. The real eigenvalues are ordered as in [12]: $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_N$. The effective resistance matrix Ω in a connected graph has $N-1$ negative eigenvalues (see e.g. [12, Theorem 34], but also Theorem 1 below) and only the spectral radius $\rho_1 = -\sum_{j=2}^{N} \rho_j > 0$. The corresponding eigenvalue equation for the Laplacian matrix is

$$Qz_k = \mu_k z_k \tag{7}$$

where μ_k is the k-th eigenvalue of Q belonging to the normalized eigenvector z_k. Any Laplacian [12, art. 102] possesses the eigenvector $z_N = \frac{u}{\sqrt{N}}$ belonging to the smallest eigenvalue $\mu_N = 0$.

Here, we explicitly express the eigenvectors $v_1, v_2, \ldots, v_N$ and eigenvalues $\rho_1, \rho_2, \ldots, \rho_N$ of the effective resistance matrix Ω in terms of the eigenvectors $z_1, z_2, \ldots, z_N = \frac{u}{\sqrt{N}}$ and eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N = 0$ of the possibly weighted, but symmetric Laplacian Q, and vice versa.

The article is outlined as follows. Section 2 starts by deriving two "quasi-eigenvalue equations" that are consequences of Fiedler's fundamental block-matrix equation (2). Section 3 introduces the Delft graph metrics as a motivation for the spectral decomposition $\Omega = \sum_{j=1}^{N} \rho_j v_j v_j^T$ of the effective resistance matrix. The remainder of the article concentrates on spectral decomposition and insights deduced from those two quasi-eigenvalue equations (9) and (10). We express the eigenvectors of Ω as a linear combination of the eigenvectors of the Laplacian matrix Q and vice versa in Section 4. In Section 5, we deduce partial fractions of the eigenvalues of Ω in terms of the spectrum (i.e. eigenvalues and eigenvectors) of the Laplacian Q and vice versa, from which the explicit characteristic polynomials are derived. The characteristic polynomial of the Laplacian Q and effective resistance matrix Ω are derived in Section 6 and 7, respectively, and the coefficients of those polynomials are explicitly deduced together with some sums of powers of eigenvalues. Section 8 focusses on the computation of eigenvalues of the effective resistance matrix Ω, given the spectrum of the Laplacian Q, and vice versa. Section 9 gives a general relation of the sum of quadratic forms and the trace of a matrix, with examples to graph metrics such as the effective graph resistance R_G. Section 10 concludes, while Appendix A proves double orthogonality of the row and column vectors of the orthogonal matrix C in (25). Appendix B complements the main Theorem 2 in Section 4 and tries to address – unsuccessfully though – the problem of determining the sign of the components of resistance curvature vector p.

2 Two quasi-eigenvalue equations

We start by deducing two different almost eigenvalues equations (9) and (10). First, we left-multiply the eigenvalue equation (6) by Q,

$$Q\Omega v_j = \rho_j Q v_j$$

Fiedler's block matrix identity (2) leads to

$$Q\Omega = 2pu^T - 2I \tag{8}$$

Using (8)

$$\rho_j Q v_j = 2\left(u^T v_j\right) p - 2v_j$$

results in, what we call, the first quasi-eigenvalue equation

$$Qv_j = -\frac{2}{\rho_j} v_j + \frac{2\left(u^T v_j\right)}{\rho_j} p \tag{9}$$

Second, the eigenvalue equation $\Omega^{-1}v_j = \rho_j^{-1}v_j$ combines with $\Omega^{-1} = \frac{1}{2\sigma^2}p.p^T - \frac{1}{2}Q$, another deduction from Fiedler's block matrix identity (2) as shown in [12, (5.17)], to a slightly different quasi-eigenvalue equation than (9),

$$Qv_j = -\frac{2}{\rho_j}v_j + \frac{p^T v_j}{\sigma^2}p \tag{10}$$

Equating the quasi-eigenvalue equations (9) and (10) indicates that the eigenvalue ρ_j of the effective resistance matrix Ω equals

$$\rho_j = 2\sigma^2 \frac{u^T v_j}{p^T v_j} \tag{11}$$

while [12, (5.27)] deduces

$$\rho_j = \frac{R_G}{N} + N\frac{\zeta^T v_j}{u^T v_j} \tag{12}$$

Clearly, if the last term in (9) or (10) vanishes, then an eigenvalue equation arises. If that last term is not too large, the quasi-eigenvalue equations and the Laplacian eigenvalue equation (7) also suggest that $\mu_j \approx -\frac{2}{\rho_j}$ plus some correction. At this point, we mention the beautiful Theorem 1, proved by Fiedler [7, Corollary 6.2.9] and later by Sun *et al.* [9]:

Theorem 1 *In a connected graph, the eigenvalues $\rho_1, \rho_2, \ldots, \rho_N$ of the effective resistance matrix Ω interlace with those of the Laplacian matrix as*

$$\rho_1 > 0 > -\frac{2}{\mu_1} \geq \rho_2 \geq -\frac{2}{\mu_2} \geq \cdots \geq -\frac{2}{\mu_{N-2}} \geq \rho_{N-1} \geq -\frac{2}{\mu_{N-1}} \geq \rho_N \tag{13}$$

If $\zeta = u\frac{R_G}{N^2}$, thus in a "resistance regular" graph, then [12, Theorem 35] states that $\rho_j = -\frac{2}{\mu_j}$ for $1 < j \leq N$ and the proof indicates that the eigenvector $v_j = z_j$ for all $1 \leq j \leq N$. Inspite of the *approximate* nature of $\mu_j \approx -\frac{2}{\rho_j}$ in Theorem 1, we show in (41) below the *exact* product

$$\prod_{m=1}^{N-1}\mu_m = -\sigma^2 N \prod_{m=1}^{N}\left(-\frac{2}{\rho_m}\right) = \frac{2\sigma^2 N}{\rho_1}\prod_{m=2}^{N}\left(-\frac{2}{\rho_m}\right)$$

We will later improve Theorem 1 and derive an exact governing equation in Corollary 2 for the eigenvalues $\rho_1, \rho_1, \ldots, \rho_N$ of the effective resistance matrix Ω.

3 Delft graph metrics

Delft graph metrics, first introduced in my book [12, Section 8.7.3], are defined as the quotient of quadratic forms for positive integers k,

$$_kD = \frac{u^T A^k \Omega A^k u}{u^T A^{2k} u} \tag{14}$$

where the denominator $N_{2k} = u^T A^{2k} u$ is the total number of walks in the graph G with $2k$ hops or of length $2k$. The Delft graph metrics $_kD$ generalize a few known graph metrics. If $k = 0$, then with $u^T u = N$, the definition (14) leads to the effective graph resistance $R_G = \frac{1}{2}u^T \Omega u$,

$$_0D = \frac{u^T \Omega u}{u^T u} = \frac{2R_G}{N}$$

and if $k = 1$, the definition (14) relates to the Kemeny constant $K_G = \frac{d^T \Omega d}{4L}$ in [14] as

$$ {}_1D = \frac{d^T \Omega d}{d^T d} = \frac{4L}{d^T d} K_G $$

Substituting the spectral decomposition $\Omega = \sum_{j=1}^{N} \rho_j v_j v_j^T$ into the definition (14) of a Delft graph metric and employing Theorem 1 telling that $\rho_1 > 0 > \rho_j$ for $2 \leq j \leq N$ yields[3]

$$ {}_kD = \frac{\sum_{j=1}^{N} \left(v_j^T A^k u \right)^2 \rho_j}{u^T A^{2k} u} = \frac{\left(v_1^T A^k u \right)^2}{u^T A^{2k} u} \rho_1 - \sum_{j=2}^{N} \frac{\left(v_j^T A^k u \right)^2}{u^T A^{2k} u} |\rho_j| \leq \frac{\left(v_1^T A^k u \right)^2}{\sum_{j=1}^{N} \left(v_j^T A^k u \right)^2} \rho_1 $$

Hence, an upper bound for the Kemeny constant K_G in a graph G is found for $k = 1$ as

$$ K_G \leq \frac{\left(v_1^T d \right)^2}{4L} \rho_1 $$

Since all components of v_1 are positive (due to the Perron-Frobenius theorem [12, p. 379] for a non-negative matrix associated with a connected graph) and $0 < (v_1)_k < 1$, it holds that $v_1^T d = \sum_{k=1}^{N} (v_1)_k d_k < \sum_{k=1}^{N} d_k = 2L$. Hence,

$$ K_G < L \rho_1 $$

Anticipating the approximation (55) for the largest eigenvalue ρ_1, we have

$$ K_G \leq L \left(\frac{R_G}{N} + \sqrt{N \zeta^T \zeta} \right) $$

that connects ${}_0D$ and ${}_1D$.

4 Eigenvectors of Ω as linear combination of those of Q

4.1 Explicit expressions

Theorem 2 exactly expresses any eigenvector of the effective resistance matrix Ω in terms of the eigenvectors and eigenvalues of the Laplacian matrix Q and vice versa:

Theorem 2 *In a connected graph G with N nodes, the j-th eigenvector v_j of the effective resistance matrix Ω is written as a linear combination of eigenvectors of the Laplacian matrix Q as*

$$ v_j = \alpha_j \left(\frac{u}{N} + \frac{2}{\rho_j} \sum_{k=1}^{N-1} \frac{\frac{\mu_k}{2} \left(\zeta^T z_k \right)}{\frac{2}{\rho_j} + \mu_k} z_k \right) \tag{15} $$

where

$$ \frac{1}{\alpha_j^2} = \frac{1}{\left(u^T v_j \right)^2} = \frac{1}{N} + \left(\frac{2}{\rho_j} \right)^2 \sum_{k=1}^{N-1} \left(\frac{\frac{\mu_k}{2} \left(\zeta^T z_k \right)}{\frac{2}{\rho_j} + \mu_k} \right)^2 \tag{16} $$

Reversively, the k-th eigenvector z_k of the Laplacian matrix is written, for $1 \leq k < N$, as a linear combination of eigenvectors of eigenvectors of the effective resistance matrix Ω as

$$ z_k = \beta_k \sum_{j=1}^{N} \frac{\frac{1}{\rho_j} \left(u^T v_j \right)}{\frac{2}{\rho_j} + \mu_k} v_j \tag{17} $$

[3] Unfortunately, there is an error in the index in the last formula at [12, p. 292].

where

$$\frac{1}{\beta_k^2} = \frac{1}{(\mu_k \zeta^T z_k)^2} = \sum_{j=1}^{N} \left(\frac{\frac{1}{\rho_j}\left(u^T v_j\right)}{\frac{2}{\rho_j} + \mu_k} \right)^2 \tag{18}$$

Before giving two proofs, one proof derived from the first quasi-eigenvalue equation (9) and the other from the second (10), we compute the scalar product of the resistance curvature vector p and the normalized eigenvector z_k of the Laplacian Q. From the definition (3), it follows that

$$z_k^T p = \frac{1}{2} z_k^T Q \zeta + z_k^T \frac{u}{N} = \frac{\mu_k}{2} z_k^T \zeta + z_k^T \frac{u}{N}$$

Thus, if $k = N$, then $\mu_N = 0$ and $z_N = \frac{u}{\sqrt{N}}$ resulting in

$$z_N^T p = \frac{1}{\sqrt{N}} \tag{19}$$

else, for $1 \leq k < N$,

$$z_k^T p = \frac{\mu_k}{2} z_k^T \zeta \tag{20}$$

We emphasize that connected graphs [12, Theorem 20, art. 115] are assumed for which the algebraic connectivity $\mu_{N-1} > 0$.

Proof 1: Left-multiplying (9) by z_k^T leads to

$$z_k^T Q v_j = -\frac{2}{\rho_j} z_k^T v_j + \frac{2\left(u^T v_j\right)}{\rho_j} z_k^T p$$

With the eigenvalue equation $Qz_j = \mu_j z_j$ in (7) of the symmetric, possibly weighted Laplacian Q,

$$\left(\mu_k + \frac{2}{\rho_j}\right) z_k^T v_j = \frac{2\left(u^T v_j\right)}{\rho_j} z_k^T p$$

leads to the scalar product between eigenvectors of Ω and Q

$$z_k^T v_j = \frac{\frac{2}{\rho_j}}{\frac{2}{\rho_j} + \mu_k} \left(u^T v_j\right)\left(p^T z_k\right)$$

Invoking the scalar product $p^T z_k$ in (19) and (20) then leads to

$$z_k^T v_j = \begin{cases} \frac{\frac{2}{\rho_j}\frac{\mu_k}{2}}{\frac{2}{\rho_j}+\mu_k}\left(u^T v_j\right)\left(\zeta^T z_k\right) & \text{for } 1 \leq k < N \\ \frac{1}{\sqrt{N}}\left(u^T v_j\right) & \text{for } k = N \end{cases} \tag{21}$$

We can now compute $v_j = \sum_{k=1}^{N} \left(z_k^T v_j\right) z_k = \sum_{k=1}^{N-1} \left(z_k^T v_j\right) z_k + \left(z_N^T v_j\right) z_N$,

$$v_j = \left(u^T v_j\right) \left(\frac{u}{N} + \frac{2}{\rho_j} \sum_{k=1}^{N-1} \frac{\frac{\mu_k}{2}}{\frac{2}{\rho_j} + \mu_k} \left(\zeta^T z_k\right) z_k \right)$$

Normalization of the eigenvector, $v_j^T v_j = 1$, leads, with the orthogonality of eigenvectors $z_k^T z_m = \delta_{mk}$, to

$$\frac{1}{\left(u^T v_j\right)^2} = \frac{1}{N} + \left(\frac{2}{\rho_j}\right)^2 \sum_{k=1}^{N-1} \left(\frac{\frac{\mu_k}{2}}{\frac{2}{\rho_j} + \mu_k} \left(\zeta^T z_k\right) \right)^2$$

from which (15) follows.

The reverse that expresses the Laplacian eigenvector $z_k = \sum_{j=1}^{N} \left(z_k^T v_j \right) v_j$ as a linear combination of the eigenvectors $v_1, v_2, \ldots, v_N$ of the effective resistance matrix Ω follows, for $1 \leq k < N$, as

$$z_k = \left(\zeta^T z_k\right) \mu_k \sum_{j=1}^{N} \frac{\frac{1}{\rho_j}}{\frac{2}{\rho_j} + \mu_k} \left(u^T v_j\right) v_j$$

After normalization $z_k^T z_k = 1$, we obtain

$$\frac{1}{\left(\mu_k \zeta^T z_k\right)^2} = \sum_{j=1}^{N} \left(\frac{\frac{1}{\rho_j}}{\frac{2}{\rho_j} + \mu_k} \left(u^T v_j\right) \right)^2$$

If $k = N$, then $z_N = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left(u^T v_j\right) v_j = \frac{u}{\sqrt{N}}$. Hence, (17) and (18) are demonstrated. $\square$

Proof 2: We repeat the same recipe of steps to the second quasi-eigenvalue equation (10). Left-multiplying (10) by z_k^T the scalar product

$$z_k^T v_j = \frac{\left(p^T v_j\right) \left(p^T z_k\right)}{\sigma^2 \left(\mu_k + \frac{2}{\rho_j}\right)}$$

Again invoking the scalar product $p^T z_k$ in (19) and (20) gives

$$z_k^T v_j = \begin{cases} \frac{\mu_k}{\left(\mu_k + \frac{2}{\rho_j}\right)} \frac{\left(p^T v_j\right)}{2\sigma^2} \left(\zeta^T z_k\right) & \text{for } 1 \leq k < N \\ \frac{1}{\sqrt{N}} \frac{\left(p^T v_j\right)}{2\sigma^2} \rho_j & \text{for } k = N \end{cases} \tag{22}$$

In summary, $v_j = \sum_{k=1}^{N} \left(z_k^T v_j\right) z_k$ equals

$$v_j = \frac{\left(p^T v_j\right)}{2\sigma^2} \left(\sum_{k=1}^{N-1} \frac{\mu_k \left(\zeta^T z_k\right)}{\mu_k + \frac{2}{\rho_j}} z_k + \frac{\rho_j}{N} u \right) \tag{23}$$

Normalization of the eigenvector, $v_j^T v_j = 1$, indicates that

$$\left(\frac{2\sigma^2}{p^T v_j} \right)^2 = \sum_{k=1}^{N-1} \left(\frac{\mu_k \left(\zeta^T z_k\right)}{\mu_k + \frac{2}{\rho_j}} \right)^2 + \frac{\rho_j^2}{N}$$

which again leads to (15). For $1 \leq k < N$, the reverse (17) follows as

$$z_k = \frac{\left(\zeta^T z_k\right) \mu_k}{2\sigma^2} \sum_{j=1}^{N} \frac{p^T v_j}{\mu_k + \frac{2}{\rho_j}} v_j$$

Finally, equating the quasi-eigenvalue equations (9) and (10) indicates that $p^T v_j = \frac{2\sigma^2}{\rho_j} u^T v_j$, again leading to (17). $\square$

The idea of Theorem 2 to deduce a scalar product between eigenvectors of two different matrices does not apply to the eigenvectors of the adjacency A and Laplacian matrix $Q = \Delta - A$ of a graph G. Indeed, starting from the eigenvalue equation (7) of the Laplacian,

$$\mu_k z_k = Q z_k = \Delta z_k - A z_k$$

left-multiplying with the m-th eigenvector x_m^T of the adjacency matrix A,

$$\mu_k x_m^T z_k = x_m^T \Delta z_k - x_m^T A z_k$$

invoking the eigenvalue equation $Ax_m = \lambda_m x_m$ of the adjacency matrix leads to the scalar product

$$x_m^T z_k = \frac{x_m^T \Delta z_k}{\mu_k + \lambda_m} \tag{24}$$

In contrast to the cases (21) and (22) for the effective resistance matrix Ω, the scalar product (24) is of a different and *not-separable* form, because $x_m^T \Delta z_k = \sum_{j=1}^N d_j (x_m)_j (z_k)_j$ is a single "scalar product" containing x_m as well as z_k, whereas both (21) and (22) are products of scalar products, where each scalar product only involves either an eigenvector v_j of Ω or an eigenvector z_k of the Laplacian Q.

4.2 Resistance regular graphs

We briefly discuss the special case of "resistance regular" graphs, where $\zeta = u\frac{R_G}{N^2}$ and $\rho_j = -\frac{2}{\mu_j}$ for $2 \leq j \leq N$. Moreover, the definition (1) indicates, with the all-one matrix $J = u.u^T$, that

$$\Omega_{\text{res. regular}} = 2\left(\frac{R_G}{N^2}J - Q^\dagger\right)$$

Since $\zeta^T z_k = 0$ for $1 \leq k < N$, the k-sum in (15) reduces to

$$\lim_{\rho_j \to -\frac{2}{\mu_j}} \sum_{k=1}^{N-1} \frac{\frac{\mu_k}{2}\left(\zeta^T z_k\right)}{\frac{2}{\rho_j} + \mu_k} z_k = \frac{\mu_j}{2} z_j \lim_{\rho_j \to -\frac{2}{\mu_j}} \frac{\zeta^T z_j}{\frac{2}{\rho_j} + \mu_j}$$

If we denote the limit $\beta_j = \lim_{\rho_j \to -\frac{2}{\mu_j}} \frac{\zeta^T z_j}{\frac{2}{\rho_j}+\mu_j}$, then the j-th eigenvector (15) belonging to the eigenvalue $\rho_j = -\frac{2}{\mu_j}$ becomes

$$v_j = \frac{\frac{u}{N} - \beta_j \frac{\mu_j^2}{2} z_j}{\sqrt{\frac{1}{N} + \left(\beta_j \frac{\mu_j^2}{2}\right)^2}}$$

which indicates that β_j must be so large[4] compared to $\frac{1}{N}$ that the all-one vector disappears, resulting in $v_j = z_j$ for all $1 \leq j \leq N$.

In the sequel, we will further ignore resistance regular graphs. Futhermore, we will confine ourselves to simple eigenvalues. In other words, for the ease of the exposition, we will omit the consideration of eigenvalues with multiplicity higher than 1.

[4] If β_j would be finite, then the eigenvalue equation $\Omega v_j = -\frac{2}{\mu_j} v_j$ in (6) for the eigenvector $v_j = \frac{u}{N} - \beta_j \frac{\mu_j^2}{2} z_j$ (without normalization) has a right-hand side equal to $-\frac{2}{\mu_j} v_j = -\frac{2}{\mu_j}\frac{u}{N} + \beta_j \mu_j z_j$, while the left-hand side, invoking $\Omega u = \zeta N + u\frac{R_G}{N}$ in [12, eq. (5.30)], gives $\Omega v_j = 2\frac{R_G}{N}\frac{u}{N} - \beta_j \frac{\mu_j^2}{2}\Omega z_j$. Equating the coefficients of the orthogonal vectors u and z_j would indicate that $\frac{1}{\mu_j} = -\frac{R_G}{N}$ (which is impossible because any Laplacian eigenvalue $\mu_j \geq 0$) and that $-\beta_j \frac{\mu_j^2}{2}\Omega z_j = \beta_j \mu_j z_j$ or $\Omega z_j = -\frac{2}{\mu_j} z_j$, implying that $v_j = z_j$!

4.3 An orthogonal matrix

The $N \times N$ matrix V contains in its colums the set of all eigenvector $v_1, v_2, \ldots, v_N$ of the effective resistance matrix Ω,

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$$

Similarly, we define the orthogonal eigenvector matrix of the Laplacian Q as

$$Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix}$$

The eigenvector (15) can be expressed [12, art.191] as

$$\begin{aligned} v_j &= \frac{\alpha_j \frac{\mu_1}{\rho_j}\left(\zeta^T z_1\right)}{\frac{2}{\rho_j} + \mu_1} \begin{bmatrix} (z_1)_1 \\ (z_1)_2 \\ \vdots \\ (z_1)_N \end{bmatrix} + \frac{\alpha_j \frac{\mu_2}{\rho_j}\left(\zeta^T z_2\right)}{\frac{2}{\rho_j} + \mu_2} \begin{bmatrix} (z_2)_1 \\ (z_2)_2 \\ \vdots \\ (z_2)_N \end{bmatrix} + \cdots + \frac{\alpha_j}{\sqrt{N}} \begin{bmatrix} (z_N)_1 \\ (z_N)_2 \\ \vdots \\ (z_N)_N \end{bmatrix} \\ &= Z \begin{bmatrix} \frac{\alpha_j \frac{\mu_1}{\rho_j}\left(\zeta^T z_1\right)}{\frac{2}{\rho_j} + \mu_1} \\ \frac{\alpha_j \frac{\mu_2}{\rho_j}\left(\zeta^T z_2\right)}{\frac{2}{\rho_j} + \mu_2} \\ \vdots \\ \frac{\alpha_j}{\sqrt{N}} \end{bmatrix} \end{aligned}$$

Substitution into the matrix V yields

$$V = Z \begin{bmatrix} \frac{\alpha_1 \frac{\mu_1}{\rho_1}\left(\zeta^T z_1\right)}{\frac{2}{\rho_1} + \mu_1} & \frac{\alpha_2 \frac{\mu_1}{\rho_2}\left(\zeta^T z_1\right)}{\frac{2}{\rho_2} + \mu_1} & \cdots & \frac{\alpha_N \frac{\mu_1}{\rho_N}\left(\zeta^T z_1\right)}{\frac{2}{\rho_N} + \mu_1} \\ \frac{\alpha_1 \frac{\mu_2}{\rho_1}\left(\zeta^T z_2\right)}{\frac{2}{\rho_1} + \mu_2} & \frac{\alpha_2 \frac{\mu_2}{\rho_2}\left(\zeta^T z_2\right)}{\frac{2}{\rho_2} + \mu_2} & \cdots & \frac{\alpha_N \frac{\mu_2}{\rho_N}\left(\zeta^T z_2\right)}{\frac{2}{\rho_N} + \mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_1}{\sqrt{N}} & \frac{\alpha_2}{\sqrt{N}} & \cdots & \frac{\alpha_N}{\sqrt{N}} \end{bmatrix}$$

Since both V and Z are orthogonal matrices, it holds that the $N \times N$ matrix

$$C = \begin{bmatrix} \frac{\alpha_1 \frac{\mu_1}{\rho_1}\left(\zeta^T z_1\right)}{\frac{2}{\rho_1} + \mu_1} & \frac{\alpha_2 \frac{\mu_1}{\rho_2}\left(\zeta^T z_1\right)}{\frac{2}{\rho_2} + \mu_1} & \cdots & \frac{\alpha_N \frac{\mu_1}{\rho_N}\left(\zeta^T z_1\right)}{\frac{2}{\rho_N} + \mu_1} \\ \frac{\alpha_1 \frac{\mu_2}{\rho_1}\left(\zeta^T z_2\right)}{\frac{2}{\rho_1} + \mu_2} & \frac{\alpha_2 \frac{\mu_2}{\rho_2}\left(\zeta^T z_2\right)}{\frac{2}{\rho_2} + \mu_2} & \cdots & \frac{\alpha_N \frac{\mu_2}{\rho_N}\left(\zeta^T z_2\right)}{\frac{2}{\rho_N} + \mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_1}{\sqrt{N}} & \frac{\alpha_2}{\sqrt{N}} & \cdots & \frac{\alpha_N}{\sqrt{N}} \end{bmatrix} \tag{25}$$

is also an orthogonal matrix. Hence, the orthogonal eigenvector matrix V equals a product of two orthogonal matrices, whose entire information can be computed from the orthogonal matrix Z and the eigenvalues $\mu_1, \mu_2, \ldots, \mu_N = 0$ of the Laplacian matrix Q,

$$V = Z.C$$

Since the determinant of orthogonal matrix is $\det C = 1$, we find that

$$1 = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \alpha_j \prod_{j=1}^{N-1} \mu_j \left(\zeta^T z_j\right) \begin{vmatrix} \frac{1}{2+\rho_1\mu_1} & \frac{1}{2+\rho_2\mu_1} & \cdots & \frac{1}{2+\rho_N\mu_1} \\ \frac{1}{2+\rho_1\mu_2} & \frac{1}{2+\rho_2\mu_2} & \cdots & \frac{1}{2+\rho_N\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

Orthogonality of a matrix implies double orthogonality [12, art. 248] in its corresponding column and row vectors, which is proved for the orthogonal matrix C in Appendix A.

5 Governing eigenvalue equations

Although the two proofs of Theorem 2 are similar, the second proof provides us with valuable colloraries:

Corollary 1 *In a connected, but not a resistance regular graph G with N nodes, the positive eigenvalues μ of the Laplacian matrix Q satisfy the governing eigenvalue partial fractions*

$$\sigma^2 = \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} \tag{26}$$

or

$$\frac{1}{4\sigma^2} = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{\rho_j \left(\rho_j \mu + 2\right)} \tag{27}$$

or

$$0 = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{2 + \rho_j \mu} \tag{28}$$

Proof: We exclude the case that $u^T v_j = p^T v_j = 0$ that happens in resistance-regular graphs. Similarly, left-multiplying $z_k = \frac{(\zeta^T z_k)\mu_k}{2\sigma^2} \sum_{j=1}^{N} \frac{p^T v_j}{\mu_k + \frac{2}{\rho_j}} v_j$ in proof 2 of Theorem 2, for $1 \leq k < N$, with the transpose of the resistance curvature vector p yields

$$p^T z_k = \frac{\left(\zeta^T z_k\right) \mu_k}{2\sigma^2} \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu_k + \frac{2}{\rho_j}}$$

The scalar product $z_k^T p = \frac{\mu_k}{2} z_k^T \zeta$ in (20) then leads, for $1 \leq k < N$, to (26) or, with $p^T v_j = \frac{2\sigma^2}{\rho_j} u^T v_j$, to (27). Alternatively, from (17) valid for $1 \leq k < N$, we obtain

$$u^T z_k = \beta_k \sum_{j=1}^{N} \frac{\frac{1}{\rho_j} \left(u^T v_j\right)^2}{\frac{2}{\rho_j} + \mu_k}$$

which leads with $z_N = \frac{u}{\sqrt{N}}$, due to orthogonality of eigenvectors $z_N^T z_k = 0$ for $1 \leq k < N$, to (28). $\square$

Even if the graph is not resistance regular, numerical evaluations can return very small values of the fundamental weight $u^T v_j$. However, this weight cannot be zero, unless the Perron eigenvector $v_1 = \frac{u}{\sqrt{N}}$, which only happens in resistance regular graphs. Moreover, orthogonality of eigenvectors $v_m^T v_j = \delta_{mj}$

indicates that all fundamentals weights $u^T v_j$ for $1 < j \leq N$ are zero if one of them is zero. Finally, the solution of *all* positive and *assumed simple* eigenvalues μ of the Laplacian requires that $u^T v_j \neq 0$ in (27) and (28). From the fractional expansions (26),(27) and (28), the interlacing Theorem 1 can be deduced. The partial fractions can be rephrased as a quadratic form using $f(\Omega) = \sum_{j=1}^{N} f(\rho_j) v_j v_j^T$ deduced in [12, eq. (A.88)], where $f(.)$ is a function defined for the eigenvalues. For example, the partial fraction (28)

$$0 = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{2 + \rho_j \mu} = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)\left(v_j^T u\right)}{2 + \rho_j \mu} = u^T \left(\sum_{j=1}^{N} \frac{v_j v_j^T}{2 + \rho_j \mu}\right) u$$

transforms to the quadratic form

$$0 = u^T \left(\frac{1}{2 + \mu\Omega}\right) u = u^T (2 + \mu\Omega)^{-1} u \tag{29}$$

The reverse of Corollary 1 is

Corollary 2 *In a connected graph G with N nodes, any eigenvalue ρ of the effective resistance matrix Ω satisfies the governing eigenvalue equation*

$$\rho = \frac{2\sigma^2}{\frac{1}{2}\sum_{k=1}^{N-1} \frac{\mu_k^2 (\zeta^T z_k)^2}{\mu_k \rho + 2} + \frac{1}{N}} \tag{30}$$

Another rewriting of (30) with $y = \frac{2}{\rho}$ is the partial fraction expansion

$$\sigma^2 = \frac{1}{4} \sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k + y} + \frac{1}{Ny} \tag{31}$$

while a characteristic polynomial of degree N in $y = \frac{2}{\rho}$ is

$$p_\Omega(y) = \prod_{m=1}^{N} (y + \mu_m) - \frac{1}{4\sigma^2} \sum_{k=1}^{N-1} \mu_k^2 \left(\zeta^T z_k\right)^2 \prod_{m=1; m\neq k}^{N} (y + \mu_m) - \frac{1}{N\sigma^2} \prod_{m=1}^{N-1} (y + \mu_m) \tag{32}$$

whose zeros, i.e. solutions of $p_\Omega(y) = 0$, are the inverse eigenvalues $\frac{2}{\rho_1}, \frac{2}{\rho_2}, \ldots, \frac{2}{\rho_N}$.

Proof : After computing the scalar product of the (unnormalized) eigenvector v_j in (23) with the resistance curvature vector p in (3), taken into account (19) and (20), we deduce that

$$1 = \frac{1}{2\sigma^2} \left(\frac{1}{2} \sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k + \frac{2}{\rho_j}} + \frac{\rho_j}{N} \right)$$

that can be rewritten as (30), because any eigenvalue ρ_j must satisfy (30), and further as (31) by letting $y = \frac{2}{\rho}$. Finally, multiplying each term in k-sum in (31) with $\prod_{m=1; m\neq k}^{N-1} (y + \mu_m)$ leads, after some manipulation, to (32). □

6 Characteristic polynomials of the Laplacian matrix Q

After multiplying the partial fraction expansions in Corollary 1 by $\prod_{m=1}^{N}(2+\rho_m\mu)$, we obtain the following characteristic polynomials for the eigenvalues of the Laplacian matrix Q:

Lemma 3 *A characteristic polynomial of the Laplacian eigenvalues of degree N is*

$$p_Q(\mu) = \sum_{j=1}^{N} \rho_j \left(p^T v_j\right)^2 \prod_{m=1;m\neq j}^{N} (2+\rho_m\mu) - \prod_{m=1}^{N} (2+\rho_m\mu)\,\sigma^2 \tag{33}$$

while a characteristic polynomial of degree $N-1$ is

$$\widetilde{p}_Q(\mu) = \sum_{j=1}^{N} \left(u^T v_j\right)^2 \prod_{m=1;m\neq j}^{N} (2+\rho_m\mu) \tag{34}$$

whose zeros, i.e. solutions of $p_Q(\mu)=0$, equal the Laplacian eigenvalues $\mu_1,\mu_2,\ldots\mu_{N-1},\mu_N=0$, while the zeros of $\widetilde{p}_Q(\mu)$ are the positive Laplacian eigenvalues $\mu_1,\mu_2,\ldots\mu_{N-1}>0$.

6.1 General properties of the polynomials $p_Q(\mu)$ and $\widetilde{p}_Q(\mu)$

The classical definition [12, art. 235] of a characteristic polynomial of a matrix A with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ is the determinant $c_A(\lambda)=\det(A-\lambda I)$, which can be expanded as a polynomial in λ of degree n,

$$c_A(\lambda) = \sum_{k=0}^{n} c_k\lambda^k = \prod_{k=1}^{n}(\lambda_k-\lambda) \tag{35}$$

from which $c_n=(-1)^n$. We wrote "a" characteristic polynomial in Lemma 3 due to the scaling of the highest order coefficient c_n. Indeed, by the fundamental property of polynomials [12, eq. (B.1) on p. 402], the polynomial (33) can be written as

$$p_Q(\mu) = \sum_{k=0}^{N} a_k(Q)\,\mu^k = a_N(Q)\prod_{m=1}^{N}(\mu-\mu_m) \tag{36}$$

while (34) as

$$\widetilde{p}_Q(\mu) = \sum_{k=0}^{N-1} \widetilde{a}_k(Q)\,\mu^k = \widetilde{a}_{N-1}(Q)\prod_{m=1}^{N-1}(\mu-\mu_m) \tag{37}$$

It follows from the spectral decomposition for any complex number z

$$x^T\Omega^z x = x^T\left(\sum_{j=1}^{N}\rho_j^z v_j v_j^T\right)x = \sum_{j=1}^{N}\rho_j^z v_j\left(v_j^T x\right)^2$$

that the evaluation for $\mu=0$ in (33) and (36) equals

$$p_Q(0) = 2^N\left(\frac{1}{2}\sum_{j=1}^{N}\rho_j\left(p^T v_j\right)^2-\sigma^2\right) = 2^N\left(\frac{1}{2}p^T\Omega p-\sigma^2\right)=0$$

and (36) indicates that

$$a_0(Q) = 0$$

while (34) and (37) show

$$\widetilde{p}_Q(0) = \widetilde{a}_0(Q) = 2^{N-1} \sum_{j=1}^{N} \left(u^T v_j\right)^2 = N 2^{N-1} \tag{38}$$

Moreover, the highest order coefficient of (33) is

$$a_N(Q) = -\sigma^2 \prod_{m=1}^{N} \rho_m \tag{39}$$

while for (34), we find

$$\begin{aligned}
\widetilde{a}_{N-1}(Q) &= \sum_{j=1}^{N} \left(u^T v_j\right)^2 \prod_{m=1; m \neq j}^{N} \rho_m = \prod_{m=1}^{N} \rho_m \sum_{j=1}^{N} \frac{1}{\rho_j} \left(u^T v_j\right)^2 \\
&= u^T \Omega^{-1} u \prod_{m=1}^{N} \rho_m
\end{aligned}$$

With $u^T \Omega^{-1} u = \frac{1}{2\sigma^2}$ (see [12, p. 182]) or from (27) for $\mu = 0$, it holds that

$$\widetilde{a}_{N-1}(Q) = \frac{1}{2\sigma^2} \prod_{m=1}^{N} \rho_m \tag{40}$$

Since $\widetilde{a}_0(Q) = \widetilde{a}_{N-1}(Q) \prod_{m=1}^{N-1} (-\mu_m)$, we find that

$$\prod_{m=1}^{N-1} \mu_m = (-1)^{N-1} \frac{\widetilde{a}_0(Q)}{\widetilde{a}_{N-1}(Q)} = \frac{(-1)^{N-1} N 2^{N-1}}{\frac{1}{2\sigma^2} \prod_{m=1}^{N} \rho_m}$$

Thus, the complexity $\xi(G) = \frac{1}{N} \prod_{m=1}^{N-1} \mu_m$ of a graph G, the number of all possible spanning trees in that graph G, equals

$$\xi(G) = \frac{1}{N} \prod_{m=1}^{N-1} \mu_m = \frac{2^N \sigma^2}{\prod_{m=1}^{N} |\rho_m|} \tag{41}$$

and

$$\widetilde{a}_{N-1}(Q) = (-1)^{N-1} \frac{2^{N-1}}{\xi(G)}$$

In conclusion, the definitions (36) and (37) show that both polynomials are related by

$$p_Q(\mu) = a_N(Q)\, \mu \prod_{m=1}^{N-1} (\mu - \mu_m) = \frac{a_N(Q)}{\widetilde{a}_{N-1}(Q)} \mu \widetilde{p}_Q(\mu)$$

and (39) and (40) lead to

$$p_Q(\mu) = -2\sigma^4 \mu \widetilde{p}_Q(\mu) \tag{42}$$

Consequently, after introducing the series in (36) and (37) into (42) and after equating corresponding powers in μ, we find, for $1 \le k \le N$, that the coefficients of the polynomials of the polynomials $p_Q(\mu)$ and $\widetilde{p}_Q(\mu)$ satisfy

$$a_k(Q) = -2\sigma^4 \widetilde{a}_{k-1}(Q) \tag{43}$$

6.2 Computation of the coefficients $\widetilde{a}_k(Q)$ of the polynomial $\widetilde{p}_Q(\mu)$

We write (34) as a product

$$\widetilde{p}_Q(\mu) = \prod_{m=1}^{N} (2 + \rho_m \mu) \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{(2 + \rho_j \mu)}$$

and expand both $\prod_{m=1}^{N} (2 + \rho_m \mu)$ and $\sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{(2+\rho_j \mu)}$ in a Taylor series around $\mu = 0$. First,

$$\begin{aligned}
\prod_{m=1}^{N} (2 + \rho_m \mu) &= 2^N \prod_{m=1}^{N} \left(1 + \frac{\rho_m}{2} \mu\right) = 2^N e^{\log \prod_{m=1}^{N} \left(1 + \frac{\rho_m}{2}\mu\right)} \\
&= 2^N e^{\sum_{m=1}^{N} \log\left(1 + \frac{\rho_m}{2}\mu\right)}
\end{aligned}$$

Taylor expansion of $\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ for $|x| < 1$ shows that

$$g(\mu) = \sum_{m=1}^{N} \log\left(1 + \frac{\rho_m}{2}\mu\right) = \sum_{m=1}^{N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\rho_m}{2}\right)^n \mu^n = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{n 2^n} \sum_{m=1}^{N} \rho_m^n \right) \mu^n$$

with Taylor coefficient $g_n = \frac{1}{n!} \left. \frac{d^n g(\mu)}{d\mu^n} \right|_{\mu=0}$ equal to $g_0 = 0$ and, for $n > 0$,

$$g_n = \frac{(-1)^{n-1}}{n 2^n} \sum_{m=1}^{N} \rho_m^n$$

Invoking our characteristic coefficients $s[k, m]$ in [10], the Taylor series

$$e^{g(\mu)} = 1 + \sum_{m=1}^{\infty} \left[\sum_{k=1}^{m} \frac{1}{k!} s_g[k, m] \right] \mu^m$$

where the combinatorial form of the characteristic coefficient

$$s_g[k, m] = \sum_{\sum_{i=1}^{k} j_i = m; j_i > 0} \prod_{i=1}^{k} g_{j_i} \tag{44}$$

which also satisfies a recursion [11] that allows to compute all characteristic coefficients exactly. Hence, we arrive at the Taylor series

$$\begin{aligned}
\prod_{m=1}^{N} (2 + \rho_m \mu) &= 2^N \left(1 + \sum_{m=1}^{\infty} \left[\sum_{k=1}^{m} \frac{1}{k!} s_g[k, m] \right] \mu^m \right) \\
&= 2^N \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} \frac{1}{k!} s_g[k, m] \right] \mu^m
\end{aligned}$$

where $s_g[k,0] = \delta_{k0} = 1_{\{k=0\}}$ and $s_g[0,m] = \delta_{0m} = 1_{\{m=0\}}$ and the indicator $1_x = 1$ if the condition x is true, else $1_x = 0$.

Second,

$$\sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{(2+\rho_j \mu)} = \frac{1}{2} \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{1+\frac{\rho_j}{2}\mu} = \frac{1}{2} \sum_{j=1}^{N} \left(u^T v_j\right)^2 \sum_{m=0}^{\infty} \left(-\frac{\rho_j}{2}\mu\right)^m$$

leads to the Taylor series

$$\begin{aligned} \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{(2+\rho_j \mu)} &= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{m+1}} \left(\sum_{j=1}^{N} \rho_j^m \left(u^T v_j\right)^2 \right) \mu^m \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{m+1}} \left(u^T \Omega^m u\right) \mu^m \end{aligned}$$

The corresponding Taylor series of the polynomial $p_Q(\mu)$ is

$$\sum_{j=1}^{N} \frac{\rho_j \left(p^T v_j\right)^2}{(2+\rho_j \mu)} - \sigma^2 = \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{m+1}} \left(p^T \Omega^{m+1} p\right) \mu^m$$

The Taylor series of the polynomial $\widetilde{p}_Q(\mu)$ is found after computing the Cauchy product,

$$\begin{aligned} \widetilde{p}_Q(\mu) &= 2^N \left(\sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} \frac{1}{k!}\, s_g[k,m] \right] \mu^m \right) \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{m+1}} \left(u^T \Omega^m u\right) \mu^m \\ &= 2^N \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \sum_{k=0}^{j} \frac{1}{k!}\, s_g[k,j] \frac{(-1)^{m-j}}{2^{m-j+1}} \left(u^T \Omega^{m-j} u\right) \right) \mu^m \end{aligned}$$

Similarly, the Cauchy product leads to

$$p_Q(\mu) = -2^{N-2} \left(p^T \Omega^2 p\right) \mu + 2^N \sum_{m=2}^{\infty} \left(\frac{(-1)^m}{2^{m+1}} \left(p^T \Omega^{m+1} p\right) + \sum_{j=0}^{m-2} \frac{(-1)^{m-j-1}}{2^{m-j}} \left(p^T \Omega^{m-j} p\right) \sum_{k=1}^{j+1} \frac{1}{k!}\, s_g[k,j+1] \right) \mu^m$$

Finally, after equating corresponding powers in μ in the above and the polynomial form $\sum_{m=0}^{N-1} \widetilde{a}_m(Q)\, \mu^m$ in (37), we arrive, for $0 \le m \le N-1$, at

$$\widetilde{a}_m(Q) = 2^{N-1} \sum_{j=0}^{m} \frac{(-1)^{m-j}}{2^{m-j}} \left(u^T \Omega^{m-j} u\right) \sum_{k=0}^{j} \frac{1}{k!}\, s_g[k,j]$$

which we rewrite, by splitting-off the $j=0$ term and using $s_g[k,0] = \delta_{k0}$ and $s_g[0,m] = \delta_{0m}$, as

$$\widetilde{a}_m(Q) = 2^{N-1} \frac{(-1)^m}{2^m} \left(u^T \Omega^m u\right) + 2^{N-1} \sum_{j=1}^{m} \frac{(-1)^{m-j}}{2^{m-j}} \left(u^T \Omega^{m-j} u\right) \sum_{k=1}^{j} \frac{1}{k!}\, s_g[k,j] \tag{45}$$

Similarly, beside $a_0(Q) = 0$ and $a_1(Q) = -2^{N-2}\left(p^T \Omega^2 p\right)$, we find the coefficient $a_m(Q)$ of the polynomial $p_Q(\mu)$ in (36) for $2 \le m \le N$ as

$$a_m(Q) = 2^N \frac{(-1)^m}{2^{m+1}} \left(p^T \Omega^{m+1} p\right) + 2^N \sum_{j=0}^{m-2} \frac{(-1)^{m-j-1}}{2^{m-j}} \left(p^T \Omega^{m-j} p\right) \sum_{k=1}^{j+1} \frac{1}{k!}\, s_g[k,j+1]$$

Invoking $a_m(Q) = -2\sigma^4 \widetilde{a}_{m-1}(Q)$ in (43) yields that

$$p^T \Omega^m p = 4\sigma^4 u^T \Omega^{m-2} u \tag{46}$$

which holds for any complex number m as follows from the eigenvalue decomposition.

We compute the polynomial coefficient $\widetilde{a}_m(Q)$ for a few values of m. First, if $m = 0$, then

$$\widetilde{a}_0(Q) = N 2^{N-1}$$

If $m = 1$, then

$$\widetilde{a}_1(Q) = 2^{N-1} \frac{-1}{2} \left(u^T \Omega u\right) + 2^{N-1} \left(u^T u\right) s_g[1,1] = -2^{N-1} R_G + 2^{N-1} N \frac{1}{2} \sum_{m=1}^{N} \rho_m$$

and, since $\sum_{m=1}^{N} \rho_m = 0$, we find

$$\widetilde{a}_1(Q) = -2^{N-1} R_G$$

We can further compute[5] any other desired value of m symbolically by the recursion of the characterstic coefficients. As an example, we list

$$\widetilde{a}_2(Q) = 2^{N-3} u^T \Omega^2 u - 2^{N-4} N \sum_{m=1}^{N} \rho_m^2$$

$$\widetilde{a}_3(Q) = -2^{N-4} u^T \Omega^3 u + 2^{N-4} R_G \sum_{m=1}^{N} \rho_m^2 + \frac{2^{N-4}}{3} N \sum_{m=1}^{N} \rho_m^3$$

$$\widetilde{a}_4(Q) = -2^{N-5} u^T \Omega^4 u + 2^{N-6} u^T \Omega^2 u \sum_{m=1}^{N} \rho_m^2 + 2^{N-8} N \left(\sum_{m=1}^{N} \rho_m^2\right)^2 - \frac{2^{N-4}}{3} R_G \sum_{m=1}^{N} \rho_m^3 - 2^{N-7} N \sum_{m=1}^{N} \rho_m^4$$

7 Characteristic polynomial of the effective resistance matrix Ω

By definition [12, art. 291] of a polynomial of degree N, the characteristic polynomial in (32) equals

$$p_\Omega(y) = \sum_{k=0}^{N} a_k(\Omega)\, y^k = a_N(\Omega) \prod_{m=1}^{N} \left(y - \frac{2}{\rho_m}\right) \tag{47}$$

The first term in $p_\Omega(y)$ in (32) is a polynomial of degree N in $y = \frac{2}{\rho}$, while the other terms contain polynomials of degree $N-1$ implying that $a_N(\Omega) = 1$.

7.1 Basic deductions from (47)

The coefficient $a_{N-1}(\Omega)$ of y^{N-1} is thus

$$\begin{aligned} a_{N-1}(\Omega) &= \sum_{m=1}^{N} \mu_m - \frac{1}{4\sigma^2} \sum_{k=1}^{N-1} \mu_k^2 \left(\zeta^T z_k\right)^2 - \frac{1}{N\sigma^2} \\ &= \sum_{m=1}^{N} \mu_m - \frac{1}{\sigma^2} \left(\frac{1}{4} \zeta^T Q^2 \zeta + \frac{1}{N}\right) \end{aligned}$$

[5] We add that, based on the coefficients $\widetilde{a}_m(Q)$, that the algebraic connectivity μ_{N-1} can be expressed as a Lagrange series (which is explicitly listed in [12, (B.68) on p. 459] up to order 5).

Relation (51) and $\sum_{m=1}^{N} \mu_m = 2L$ (see e.g. [12, art. 105]) shows that

$$a_{N-1}(\Omega) = 2L - \frac{p^T p}{\sigma^2}$$

The Newton identities [12, art. 294] indicates that the coefficient $a_{N-1}(\Omega) = -\sum_{m=1}^{N} y_m = -2\sum_{m=1}^{N} \frac{1}{\rho_m}$, leading to

$$\sum_{m=1}^{N} \frac{1}{\rho_m} = -\left(L - \frac{p^T p}{2\sigma^2}\right) \tag{48}$$

Furthermore, $p_\Omega(0) = \prod_{m=1}^{N}\left(-\frac{2}{\rho_m}\right) = -\frac{2^N}{\prod_{m=1}^{N}|\rho_m|}$. Recalling that $\mu_N = 0$, the characteristic polynomial in (32) shows that

$$p_\Omega(0) = -\frac{1}{N\sigma^2}\prod_{m=1}^{N-1}\mu_m$$

Equating both expressions for $p_\Omega(0)$ again leads to the complexity $\xi(G)$ formula in (41). The computation of the coefficient $a_k(\Omega) = \frac{1}{k!}\left.\frac{d^k p_\Omega(y)}{dy^k}\right|_{y=0}$ for $k = 1$ leads, after equating the two different ways, to the definition (4) of σ^2. After tedious manipulations, the computation of the coefficient $a_2(\Omega)$ leads to

$$\sum_{m=1}^{N} \rho_m^2 = 4\sum_{m=1}^{N-1}\frac{1}{\mu_m^2} + 2\frac{R_G^2}{N^2} + 2N\zeta^T\zeta \tag{49}$$

7.2 Computation of the coefficients $a_k(\Omega)$ of the polynomial $p_\Omega(y)$

We mimic the method of Section 6.2 to compute only the coefficients $a_m(\Omega) = \frac{1}{m!}\left.\frac{d^m p_\Omega(y)}{dy^m}\right|_{y=0}$ for any integer $0 \le m \le N$ and write (32) as

$$p_\Omega(y) = \prod_{m=1}^{N-1}(y + \mu_m)\left(y - \frac{1}{4\sigma^2}\sum_{k=1}^{N-1}\frac{y\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k + y} - \frac{1}{N\sigma^2}\right)$$

We expand the product $\prod_{m=1}^{N-1}(y+\mu_m) = \prod_{m=1}^{N-1}\mu_m \prod_{m=1}^{N-1}\left(1 + \frac{y}{\mu_m}\right)$, using our characteristic coefficients with

$$h(y) = \log\prod_{m=1}^{N-1}\left(1 + \frac{y}{\mu_m}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sum_{m=1}^{N-1}\frac{1}{\mu_m^n}\right)y^n$$

into a Taylor series around $y = 0$ in terms of the complexity $\xi(G) = \frac{1}{N}\prod_{m=1}^{N-1}\mu_m$,

$$\prod_{m=1}^{N-1}(y + \mu_m) = N\xi(G)\left(1 + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\frac{1}{k!}s_h[k,m]\right]y^m\right)$$

The Taylor series of the partial fraction expansion is

$$F(y) = y - \frac{1}{4\sigma^2}\sum_{k=1}^{N-1}\frac{y\mu_k\left(\zeta^T z_k\right)^2}{\left(1+\frac{y}{\mu_k}\right)} - \frac{1}{N\sigma^2} = y - \frac{1}{4\sigma^2}\sum_{n=1}^{\infty}(-1)^n\left(\sum_{k=1}^{N-1}\left(\zeta^T z_k\right)^2\frac{1}{\mu_k^{n-1}}\right)y^{n+1} - \frac{1}{N\sigma^2}$$

$$= -\frac{1}{N\sigma^2} + y + \frac{1}{4\sigma^2}\left(\zeta^T\zeta - \left(\zeta^T z_N\right)^2\right)y^2 - \frac{1}{4\sigma^2}\sum_{n=2}^{\infty}(-1)^n\left(\zeta^T\left(Q^\dagger\right)^{n-1}\zeta\right)y^{n+1}$$

where the spectral decomposition [12, eq. (4.30)] of the pseudoinverse of the Laplacian $Q^\dagger = \sum_{k=1}^{N-1}\frac{1}{\mu_k}z_k z_k^T$ has been used. Denoted as $F(y) = \sum_{n=0}^{\infty}F_n y^n$, the Taylor coefficients are

$$F_0 = -\tfrac{1}{N\sigma^2} \quad F_1 = 1 \quad F_2 = \tfrac{1}{4\sigma^2}\left(\zeta^T\zeta - \tfrac{R_G^2}{N^3}\right)$$

and for $n > 2$,

$$F_n = \frac{(-1)^n}{4\sigma^2}\left(\zeta^T\left(Q^\dagger\right)^{n-2}\zeta\right)$$

Executing the Cauchy product results in

$$p_\Omega(y) = N\xi(G)\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\frac{1}{k!}s_h[k,m]\right]y^m\sum_{n=0}^{\infty}F_n y^n$$

$$= N\xi(G)\sum_{m=0}^{\infty}\sum_{j=0}^{m}\left[F_{m-j}\sum_{k=0}^{j}\frac{1}{k!}s_h[k,j]\right]y^m$$

Equating corresponding powers in y in the above and (47) finally yields, for $0 \leq m \leq N$, the general expression of the coefficients

$$a_m(\Omega) = N\xi(G)\left(F_m + \sum_{j=1}^{m}F_{m-j}\sum_{k=1}^{j}\frac{1}{k!}s_h[k,j]\right) \tag{50}$$

Again, as an example, we list

$$\frac{a_0(\Omega)}{\xi(G)} = -\frac{1}{\sigma^2}$$

$$\frac{a_1(\Omega)}{\xi(G)} = N + \frac{R_G}{N\sigma^2}$$

$$\frac{a_2(\Omega)}{\xi(G)} = -R_G - \frac{3R_G^2}{4N^2\sigma^2} + \frac{N\zeta^T\zeta}{4\sigma^2} - \frac{1}{2\sigma^2}\sum_{m=1}^{N-1}\frac{1}{\mu_m^2}$$

$$\frac{a_3(\Omega)}{\xi(G)} = \frac{R_G^2}{2N} + \frac{5R_G^3}{12N^3\sigma^2} - R_G\frac{\zeta^T\zeta}{4\sigma^2} - N\frac{\zeta^T Q^\dagger\zeta}{4\sigma^2} + \left(\frac{N}{2} + \frac{R_G}{2N\sigma^2}\right)\sum_{m=1}^{N-1}\frac{1}{\mu_m^2} + \frac{1}{3\sigma^2}\sum_{m=1}^{N-1}\frac{1}{\mu_m^3}$$

8 Eigenvalues

8.1 The eigenvalues of the effective resistance matrix Ω

For $\rho > 0$ and for a simplex-altitude vector $\zeta \neq \alpha u$ not proportional to the all-one vector u, the sum $S(\rho) = \frac{1}{2}\sum_{k=1}^{N-1}\frac{\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k\rho+2} > 0$ in (30) in Corollary 2 and $S(\rho)$ decreases monotonically in ρ

towards zero at $\rho \to \infty$, implying that $S(0) = \frac{1}{4}\sum_{k=1}^{N-1} \mu_k^2 \left(\zeta^T z_k\right)^2 > S(\rho) > 0 = \lim_{\rho\to\infty} S(\rho)$ and the eigenvalue equation (30) shows that there exists only one positive eigenvalue $\rho_1 > 0$ at the crossing of the bisectrice, i.e. the straight line at 45 degrees with the ρ-axis, and $S(\rho)$. It follows from $Q^m = \sum_{k=1}^{N-1} \mu_k^m z_k z_k^T$ for a positive real m that $S(0) = \frac{1}{4}\zeta^T Q^2 \zeta = \frac{1}{4}\|Q\zeta\|_2^2$. The definition (3) of the resistance curvature vector $p = \frac{1}{2}Q\zeta + \frac{u}{N}$ shows that

$$p^T p = \|p\|_2^2 = \frac{1}{4}\zeta^T Q^2 \zeta + \frac{1}{N} \tag{51}$$

The right-hand side of (30) for $\rho = 0$ equals, with σ^2 in (4),

$$\frac{2\sigma^2}{\frac{1}{4}\sum_{k=1}^{N-1} \mu_k^2 \left(\zeta^T z_k\right)^2 + \frac{1}{N}} = \frac{2\sigma^2}{\|p\|_2^2} = \frac{\frac{1}{2}\zeta^T Q\zeta + 2R_G}{\frac{1}{4}\zeta^T Q^2\zeta + \frac{1}{N}}$$

Hence, we conclude that the largest eigenvalue satisfies the lower bound

$$\rho_1 > 2\left(\frac{\sigma}{\|p\|_2}\right)^2 \tag{52}$$

while an upper bound follows from $\lim_{\rho\to\infty} S(\rho) = 0$, so that (30) becomes

$$\rho_1 < 2\sigma^2 N \tag{53}$$

The lower bound (52) also follows from the Rayleigh inequality [12, art. 251], stating that $\frac{w^T \Omega w}{w^T w} \leq \rho_1$ for any vector w and that equality only is achieved if $w = v_1$, the eigenvector of Ω belonging to eigenvalue ρ_1. If $w = p$ and recalling, as mentioned in the Introduction (Section 1), that $p^T \Omega p = 2\sigma^2$, then the Rayleigh inequality indicates that $\rho_1 \geq \frac{2\sigma^2}{p^T p}$. Finally, (23) shows that, generally[6], $p \neq v_1$, else all v_j would be zero. Hence, the strict inequality in (52) is demonstrated for *non-resistance* regular graphs, where $\zeta \neq \frac{R_G}{N^2}u$.

By Taylor expansion of (31) in $w = y + \mu_j$ around $w = 0$, a tedious computation (neglecting[7] $O(w)$ terms) shows that

$$\rho_j < -\frac{2}{\mu_j}\left(1 + \frac{\mu_j \left(\zeta^T z_j\right)^2}{4\left(\sigma^2 + \frac{1}{N\mu_j}\right) - \sum_{k=1;k\neq j}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k - \mu_j}}\right)^{-1}$$

which improves upon the estimate of Theorem 1.

For $\rho < 0$, it is instructive to rewrite eigenvalue equation (31) as the function

$$f(\rho) = 2\sigma^2 - \frac{\rho}{N} - \frac{1}{2}\sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k + \frac{2}{\rho}}$$

Since

$$\frac{df(\rho)}{d\rho} = -\frac{1}{N} - \sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\left(\rho\mu_k + 2\right)^2} < 0$$

---

[6] Only in resistance regular graphs, where $\zeta = \frac{R_G}{N^2}u$, then $p = \frac{u}{N} = \frac{v_1}{\sqrt{N}}$.

[7] If we neglect $O\left(w^2\right)$, then also an analytic result as roots of a quadratic polynomial can be found, which is, however, unwieldy and omitted.

the function $f(\rho)$ is monotonously decreasing for all ρ. The sum, which equals $\rho S(\rho)$, possesses poles at $\rho = -\frac{2}{\mu_k}$ and the sum $S(\rho)$ travels from ∞ to $-\infty$ between two poles (at which $f(\rho)$ has an asymptote, i.e. vertical line), illustrating that between those poles, there is a zero of the function $f(\rho)$, in agreement with the inter-lacing Theorem 1 and implying that there are $N-1$ real, negative zeros (i.e. eigenvalues) of the function $f(\rho)$.

Largest eigenvalue ρ_1 For large ρ, the Taylor expansion of the function $f(\rho)$ is, using $\frac{1}{\mu_k+\frac{2}{\rho}} = \frac{1}{\mu_k}\sum_{m=0}^{\infty}\left(-\frac{2}{\mu_k}\right)^m \frac{1}{\rho^m}$ valid for $\left|\frac{2}{\mu_k \rho}\right| < 1$ for any $1 \leq k \leq N-1$,

$$\begin{aligned} f(\rho) &= 2\sigma^2 - \frac{\rho}{N} - \frac{1}{2}\sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2 \sum_{m=0}^{\infty}\left(-\frac{2}{\mu_k}\right)^m \frac{1}{\rho^m} \\ &= 2\sigma^2 - \frac{\rho}{N} - \frac{1}{2}\sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2 + \sum_{k=1}^{N-1}\left(\zeta^T z_k\right)^2\frac{1}{\rho} \\ &\quad + \sum_{m=2}^{\infty}(-1)^{m-1}2^{m-1}\sum_{k=1}^{N-1}\frac{\left(\zeta^T z_k\right)^2}{\mu_k^{m-1}}\frac{1}{\rho^m} \end{aligned}$$

Using spectral decomposition, i.e. $\frac{1}{2}\sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2 = \frac{1}{2}\zeta^T Q\zeta$, $\sum_{k=1}^{N-1}\left(\zeta^T z_k\right)^2 = \zeta^T\zeta - \frac{\left(\zeta^T u\right)^2}{N} = \zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2$ and, for $m>1$, $\sum_{k=1}^{N-1}\frac{\left(\zeta^T z_k\right)^2}{\mu_k^{m-1}} = \zeta^T\left(Q^\dagger\right)^{m-1}\zeta > 0$, we find, for large $\rho > \frac{2}{\mu_{N-1}}$, that

$$f(\rho) = -\frac{\rho}{N} + 2\left(\sigma^2 - \frac{1}{4}\zeta^T Q\zeta\right) + \left(\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}2^{m-1}\zeta^T\left(Q^\dagger\right)^{m-1}\zeta}{\rho^m}$$

Invoking the definition (4) of σ^2 leads to the Taylor series of $f(\rho)$, valid for $\rho > \frac{2}{\mu_{N-1}}$,

$$f(\rho) = -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}2^{m-1}\zeta^T\left(Q^\dagger\right)^{m-1}\zeta}{\rho^m} \tag{54}$$

Since $0 = f(\rho_1)$ and $\rho_1 > \frac{2}{\mu_{N-1}}$, the first order in (54), ignoring the $O\left(\frac{1}{\rho}\right)$ contribution, is $f(\rho) \approx \frac{1}{N}\left(-\rho + \frac{2R_G}{N}\right)$ and leads to the estimate $\rho_1 \approx \frac{2R_G}{N}$, which is rather weak, because the bound $\rho_1 \geq \frac{2R_G}{N}$ holds [12, (5.33)]. After ignoring $\frac{-2\zeta^T Q^\dagger\zeta}{\rho^2} + O\left(\frac{1}{\rho^3}\right)$ terms, which are negative for sufficiently large ρ, the Taylor expansion (54) becomes

$$f(\rho) \gtrapprox -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho}$$

which translates to a quadratic equation in ρ,

$$\rho^2 - \frac{2R_G}{N}\rho - \left(N\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right) = 0$$

with roots $\rho_\pm = \frac{R_G}{N} \pm \sqrt{N\zeta^T\zeta}$. The smaller root $\rho_- \approx \frac{R_G}{N} - \sqrt{N\zeta^T\zeta}$ violates the condition $\rho > \frac{2}{\mu_{N-1}}$, but the largest root

$$\rho_1 \lessapprox \frac{R_G}{N} + \sqrt{N\zeta^T\zeta} \tag{55}$$

seems an accurate upper bound, sharper than (53). After ignoring $\frac{4\zeta^T(Q^\dagger)^2\zeta}{\rho^3} + O\left(\frac{1}{\rho^4}\right) > 0$, we have a lower bound

$$f(\rho) < -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} - \frac{2\zeta^T Q^\dagger \zeta}{\rho^2}$$

which translates to a cubic polynomial in ρ,

$$\rho^3 - \frac{2R_G}{N}\rho^2 - \left(N\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\rho + 2N\zeta^T Q^\dagger \zeta = 0$$

Cardano's solution, that is involved and here omitted, can be symbolically computed. Increasing one order,

$$f(\rho) > -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} - \frac{2\zeta^T Q^\dagger \zeta}{\rho^2} + \frac{4\zeta^T\left(Q^\dagger\right)^2\zeta}{\rho^3}$$

is again an upper bound, that results in a quartic polynomial in ρ,

$$\rho^4 - \frac{2R_G}{N}\rho^3 - \left(N\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\rho^2 + \left(2N\zeta^T Q^\dagger \zeta\right)\rho - 4N\zeta^T\left(Q^\dagger\right)^2\zeta = 0$$

the last polynomial whose exact zeros are still generally possible to express in closed form (e.g. by Mathematica).

Smallest eigenvalue ρ_N We aim to find the smallest eigenvalue $\rho_N < 0$ and we consider $f(-|\rho|)$, which is strict increasing in $|\rho|$,

$$f(-|\rho|) = 2\sigma^2 + \frac{|\rho|}{N} - \frac{1}{2}\sum_{k=1}^{N-2}\frac{\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k - \frac{2}{|\rho|}} - \frac{1}{2}\frac{\mu_{N-1}^2\left(\zeta^T z_{N-1}\right)^2}{\mu_{N-1} - \frac{2}{|\rho|}}$$

For large $|\rho|$, (54) illustrates that

$$f(-|\rho|) = \frac{|\rho|}{N} + \frac{2R_G}{N^2} + O\left(\frac{1}{|\rho|}\right)$$

which leads, for $|\rho| > \frac{2}{\mu_{N-1}}$, to the estimate

$$f(-|\rho|) \simeq \frac{|\rho|}{N} + \frac{2R_G}{N^2} - \frac{1}{2}\frac{\mu_{N-1}^2\left(\zeta^T z_{N-1}\right)^2}{\mu_{N-1} - \frac{2}{|\rho|}}$$

The solution $f(-|\rho|) = 0$ then approximates $-|\rho_N|$. Thus the positive zero of

$$\frac{|\rho|}{N} + \frac{2R_G}{N^2} - \frac{1}{2}\frac{\mu_{N-1}^2\left(\zeta^T z_{N-1}\right)^2}{\mu_{N-1}|\rho| - 2}|\rho| = 0$$

is

$$|\rho|_N = \frac{\frac{2}{\mu_{N-1}} - \frac{2R_G}{N} + \frac{1}{2}N\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2 + \sqrt{\left(\frac{2}{\mu_{N-1}} - \frac{2R_G}{N} + \frac{1}{2}N\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2\right)^2 + \frac{16R_G}{N\mu_{N-1}}}}{2}$$

Hence, our estimate for the smallest eigenvalue ρ_N of the effective resistance matrix Ω is

$$\rho_N \simeq -\left(\frac{1}{\mu_{N-1}} - \frac{R_G}{N} + \frac{1}{4}N\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2\right) - \sqrt{\left(\frac{1}{\mu_{N-1}} - \frac{R_G}{N} + \frac{1}{4}N\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2\right)^2 + \frac{4R_G}{N\mu_{N-1}}} \tag{56}$$

which is a sharp estimate, slightly upper bounding the exact eigenvalue ρ_N (i.e. lower bounding $|\rho_N|$).

Since $\rho_1 = -\sum_{j=2}^{N} \rho_j$, the upper bound (53) implies that all eigenvalues of Ω lie in the interval $\left[-2\sigma^2 N, 2\sigma^2 N\right]$, but a more precise determination of the interval $[\rho_N, \rho_1]$ is obtained with the upper bound (55) for ρ_1 and the accurate (56), although (56) is an upper bound, while a lower bound would have been "safer".

8.2 The eigenvalues of the Laplacian matrix Q

Similarly as in Section 8.1, we define from $\sigma^2 = \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}}$ in (26) the monotonously increasing function in μ,

$$h(\mu) = \sigma^2 - \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}}$$

satisfies

$$h'(\mu) = \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\left(\mu + \frac{2}{\rho_j}\right)^2} > 0$$

Similarly as the transformation of the partial fraction (28) into the quadratic form (29), we find

$$h(\mu) = \sigma^2 - p^T\left(\mu I + 2\Omega^{-1}\right)^{-1} p \tag{57}$$

For example, if $\mu = 0$, then (57) simplifies to

$$h(0) = \sigma^2 - \frac{1}{2}p^T \Omega p$$

Since $2\sigma^2 = p^T\Omega p$, we find $h(0) = 0$ and, indeed, $\mu = 0$ is an eigenvalue of the Laplacian Q. Thus, we can reform (57) as

$$h(\mu) = \frac{\mu}{4}p^T\left(\frac{\Omega^2}{\frac{\mu}{2}\Omega + I}\right)p$$

For large μ, it holds that $\lim_{\mu\to\infty} h(\mu) = \sigma^2$. Without poles on the positive real axis, the function $h(\mu)$ increases continuously from 0 to σ^2. However, at $\mu = -\frac{2}{\rho_k}$ for an integer k satisfying $2 \leq k \leq N$, the function $h(\mu)$ has a simple[8] pole at which $h(\mu)$ switches from $+\infty$ for $\mu = -\frac{2}{\rho_k} - \varepsilon$ to $-\infty$ for $\mu = -\frac{2}{\rho_k} + \varepsilon$ for arbitrary small $\varepsilon > 0$. The increasing nature of $h(\mu)$ indicates that $h(\mu)$ can only have one zero in the interval $\left(-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}}\right)$. For example, the function $h(\mu)$ increases from $h(\mu) = 0$ at $\mu = 0$ until $h(\mu) > 0$ for $\mu < -\frac{2}{\rho_2}$. At $\mu = -\frac{2}{\rho_2}$, the function $h(\mu)$ switches from $+\infty$ for $\mu = -\frac{2}{\rho_2} - \varepsilon$ to $-\infty$ for $\mu = -\frac{2}{\rho_2} + \varepsilon$ and $\varepsilon > 0$ is arbitrary small. Hence, $h(\mu)$ can only have a zero for $\mu > -\frac{2}{\rho_2}$, implying that the algebraic connectivity $\mu_{N-1} > -\frac{2}{\rho_2}$, which is in agreement with the Interlacing Theorem 1.

[8] Recall that we assume simple eigenvalues.

8.2.1 Estimate of a Laplacian eigenvalue

We write

$$h(\mu) = r_k(\mu) - \frac{\left(p^T v_k\right)^2}{\mu + \frac{2}{\rho_k}} - \frac{\left(p^T v_{k+1}\right)^2}{\mu + \frac{2}{\rho_{k+1}}}$$

where

$$\begin{aligned} r_k(\mu) &= \sigma^2 - \sum_{j=1}^{k-1} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} - \sum_{j=k+2}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} \\ &= \frac{\mu}{4} p^T \left(\frac{\Omega^2}{\frac{\mu}{2}\Omega + I} \right) p + \frac{\left(p^T v_k\right)^2}{\mu + \frac{2}{\rho_k}} + \frac{\left(p^T v_{k+1}\right)^2}{\mu + \frac{2}{\rho_{k+1}}} \end{aligned}$$

is regular at any point μ in the interval $\left[-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}}\right]$. Moreover, $r_k(\mu)$ is a positive, strict increasing function in μ, so that $0 < r_k\left(-\frac{2}{\rho_k}\right) < r_k(\mu) < r_k\left(-\frac{2}{\rho_{k+1}}\right)$. The zero $\mu^* \in \left(-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}}\right)$ satisfies

$$r_k(\mu^*) - \frac{\left(p^T v_k\right)^2}{\mu^* + \frac{2}{\rho_k}} - \frac{\left(p^T v_{k+1}\right)^2}{\mu^* + \frac{2}{\rho_{k+1}}} = 0$$

where $-\frac{\left(p^T v_k\right)^2}{\mu^* + \frac{2}{\rho_k}} < 0$ and $r_k(\mu^*) - \frac{\left(p^T v_{k+1}\right)^2}{\mu^* + \frac{2}{\rho_{k+1}}} > 0$ and which is equivalent to

$$r_k(\mu^*)\left(\mu^* + \frac{2}{\rho_k}\right)\left(\mu^* + \frac{2}{\rho_{k+1}}\right) - \left(p^T v_k\right)^2\left(\mu^* + \frac{2}{\rho_{k+1}}\right) - \left(\mu^* + \frac{2}{\rho_k}\right)\left(p^T v_{k+1}\right)^2 = 0 \qquad (58)$$

Let $y = \mu^* + \frac{2}{\rho_k}$, then

$$r_k\left(y - \frac{2}{\rho_k}\right) y \left(y + \frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) - \left(p^T v_k\right)^2\left(y + \frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) - y\left(p^T v_{k+1}\right)^2 = 0$$

After simplification, we find the quadratic equation in y,

$$r_k\left(y - \frac{2}{\rho_k}\right) y^2 + \left\{ r_k\left(y - \frac{2}{\rho_k}\right)\left(\frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) - \left(p^T v_k\right)^2 - \left(p^T v_{k+1}\right)^2 \right\} y - \left(p^T v_k\right)^2\left(\frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) = 0$$

Since $r_k\left(y - \frac{2}{\rho_k}\right)$ does not contain a zero for $y \in \left[0, -\frac{2}{\rho_{k+1}} + \frac{2}{\rho_k}\right]$, we arrive, with $\delta_k = -\frac{2}{\rho_{k+1}} + \frac{2}{\rho_k} > 0$ at the quadratic equation in y

$$y^2 - \left\{ \delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)} \right\} y + \left(p^T v_k\right)^2 \frac{\delta_k}{r_k\left(y - \frac{2}{\rho_k}\right)} = 0 \qquad (59)$$

with roots

$$y_\pm = \frac{\left(\delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}\right) \pm \sqrt{\left(\delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}\right)^2 - \frac{4\left(p^T v_k\right)^2 \delta_k}{r_k\left(y - \frac{2}{\rho_k}\right)}}}{2}$$

Since $r_k\left(y-\frac{2}{\rho_k}\right)>0$ in $\left[0,-\frac{2}{\rho_{k+1}}+\frac{2}{\rho_k}\right]$, the discriminant

$$\begin{aligned}\Lambda &= \left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2-\frac{4\left(p^Tv_k\right)^2\delta_k}{r_k\left(y-\frac{2}{\rho_k}\right)} \\ &= \left(\delta_k+\frac{\left(p^Tv_{k+1}\right)^2-\left(p^Tv_k\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2+\frac{4\left(p^Tv_k\right)^2\left(p^Tv_{k+1}\right)^2}{r_k^2\left(y-\frac{2}{\rho_k}\right)}>0\end{aligned}$$

is positive and the quadratic equation has two positive zeros,

$$y_\pm=\frac{\left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)\pm\sqrt{\left(\delta_k+\frac{(p^Tv_{k+1})^2-(p^Tv_k)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2+\frac{4(p^Tv_k)^2(p^Tv_{k+1})^2}{r_k^2\left(y-\frac{2}{\rho_k}\right)}}}{2}$$

It remains to determine the zero lying in the interval $\left[0,\delta_k=-\frac{2}{\rho_{k+1}}+\frac{2}{\rho_k}\right]$. It holds that $\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}>\delta_k$, but $\delta_k+\frac{\left(p^Tv_{k+1}\right)^2-\left(p^Tv_k\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}$ can be smaller than δ_k, in which case there are two zeros in the interval, which is not possible. We are certain that the minus sign always lies in $[0,\delta_k]$. Hence, we translate back with $y=\mu^*+\frac{2}{\rho_k}$ to

$$\mu^*=-\frac{2}{\rho_k}+\frac{\left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k(\mu^*)}\right)-\sqrt{\left(\delta_k+\frac{(p^Tv_{k+1})^2-(p^Tv_k)^2}{r_k(\mu^*)}\right)^2+\frac{4(p^Tv_k)^2(p^Tv_{k+1})^2}{r_k^2(\mu^*)}}}{2}$$

which is a kind of self-consistent equation in μ^*. We now invoke the bounds $r\left(-\frac{2}{\rho_k}\right)<r(\mu)<r\left(-\frac{2}{\rho_{k+1}}\right)$ and arrive at

$$\mu^*<-\frac{2}{\rho_k}+\frac{\left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(-\frac{2}{\rho_k}\right)}\right)-\sqrt{\left(\delta_k+\frac{(p^Tv_{k+1})^2-(p^Tv_k)^2}{r_k\left(-\frac{2}{\rho_k}\right)}\right)^2+\frac{4(p^Tv_k)^2(p^Tv_{k+1})^2}{r_k^2\left(-\frac{2}{\rho_k}\right)}}}{2} \tag{60}$$

and

$$\mu^*>-\frac{2}{\rho_k}+\frac{\left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(-\frac{2}{\rho_{k+1}}\right)}\right)-\sqrt{\left(\delta_k+\frac{(p^Tv_{k+1})^2-(p^Tv_k)^2}{r_k\left(-\frac{2}{\rho_{k+1}}\right)}\right)^2+\frac{4(p^Tv_k)^2(p^Tv_{k+1})^2}{r_k^2\left(-\frac{2}{\rho_{k+1}}\right)}}}{2} \tag{61}$$

Numerical results indicate that the upper (60) and lower bound (61) are quite tight and improve the bounds of the Interlacing Theorem 1 A slightly easier, but less accurate estimate of a Laplacian eigenvalue μ^* is obtained, for y is small by ignoring y^2 in the quadratic equation (59) leading to the inequality,

$$-\left\{\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right\}y+\left(p^Tv_k\right)^2\frac{\delta_k}{r_k\left(y-\frac{2}{\rho_k}\right)}=-y^2<0$$

from which

$$y < \delta_k \frac{\frac{\left(p^T v_k\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}}{\delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}} < \delta_k$$

8.2.2 Lagrange series for a Laplacian eigenvalue μ

We can proceed further to compute any Laplacian eigenvalue μ in the interval $\left[-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}}\right]$ to any desired accuracy via Lagrange series. First, the exact Taylor expansion of $r_k(\mu)$ around μ_0, different from a simple pole, is

$$r_k(\mu) = r_k(\mu_0) + \sum_{m=1}^{\infty} (-1)^{m+1} \left(\sum_{j=1; j \neq \{k,k+1\}}^{N} \frac{\left(p^T v_j\right)^2}{\left(\frac{2}{\rho_j} + \mu_0\right)^{m+1}} \right) (\mu - \mu_0)^m \tag{62}$$

that converges for $|\mu - \mu_0| < \min_{1 \leq j \leq N} \left| \left(\frac{2}{\rho_j} + \mu_0\right) \right|$. Second, we introduce the Taylor series (62) into the quadratic equation (58), which results in a new Taylor series $\widetilde{r}_k(\mu^*) = \sum_{m=0}^{\infty} (\widetilde{r}_k(\mu_0))_m (\mu^* - \mu_0)^m$ with Taylor coefficients $(\widetilde{r}_k(\mu_0))_m$ of the function $\widetilde{r}_k(\mu^*) = w$. The Lagrange series for the inverse function $\mu^* = (\widetilde{r}_k)^{-1}(w)$ around μ_0 can be computed symbolically to any finite desired order via our characteristic coefficients [11], [12, p. 459]. The Laplacian eigenvalue μ is a zero of $\widetilde{r}_k(\mu) = 0 = w$ and the Lagrange series for inverse function $\mu = (\widetilde{r}_k)^{-1}(0)$ around μ_0 converges around μ_0, provided that μ_0 lies sufficiently close to μ_k. The art lies in finding an expansion point μ_0 that lies sufficiently close to a Laplacian eigenvalue μ_k, so that the Lagrange series converges fast enough. Since the lower (61) and upper (60) bound are numerically close the zero μ, by choosing expansion point μ_0 equal to the lower bound for (61), we expected a fast converge of the Lagrange series. We have not further explored this route because the method is numerical.

9 Sum of quadratric forms and the trace of matrix

We insert here an intermezzo. It follows from $p = \sum_{k=1}^{N} \left(p^T v_k\right) v_k$ that $p^T p = \sum_{j=1}^{N} \left(p^T v_j\right)^2$. Using (72) in Appendix B leads to

$$\frac{p^T p}{2\sigma^2} = \frac{1}{2} \sum_{j=1}^{N} v_j^T Q v_j + \sum_{j=1}^{N} \frac{1}{\rho_j} \tag{63}$$

Comparison with (48) indicates that the number of links in the graph G equals

$$L = \frac{1}{2} \sum_{j=1}^{N} v_j^T Q v_j \tag{64}$$

Since $L = \frac{1}{2} \sum_{j=1}^{N} d_j$, where d_j is the degree of node j, it might be suggestive to propose that $d_j = v_j^T Q v_j$, which is unfortunately not true. However, if $[x]$ denotes "rounding up to the nearest integer", we found numerically that $\left[v_1^T Q v_1\right] = 0$, while $\left[v_j^T Q v_j\right]$ for $2 \leq j \leq N$ is close to a degree in the graph G within an error of ± 1.

We now show that formula (64) is an instance of a more general property of the trace of a matrix. Indeed, we rewrite

$$\sum_{j=1}^{N} v_j^T Q v_j = \text{trace}\left(V^T Q V\right)$$

and we apply the cyclic permutation property of the trace-operator [12, eq, (4,15)],

$$\text{trace}\left(V^T Q V\right) = \text{trace}\left(Q V V^T\right)$$

Since V is an orthogonal matrix that satisfies $VV^T = I$, we obtain $\text{trace}\big(V^T Q V\big) = \text{trace}(Q)$. The general relation between a sum of quadratric forms and the trace is

Theorem 4 *Let* $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}$ *be an* $n \times n$ *orthogonal matrix, which satisfies* $X^T X = XX^T = I$ *and let the* $n \times 1$ *vector* x_j *contain as components the column* j *elements of the matrix* X*. Then, for any* $n \times n$ *square matrix* M *it holds that*

$$\sum_{j=1}^{n} x_j^T M x_j = \ \textit{trace}\left(X^T M X\right) = \textit{trace}\left(M\right) \tag{65}$$

where the quadratic form is $x_j^T M x_j = \sum_{k=1}^{n} \sum_{l=1}^{n} m_{kl} \left(x_j\right)_k \left(x_j\right)_l$*.*

Another direct proof of (65) follows from the second orthogonality relation $\sum_{j=1}^{n} \left(x_j\right)_k \left(x_j\right)_l = \delta_{kl}$ in [12, eq. (A.126)].

Example If the orthogonal matrix X is the eigenvector matrix of an $n \times n$ symmetric matrix A and x_j is the normalized eigenvector belonging to eigenvalue λ_j, then (65) reduces to the well-known eigenvalue formula [12, eq. (A.99)]

$$\sum_{j=1}^{n} \lambda_j = \text{trace}\left(A\right) \tag{66}$$

Example Another example of Theorem 4 than (64) applies to $\frac{R_G}{N} = N \ \text{trace}\big(Q^\dagger\big)$ as

$$R_G = N \sum_{j=1}^{N-1} x_j^T Q^\dagger x_j$$

If we choose $x_j = z_j$, for which $z_j^T Q^\dagger z_j = \frac{1}{\mu_j}$, we find the known instance of (66), $R_G = N \sum_{j=1}^{N-1} \frac{1}{\mu_j}$. If $x_j = v_j$, then we find $R_G = N \sum_{j=1}^{N-1} v_j^T Q^\dagger v_j$.

Example For any set $\{z_j\}_{1 \le j \le n}$ of eigenvectors of a symmetric Laplacian matrix, and not only for the Laplacian Q associated to the graph G, (65) becomes $\sum_{j=1}^{n} z_j^T M z_j$ =trace(M). Since the eigenvector $z_n = \frac{u}{\sqrt{n}}$ belonging to eigenvalue $\mu_n = 0$ for any $n \times n$ symmetric Laplacian matrix, we find that

$$\text{trace}\left(M\right) - \frac{u^T M u}{n} = \sum_{j=1}^{n-1} z_j^T M z_j \tag{67}$$

where $u^T M u = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij}$ equals the sum of all elements in the matrix M. For example, if M is the effective resistance matrix Ω, whose trace(Ω) = 0, then (67) reduces, with the effective graph resistance $R_G = \frac{1}{2} u^T \Omega u$, to

$$\frac{2R_G}{N} = -\sum_{j=1}^{N-1} z_j^T \Omega z_j \tag{68}$$

If we choose for $\{z_j\}_{1\leq j\leq n}$ the Laplacian set of eigenvectors of the cycle graph, whose orthogonal matrix is the Fourier matrix [12, p. 196], then (68) provides a discrete Fourier type of expansion of the effective graph resistance, where z_k belongs to the k-th largest eigenvalue μ_k, which can be associated to a frequency[9] of a signal. We refer further to the graph signal processing community for these kind of expansions of signals. Similar for the adjacency matrix A^k, whose $\text{trace}\left(A^k\right) = W_k$ is the number of closed walks with k hops and $u^T A^k u = N_k$ is the number of walks with k hops, (67) simplifies to a spectral decomposition in the basis of Laplacian eigenvalues of any other graph on N nodes

$$W_k - \frac{N_k}{N} = \sum_{j=1}^{N-1} z_j^T A^k z_j$$

which simplifies for $k = 1$ to the average degree

$$E[D] = \frac{2L}{N} = -\sum_{j=1}^{N-1} z_j^T A z_j$$

For the Laplacian Q of the graph G, for which $\text{trace}(Q) = 2L$ and $Qu = 0$, (67) gives us

$$2L = \sum_{j=1}^{N-1} \widetilde{z}_j^T Q \widetilde{z}_j$$

where $\{\widetilde{z}_j\}_{1\leq j\leq n}$ are the Laplacian eigenvectors of a graph $\widetilde{G}$. Only if $G = \widetilde{G}$, then $\widetilde{z}_j^T Q \widetilde{z}_j = \mu_j$, which reduces to an instant of (66), see e.g. [12, Eq. (4.7)].

10 Conclusion

Given the spectrum (i.e. eigenvalues and eigenvectors) of either the effective resistance matrix Ω or the Laplacian matrix Q of a graph G allows the computation of the spectrum of the other matrix. Hence, there is a nice one-to-one relation, which is, unfortunately, not existent for the adjacency matrix and the Laplacian. Several new formulae, e.g. those in Theorem 2 and 4, Corollary 2 and 1 and several bounds on eigenvalues in Section 8, are derived from the one-to-one correspondence between Ω and Q. Finally, we improve the Interlacing Theorem 1.

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A Double orthogonality of the row and column vectors of the orthogonal matrix C in (25)

Orthogonality of a matrix implies double orthogonality [12, art. 248] in its corresponding column vectors, which applied to the matrix C in (25) implies that

$$v_k^T v_m = \alpha_k \alpha_m \left(\sum_{j=1}^{N-1} \frac{\left(\mu_j \left(\zeta^T z_j\right)\right)^2}{\left(2+\rho_k \mu_j\right)\left(2+\rho_m \mu_j\right)} + \frac{1}{N} \right) = \delta_{km}$$

and for the row vectors

$$\mu_k \left(\zeta^T z_k\right) \mu_m \left(\zeta^T z_m\right) \sum_{j=1}^{N} \frac{\alpha_j^2}{\left(2+\rho_j \mu_k\right)\left(2+\rho_j \mu_m\right)} = \delta_{km} \text{ for } m \neq N$$

$$\frac{\mu_k \left(\zeta^T z_k\right)}{\sqrt{N}} \sum_{j=1}^{N} \frac{\alpha_j^2}{2+\rho_j \mu_k} = 0 \text{ for } k \neq N$$

In the sequel, we prove both orthogonality relations. First, for the columns of the orthogonal matrix C, partial fraction expansion yields

$$\frac{1}{\left(2+\rho_k x\right)\left(2+\rho_m x\right)} = \frac{\frac{\rho_k}{2}}{\left(\rho_k-\rho_m\right)\left(2+\rho_k x\right)} + \frac{\frac{\rho_m}{2}}{\left(\rho_m-\rho_k\right)\left(2+\rho_m x\right)} \tag{69}$$

so, for $k \neq m$,

$$v_k^T v_m = \alpha_k \alpha_m \left(\frac{\frac{\rho_k}{2}}{(\rho_k - \rho_m)} \sum_{j=1}^{N-1} \frac{\left(\mu_j \left(\zeta^T z_j\right)\right)^2}{(2 + \rho_k \mu_j)} - \frac{\frac{\rho_m}{2}}{(\rho_k - \rho_m)} \sum_{j=1}^{N-1} \frac{\left(\mu_j \left(\zeta^T z_j\right)\right)^2}{(2 + \rho_m \mu_j)} + \frac{1}{N} \right)$$

Each eigenvalue ρ_m obeys the eigenvalue partial fraction (31)

$$\frac{\rho_m}{2} \sum_{j=1}^{N-1} \frac{\mu_j^2 \left(\zeta^T z_j\right)^2}{\rho_m \mu_j + 2} = 2\sigma^2 - \frac{\rho_m}{N}$$

showing that $v_k^T v_m = 0$.

Similarly, for the rows of the orthogonal matrix C, partial fraction expansion

$$\frac{1}{(2 + x\mu_k)(2 + x\mu_m)} = \frac{\frac{\mu_k}{2}}{(\mu_k - \mu_m)(2 + \mu_k x)} + \frac{\frac{\mu_m}{2}}{(\mu_m - \mu_k)(2 + \mu_m x)}$$

yields

$$\sum_{j=1}^{N} \frac{\alpha_j^2}{(2 + \rho_j \mu_k)(2 + \rho_j \mu_m)} = \frac{\frac{\mu_k}{2}}{(\mu_k - \mu_m)} \sum_{j=1}^{N} \frac{\alpha_j^2}{(2 + \mu_k \rho_j)} - \frac{\frac{\mu_m}{2}}{(\mu_k - \mu_m)} \sum_{j=1}^{N} \frac{\alpha_j^2}{(2 + \mu_m \rho_j)}$$

and the eigenvalue partial fraction (28) shows that the right-hand side equals zero.

B The resistance curvature vector p

It seems interesting[10] to determine the sign of the components of the resistance curvature vector $p = (p_1, p_2, \ldots, p_N)$. We derive several decompositions of the vector p, apart from

$$p = \frac{1}{2} \sum_{k=1}^{N-1} \left(\mu_k \zeta^T z_k\right) z_k + \frac{u}{N}$$

which directly follows from the definition $p = \frac{1}{2}Q\zeta + \frac{u}{N}$ in (3) and spectral decomposition $Q = \sum_{k=1}^{N-1} \mu_k z_k z_k^T$.

B.1 Signs of scalar products

(a) We use $p^T v_j = 2\sigma^2 \frac{u^T v_j}{\rho_j}$ in (11) into $p = \sum_{j=1}^{N} \left(p^T v_j\right) v_j$, then

$$\begin{aligned} p &= 2\sigma^2 \sum_{j=1}^{N} \frac{u^T v_j}{\rho_j} v_j \\ &= 2\sigma^2 \frac{u^T v_1}{\rho_1} v_1 - 2\sigma^2 \sum_{j=2}^{N} \frac{u^T v_j}{|\rho_j|} v_j \end{aligned}$$

Thus, only if $j = 1$, the $\text{sign}\left(p^T v_1\right) = \text{sign}\left(u^T v_1\right) > 0$ else $\text{sign}\left(p^T v_j\right) = -\,\text{sign}\left(u^T v_j\right)$. Introducing the definition $p = \frac{1}{2}Q\zeta + \frac{u}{N}$ in (3) shows that

$$p^T v_j = \frac{1}{2} \zeta^T Q v_j + \frac{u^T v_j}{N}$$

[10] Private communication with Karel Devriendt.

and (11) then indicates that

$$\zeta^T Q v_j = 2\left(\frac{2\sigma^2}{\rho_j} - \frac{1}{N}\right) u^T v_j$$

The upper bound $\frac{2\sigma^2}{\rho_1} > \frac{1}{N}$ in (53) illustrates that $\operatorname{sign}\left(\zeta^T Q v_1\right) > 0$, while $\operatorname{sign}\left(\zeta^T Q v_j\right) = -\operatorname{sign}\left(u^T v_j\right)$ for $2 \le j \le N$.

(b) Combining $\frac{\rho_j}{2\sigma^2} p^T v_j = u^T v_j$ in (11) and $\left(\rho_j - \frac{R_G}{N}\right) u^T v_j = N \zeta^T v_j$ in (12) shows that

$$\left(\rho_j - \frac{R_G}{N}\right) \rho_j = 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j}$$

We deduce for any $1 \le j \le N$ that $\operatorname{sign}\left(\zeta^T v_j\right) = \operatorname{sign}\left(p^T v_j\right)$ and $\frac{\zeta^T v_j}{p^T v_j} > 0$. Moreover, the equation also equals

$$\rho_j^2 - \frac{R_G}{N} \rho_j - 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j} = 0$$

The solution of the quadratic equation is

$$\rho_j = \frac{R_G}{2N} \pm \sqrt{\left(\frac{R_G}{2N}\right)^2 + 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j}}$$

Since $\frac{\zeta^T v_j}{p^T v_j} > 0$, only for $j = 1$ we must choose the positive sign, while for $j > 2$, the negative sign. Hence,

$$\rho_1 = \frac{R_G}{2N} + \sqrt{\left(\frac{R_G}{2N}\right)^2 + 2\sigma^2 N \frac{\zeta^T v_1}{p^T v_1}}$$

and, for $2 \le j \le N$,

$$\rho_j = -\sqrt{\left(\frac{R_G}{2N}\right)^2 + 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j}} + \frac{R_G}{2N}$$

B.2 Decomposition of p in the basis of eigenvectors of Ω

We further derive expressions for the scalar product $v_k^T p$. Left-multiplying the first quasi-eigenvalue equation (9) by v_k^T

$$v_k^T Q v_j = -\frac{2}{\rho_j} v_k^T v_j + \frac{2\left(u^T v_j\right)}{\rho_j} v_k^T p$$

leads, after invoking orthogonality of eigenvectors $v_k^T v_j = \delta_{kj} = 1_{\{k=j\}}$, to

$$v_k^T p = \frac{\rho_j v_k^T Q v_j}{2\left(u^T v_j\right)} + \frac{1_{\{k=j\}}}{u^T v_j}$$

Thus, if $k = j$, then the scalar product is

$$v_j^T p = \frac{\frac{1}{2}\rho_j v_j^T Q v_j + 1}{u^T v_j} \tag{70}$$

else

$$v_k^T p = \frac{\rho_j v_k^T Q v_j}{2\left(u^T v_j\right)} \tag{71}$$

Either after using (11) in (71) and (71) or after left-multiplying the second quasi-eigenvalue equation (10) by v_k^T, that leads to

$$v_k^T Q v_j = -\frac{2}{\rho_j} v_k^T v_j + \frac{1}{\sigma^2} \left(p^T v_j\right) \left(v_k^T p\right)$$

we obtain, for $k = j$,

$$\left(v_j^T p\right)^2 = \sigma^2 \left(v_j^T Q v_j + \frac{2}{\rho_j}\right) \tag{72}$$

else

$$v_k^T Q v_j = \frac{1}{\sigma^2} \left(p^T v_j\right) \left(v_k^T p\right) \tag{73}$$

Equation (73) determines the sign of $p^T v_j$ and $p^T v_k$ in relation to that of $v_k^T Q v_j$.

The positive semi-definiteness [12, art. 102] of the Laplacian Q states that $z^T Q z = \sum_{l \in \mathcal{L}} \left(z_{l^+} - z_{l^-}\right)^2 \geq 0$ for any vector z. Thus, the quadratic form $v_j^T Q v_j \geq 0$ is non-negative for any eigenvector v_j of the effective resistance matrix Ω. We know [12, p. 181-182] that the resistance curvature vector p is the eigenvector of the matrix $Q\Omega$ belonging to the zero eigenvalue.

Lemma 5 *The resistance curvature vector p is not an eigenvector of the effective resistance matrix Ω of a non-resistance regular graph.*

Proof: The condition $v_k^T p = 0$ for a certain integer k implies, by orthogonality of the eigenvectors $v_k^T v_j = \delta_{kj}$, that the vector p must be an eigenvector of the effective resistance matrix Ω belonging to an eigenvalue $j \neq k$. It suffices to show that $v_j^T p = 0$ is not possible in a non-resistance regular graph. For a non-resistance regular graph, the condition that $v_j^T p = 0$ in (70) or (72) implies that $\rho_j = -\frac{2}{v_j^T Q v_j} < 0$, which is only positive if $j > 1$, because then the eigenvalues $\rho_j < 0$. It also means that $v_j^T Q v_j = -\frac{2}{\rho_j} = v_j^T \Omega v_j$, which is only possible in "resistance regular graphs". Alternatively, $v_k^T p = 0$ in (71) is only possible if $v_k^T Q v_j = 0$, which means that either $v_k = \frac{u}{\sqrt{N}}$ or $v_j = \frac{u}{\sqrt{N}}$. The latter is, again, only possible in "resistance regular graphs". $\square$

Since the vector p can be written as a linear combination of the eigenvectors of the effective resistance matrix Ω,

$$p = \sum_{k=1}^{N} \left(p^T v_k\right) v_k = \left(v_j^T p\right) v_j + \sum_{k=1; k \neq j}^{N} \left(p^T v_k\right) v_k$$

we obtain, for any integer $1 \leq j \leq N$, the decomposition of the resistance curvature vector p in the basis of eigenvectors $v_1, v_2, \ldots, v_N$,

$$p = \left(\frac{\frac{1}{2}\rho_j v_j^T Q v_j + 1}{u^T v_j}\right) v_j + \frac{\rho_j}{2\left(u^T v_j\right)} \sum_{k=1; k \neq j}^{N} \left(v_k^T Q v_j\right) v_k \tag{74}$$

Since $\rho_1 > 0 > \rho_2$, it is convenient to choose $j = 1$ and (74) becomes

$$p = \left(\frac{\frac{1}{2}\rho_1 v_1^T Q v_1 + 1}{u^T v_1}\right) v_1 + \frac{\rho_1}{2\left(u^T v_1\right)} \sum_{k=2}^{N} \left(v_k^T Q v_1\right) v_k \tag{75}$$

where the prefactors $\frac{\frac{1}{2}\rho_1 v_1^T Q v_1 + 1}{u^T v_1}$ and $\frac{\rho_1}{2(u^T v_1)}$ are positive, because the Perron-Frobenius Theorem [12, art. 269] of non-negative matrices states the components of the principal eigenvector v_1 are all positive in a connected graph G.

We end with the computation of we $v_m^T Q^l v_j$ for any positive real l with (15),

$$Q^l v_j = \alpha_j \sum_{k=1}^{N-1} \frac{\mu_k^{1+l} \left(\zeta^T z_k\right)}{2 + \rho_j \mu_k} z_k$$

Then, with $v_m^T z_k = \frac{\frac{2}{\rho_m}\frac{\mu_k}{2}}{\frac{2}{\rho_m}+\mu_k} \left(u^T v_m\right)\left(\zeta^T z_k\right)$ from (21), we find that[11]

$$v_m^T Q^l v_j = \left(u^T v_m\right) \alpha_j \sum_{k=1}^{N-1} \frac{\mu_k^{2+l} \left(\zeta^T z_k\right)^2}{\left(2 + \rho_j \mu_k\right)\left(2 + \rho_m \mu_k\right)}$$

[11] For $m = j$, it follows that $v_j^T Q v_j = \left(u^T v_j\right)^2 \sum_{k=1}^{N-1} \frac{\mu_k^3 \left(\zeta^T z_k\right)^2}{\left(2+\rho_j \mu_k\right)^2} > 0$, because $\left(u^T v_j\right) \alpha_j = \alpha_j^2 \geq 0$ and that with (64)

$$2L = \sum_{j=1}^{N} v_j^T Q v_j = \sum_{j=1}^{N} \sum_{k=1}^{N-1} \frac{\mu_k^3 \left(u^T v_j\right)^2 \left(\zeta^T z_k\right)^2}{\left(2 + \rho_j \mu_k\right)^2}$$