Stochastic Model for the Number of Traversed Routers in Internet

P. Van Mieghem, G. Hooghiemstra and R. W. van der Hofstad

Abstract— **Previously we have computed the probability distribution of the hopcount in the class of random graphs $G_p(N)$ with exponential link weights. The applicability of that asymptotic law of the hopcount has been extended to other classes of random graphs and other link weight distributions. In addition, the asymptotic law agrees well with Internet measurements. After a short review of the model that lead to the asymptotic law for the hopcount, we demonstrate that the asymptotic law possesses the remarkable property of *almost sure behavior*. Almost sure behavior means that each histogram obtained by measuring the hopcount from a source to several (but enough) destinations will closely resemble the asymptotic law.**

Keywords— **hopcount, asymptotic law, Internet, graph.**

I. Introduction

In an abundant amount of recent articles on networking, authors feel the necessity to use or to introduce the concept of Quality of Service (QoS) as a justification for their work. Anybody familiar with networking knows how intricate QoS problems are and how few proposals really contribute to their solutions. Since the introduction of QoS in ATM about ten years ago, the QoS problem seems to have shifted to and penetrated in the Internet, where, initially, the multiple parameter QoS problem was absent. This articles aims to enhance the understanding of the end-to-end variations in QoS qualifiers such as delay, jitter and packet loss. These QoS qualifiers can be argued as dependent on the number of routers traversed since most of the QoS degradation occurs in routers (apart from mobile networks that suffer from anomalies of the transmission medium such as fading, reflections, etc.). Hence, we consider a good understanding of the distribution of the hopcount, defined as the number of traversed routers along the shortest path between a source and destination, as a first step to estimate end-to-end QoS behavior.

Perhaps the determination of the hopcount in the Internet can be regarded as one of the simplest measurements due to the large availability in unix-kernels of the `traceroute` utility. Unfortunately, as shown in [3], precise and reliable determination of the hopcount seems less obvious. In addition, up till now, no underlying theoretical model for the distribution of the hopcount is known. Here we present a model for the hopcount and demonstrate the applicability of that model to the Internet by illustrating that the distribution (1) possesses the remarkable property of almost sure behavior. The latter implies that (1) features a high degree of robustness.

Delft University of Technology, Information Technology and Systems, P.O. Box 5031, 2600 GA Delft, The Netherlands. Email: P.VanMieghem@its.tudelft.nl, G. Hooghiemstra@its.tudelft.nl and R.W.vanderHofstad@its.tudelft.nl

II. Model for the hopcount in Internet

Previously in [6] and [7], we have presented an asymptotic expression (1) for the probability density function (pdf) of the hopcount of the shortest path derived from random graphs (see e.g. [2] and [5]) of the class $G_p(N)$ with exponentially or uniformly distributed link weights. The class of random graphs $G_p(N)$ consists of all graphs with N nodes in which the links are chosen independently and with probability p. Hence, $p = \frac{2E[L]}{N(N-1)}$ is link density where L is the (variable) number of links and $p(N-1)$ is the average number of links per node. In order to compute the shortest path between two nodes, the weight of each link must be specified and exponentially distributed link weights (with mean 1) have been chosen initially. Under these model assumptions, we have shown that the distribution of the hopcount can be well represented by the distribution of the depth in a uniform recursive tree and that it is given, for large N, by

$$\Pr[h_N = k] = \frac{(1+o(1))}{N} \sum_{m=0}^{k} c_{m+1} \frac{\ln^{k-m} N}{(k-m)!}, \qquad (1)$$

where c_m are the Taylor coefficients of $\frac{1}{\Gamma(z)}$ listed in [1, 6.1.34].

We present a brief explanation of this basic result. We have demonstrated in [7] that the shortest path problem in the complete graph K_N (the case where $p = 1$ in $G_p(N)$) with exponentially distributed link weights can be exactly reformulated into a Markov discovery process. The hopcount of the shortest path, deduced from the Markov discovery process, is precisely equal to the depth of a node in a uniform recursive tree (see Figure 2) rooted at the source. Denote by $X_N^{(k)}$ the number of nodes with hopcount k in the uniform tree of size N and by y_N the hopcount of a randomly chosen node, possibly equal to the root. Denote by $p_N^{(k)}$ the probability that a randomly chosen node has hopcount k, then

$$\Pr[y_N = k] = p_N^{(k)} = \frac{E\left[X_N^{(k)}\right]}{N}. \qquad (2)$$

We prove in [7] that, for $N \geq 1$ and $1 \leq k \leq N-1$, the probability $p_N^{(k)}$ satisfies the recursion

$$p_N^{(k)} = \frac{1}{N} \sum_{m=k}^{N-1} p_m^{(k-1)}$$

and that the corresponding generating function equals

$$\varphi_N(x) = \sum_{k=0}^{N-1} p_N^{(k)} x^k = \frac{\Gamma(N+x)}{\Gamma(N+1)\Gamma(x+1)}.$$

From this generating function, the probability that a uniformly chosen node in that tree has hopcount k can be written as

$$p_N^{(k)} = \frac{(-1)^{N-1-k} S_N^{(k+1)}}{N!} \quad (3)$$

where $S_N^{(k)}$ denotes the Stirling number of the first kind [1, 24.1.3]. Finally, we are interested in the hopcount y_N excluding the event $y_N = 0$, which means that, for $k \geq 1$,

$$\begin{aligned} \Pr[h_N = k] &= \Pr[y_N = k | y_N \neq 0] \\ &= \frac{N}{N-1} \Pr[y_N = k] \end{aligned}$$

with corresponding generating function,

$$\begin{aligned} \varphi_N(x) &= \sum_{k=1}^{N-1} \Pr[h_N = k]\, x^k \\ &= \frac{N}{N-1} \left(\varphi_N(x) - \frac{1}{N} \right) \end{aligned}$$

Expanding $\varphi_N(x)$ in a Taylor series around $x = 0$, finally leads, for large N, to the asymptotic law (1).

In [6] and [7], we have extended this result obtained for the complete graph to almost all connected graphs of the class $G_p(N)$. In particular, we have shown that, for large N, the hopcount is independent of the link density p, insensitive to the precise details of the topology (as the results also hold for Waxman graphs, another class of random graphs) but it varies with the link weight distribution. Besides uniform and exponentially distributed link weights, the asymptotic law can be extended to polynomially distributed link weights. Furthermore, the model has been extended to multiple parameter routing which is the basis for QoS routing, have been obtained. In the latter, every link is specified by a link weight vector with several components, each independent and uniformly distributed.

The degree (or number of neighbors) per node in the Internet is very low (about 3 to 5 on average). The extension of the asymptotic law (1) in [6] to almost every connected graph of $G_p(N)$ has been rigorously proved under the restriction that $\frac{Np_N}{\log^3 N} \to \infty$, where $p = p_N$ depends on the number of nodes N. This condition still implies that, in the limit $N \to \infty$, the average number of neighbors Np_N grows unboundedly. While this result is apparently not realistic, we note that if $Np_N \sim \log N$, the average degree per node returns 'realistic' numbers around 10 for the currently estimated size of the Internet ($N \approx 10^5$). By simulation in [7], we have shown that the asymptotic law (1) still holds if the average degree is further pushed down: $Np_N \to \lambda$, with λ a large constant. Since the number of router ports is limited even if the number of routers (nodes) N increases, the asymptotic law (1) with constant λ can serve to model shortest paths between source and arbitrary destination in the Internet. Finally, the asymptotic expression (1) has been compared with Internet measurements in [7] and in [3]. The agreement was surprisingly good.

Although the Internet is not a random graph of the class $G_p(N)$, the results deduced in the way presented here can be understood as follows. First of all, any graph is a subgraph of the complete graph, also the graph of the Internet. Next, we have randomly thinned the complete graph in two different ways: by altering the structure via p and by superimposing weights on the links. Any graph where communication takes place between arbitrary nodes using a shortest path algorithm can be obtained in this way, by erasing from the complete graph the appropriate links that are not present in the real graph, and putting the right weights on the available links. Next, the focus on the shortest paths starting from a destination node A towards an arbitrary node B in the network leads us to consider a shortest path tree. Only links of this shortest path tree matter for the hopcount and a large number of links in the topology seems superfluous. Thus, by confining to the shortest path, we filter the actual topology to a tree rooted at A that is dependent on the link weights. This explains the apparent negligible influence on the details of the topology and underlines the importance of the link weight distribution. It also shows that information about the hopcount alone is insufficient to construct the Internet topology: not the number of links from a given node matters but the number of links *with small weights*.

In [8], we have proposed a general theory to compute the efficiency of multicast over unicast, defined as the number of links in a shortest path tree to m multicast group members. Again for random graphs, exact formulas for the efficiency were derived. Comparison with Internet measurements have shown that shortest path trees computed via random graphs are quite accurate. This can be regarded as another, independent verification of the quality of shortest path result based on random graph models.

III. Almost sure behavior.

The remarkable agreement with Internet measurements is surprising as explained below, not because the model would be speculative, but, because it suggests an additional property of the hopcount in Internet, namely, almost sure behavior. There are two different approaches to compute the probability distribution function (pdf) of the hopcount of a path from A to B. Either we fix the topology and vary the source and destination over all possible couples, or we choose a particular source and destination, and let the topology change over all possible graphs (e.g. all graphs in $G_p(N)$). The first approach suffers from the fact that the paths are not independent because of overlap. Since the influence of the correlation structure of this overlap on the hopcount is a priori difficult to estimate, in simulations and computation, the second approach had been followed.

However, the experimental trace-route measurements [3] are all performed on one fixed graph (the Internet graph) starting from one source (at Delft University of Technol-

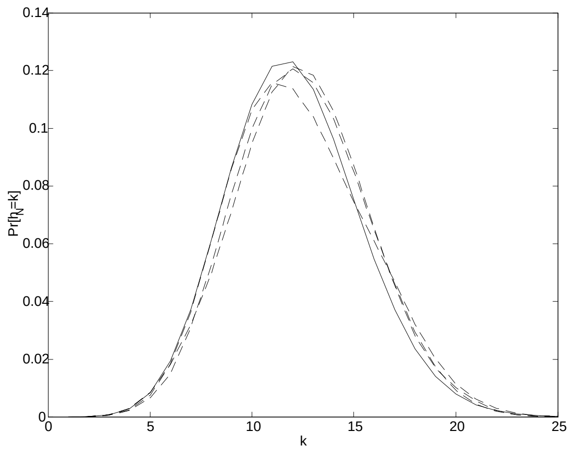

Fig. 1. Hopcount distributions obtained from a single instance of the class of random graphs (dashed line) together with the theoretical distribution law (1) (full line) for $N = 10^5$.

ogy) to a finite number of randomly chosen destinations (about 200). The good agreement between the model and the measurements points towards almost sure behavior. Almost sure behavior means that, for sufficiently large N, the distribution of the hopcount deduced from a source to a set of destinations in *any* random graph of the class $G_p(N)$ with exponentially (or uniformly) distributed weights is the same.

A. Simulations.

In order to motivate the theoretical results in the next section, we present here simulation results of the almost sure behavior. In each of the randomly generated graphs of the class $G_p(N)$ with exponentially distributed link weights, the distribution of the hopcount from an arbitrarily chosen source to each other node in that same graph has been computed.

In Figure 1, the pdf of the hopcount for three random graphs has been plotted for $N = 10^5$, which is estimated as roughly the size of the current Internet. The full line depicts the theoretical law (1). The correspondence of each distribution with the asymptotic law (1) is convincing.

B. Theory.

We will confine ourselves to analyzing the almost sure behavior in the complete graph with exponential weights. We expect that the almost sure behavior then can be extended in similar way as outlined above to almost all connected graphs of the class $G_p(N)$.

As an estimator for the average number of nodes at k hops from the source, we use

$$\hat{p}_N^{(k)} = \frac{X_N^{(k)}}{N}.$$

Clearly with (2), the mean of this estimator is $E\left[\hat{p}_N^{(k)}\right] = p_N^{(k)}$ so that the estimator is unbiased. From (3) or (1), we observe that, for any k, $p_N^{(k)} \to 0$, if $N \to \infty$. Almost sure behavior would follow if, for large N, the variance of the estimator $\hat{p}_N^{(k)}$, $E\left[\left(\hat{p}_N^{(k)}\right)^2\right] - \left(p_N^{(k)}\right)^2$ tends to zero more rapidly than $\left(p_N^{(k)}\right)^2$. Equivalently, the condition for almost sure behavior is $var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right] = o(1)$ so that

$$\lim_{N\to\infty} \frac{\hat{p}_N^{(k)}}{p_N^{(k)}} = 1$$

in probability.

In order to compute the variance $var\left[\hat{p}_N^{(k)}\right]$, we will take benefit of properties of the uniform recursive tree.

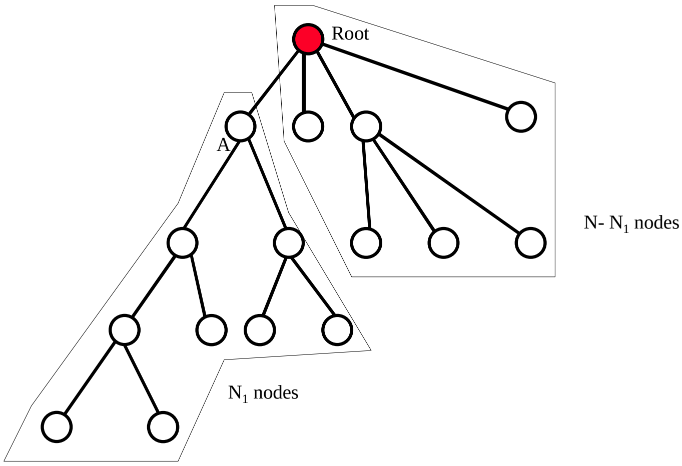

Fig. 2. The recursion relation for a uniform recursive tree.

As illustrated in Figure 2, we grow uniform recursive trees by starting with a single node (the root). During the construction procedure, each new (i.e. not yet placed) node of the set of $N-1$ nodes, has equal probability to be connected to any of the already attached nodes. We observe that the cluster connected to the root (via node A in Figure 2) is again a uniform recursive tree of size N_1. This size N_1 is a uniform random variable in $1, \cdots, N-1$. Moreover, also the tree connected to the root obtained by deleting this cluster of size N_1 is a uniform recursive tree of size $N - N_1$. These observations lead to the relation that

$$X_N^{(k)} = X_{N_1}^{(k-1)} + X_{N-N_1}^{(k)}, \tag{4}$$

where the latter two random variables are conditionally independent given N_1. Hence, we arrive at

$$E\left[\left(X_N^{(k)}\right)^2\right] = \frac{1}{N-1}\sum_{m=1}^{N-1} E\left[\left(X_m^{(k-1)} + X_{N-m}^{(k)}\right)^2\right].$$

The cross term can be computed analytically

$$a[k,N] \quad = \quad \frac{2}{N-1}\sum_{m=1}^{N-1} E\left[X_m^{(k-1)}\right] E\left[X_{N-m}^{(k)}\right]$$

$$= \frac{2}{N-1}\sum_{m=1}^{N-1}\frac{(-1)^{m-k}S_m^{(k)}}{(m-1)!}\frac{(-1)^{N-m-1-k}S_{N-m}^{(k+1)}}{(N-m-1)!}$$
$$= \frac{2(-1)^{N-1}}{(N-1)!}\sum_{m=1}^{N-1}\binom{N-2}{m-1}S_m^{(k)}S_{N-m}^{(k+1)}.$$

Denoting $s[k,N] = E\left[\left(X_N^{(k)}\right)^2\right]$ and taking into account that $X_N^{(k)} = 0$ if $k \geq N$, the following recursion must be solved

$$s[k,N] = \frac{1}{N-1}\sum_{m=k}^{N-1}s[k-1,m] + \frac{1}{N-1}\sum_{m=k+1}^{N-1}s[k,m] + a[k,N],$$

with initial condition $s[0,N] = 1$. By inspection, we find as solution

$$s[k,N] = \frac{(-1)^{N-k-1}}{(N-1)!}\sum_{j=0}^{\min[k,N-k-1]}(-1)^j\binom{2j}{j}S_N^{(k+1+j)}. \tag{5}$$

Explicitly, for the highest values of k, we have

$$s[N-1,N] = \frac{1}{(N-1)!}$$
$$s[N-2,N] = \frac{\binom{N}{2}+2}{(N-1)!} \qquad N \geq 3$$

The solution (5) can be rewritten with (3) as

$$E\left[\left(X_N^{(k)}\right)^2\right] = N\sum_{j=0}^{\min[k,N-k-1]}\binom{2j}{j}p_N^{(j+k)} \tag{6}$$

which shows that all terms in the summation are positive. Using (6), the variance

$$var\left[\hat{p}_N^{(k)}\right] = \frac{1}{N^2}E\left[\left(X_N^{(k)}\right)^2\right] - \left(p_N^{(k)}\right)^2$$

gives

$$var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right] = \frac{1}{N}\sum_{j=0}^{\min[k,N-k-1]}\binom{2j}{j}\frac{p_N^{(j+k)}}{\left(p_N^{(k)}\right)^2} - 1 \tag{7}$$

It remains to demonstrate that the right hand side of (7) tends to zero as $N \to \infty$. Numerical data indeed shows that $var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right]$ decreases monotonous for any computed large N (verified up to $N = 1000$) and fixed k. Also if $k = [\log N]$ and N large , $var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right]$ decreases as a function of N per region where $k = [\log N]$ is constant. Note that $[x]$ denotes the largest integer smaller than or equal to x.

For extreme values of k, such as $k = N-1$, we find that

$$var\left[\frac{\hat{p}_N^{(N-1)}}{p_N^{(N-1)}}\right] = (N-1)! - 1$$

which demonstrate that almost sure behavior characterized via $var\left[\frac{\hat{p}_N^{(N-1)}}{p_N^{(N-1)}}\right] = o(1)$, does not hold for all k. The precise k-region for which $var\left[\frac{\hat{p}_N^{(N-1)}}{p_N^{(N-1)}}\right]$ decreases towards zero as $N \to \infty$ is still unknown. The k-region of practical interest, however, extends roughly from $k = 1$ to $k = E[h_N] + c\sqrt{var[h_N]} = \log N + c\sqrt{\log N}$, where c is a small constant, say $c < 10$.

For $k = o(\log N)$, we can apply the asymptotic expression [1, 24.1.3.III],

$$\left|S_N^{(k)}\right| \sim (N-1)!\frac{(\gamma + \log N)^{k-1}}{(k-1)!} \tag{8}$$

into (7),

$$var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right] = \sum_{j=0}^{\min[k,N-k-1]}\binom{2j}{j}\frac{\left|S_N^{(k+1+j)}\right|(N-1)!}{\left(S_N^{(k+1)}\right)^2} - 1$$
$$\sim \frac{(k!)^2}{(\gamma+\log N)^k}\sum_{j=0}^{k}\binom{2j}{j}\frac{(\gamma+\log N)^j}{(k+j)!} - 1$$
$$= \frac{(k!)^2}{(\gamma+\log N)^k}\sum_{j=0}^{k-1}\binom{2j}{j}\frac{(\gamma+\log N)^j}{(k+j)!}$$
$$= \sum_{q=1}^{k}\binom{2(k-q)}{k-q}\frac{(k!)^2}{(2k-q)!}\frac{1}{(\gamma+\log N)^q}$$

Clearly, if k is fixed, the latter shows that $var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right] \sim O\left(\frac{1}{\log N}\right)$. Let $b_k = \binom{2(k-q)}{k-q}\frac{(k!)^2}{(2k-q)!}$. The ratio of the two successive coefficients in the q-series equals

$$\frac{b_q}{b_{q+1}} = 2\frac{2k-2q-1}{2k-q} < 2$$

For sufficiently large k, we have $\frac{b_q}{b_{q+1}} \sim 2$ or $b_q \sim b_1\left(\frac{1}{2}\right)^{q-1}$ with $b_1 = \binom{2(k-1)}{k-1}\frac{(k!)^2}{(2k-1)!} = \frac{k^2}{2k-1}$

$$var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right] \sim b_1\sum_{q=1}^{\infty}\frac{1}{(2(\gamma+\log N))^q}$$
$$= \frac{b_1}{2(\gamma+\log N)-1} \sim \frac{k}{4\log N}$$

Since the asymptotic expression (8) is only valid for $k = o(\log N)$, we arrive at $var\left[\frac{\hat{p}_N^{(k)}}{p_N^{(k)}}\right] = o(1)$. So far, we have demonstrated almost sure behavior for all $k = o(\log N)$. The remaining region (from $k = o(\log N)$ to $k = \log N + c\sqrt{\log N}$) is expected to obey the condition for almost sure behavior. Simulations up to $N = 1000$ are confirming the trend, but a rigorous proof is still lacking.

IV. Conclusion.

A model for the number of traversed routers in the Internet has been reviewed. The good agreement with various measurement of the hopcount has suggested that the asymptotic probability density function of the hopcount (1) is likely to possess the remarkable property of almost sure behavior. Such behavior implies that *any* hopcount distribution in the Internet deduced by measurement from a source to various destinations will satisfy (1) closely. In this sense, the asymptotic law can be regarded as robust against small changes in the model assumptions.

The major contribution of this article lies in the theoretical demonstration that the probability distribution of the hopcount indeed possesses the property of almost sure behavior. The result has been shown for hopcounts $k = o(\log N)$. In addition, almost sure behavior has only been demonstrated for the complete graph with exponentially distributed link weights, but we claim that the results also holds for almost all connected random graphs of the class $G_p(N)$ and that by a similar coupling argument in [6] the claim is very likely to be proved.

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