# Spectral Graph Theory for Dynamic Processes on Networks

Piet Van Mieghem

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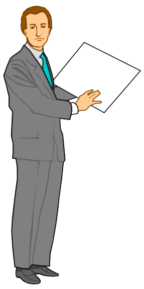
## Outline



Dynamic Processes on Networks

Modifying the spectral radius

Principal eigenvector

New (?) graph theoretic results

Summary

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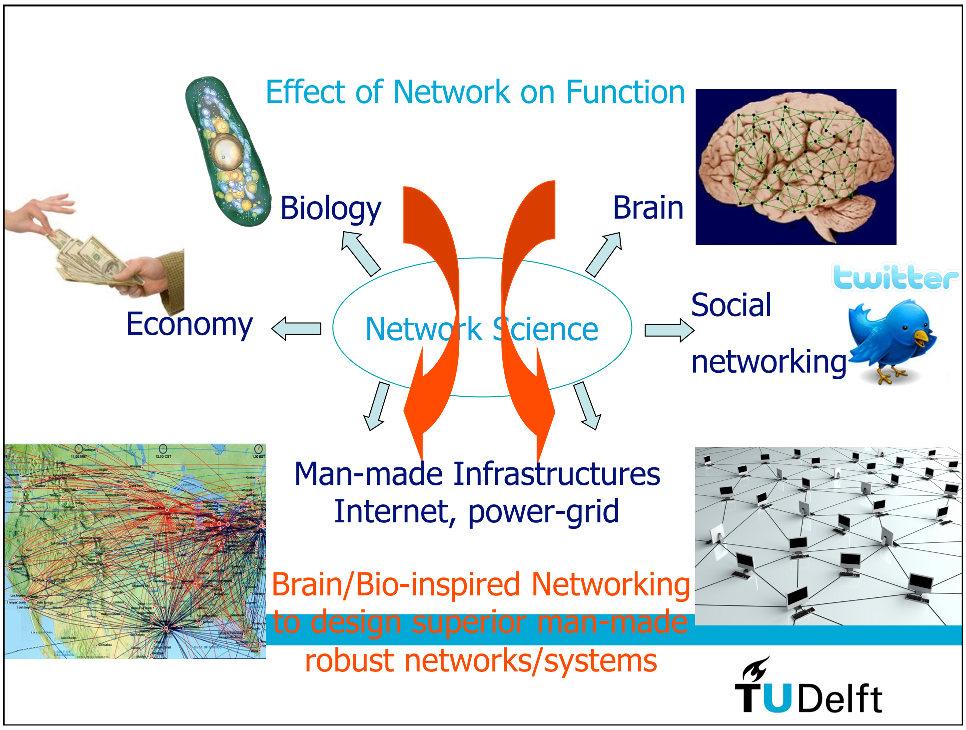
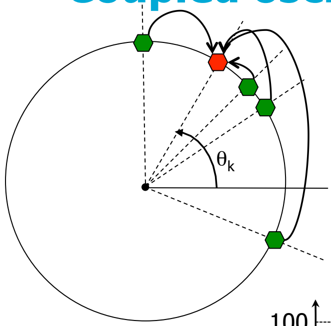



# Coupled oscillators (Kuramoto model)

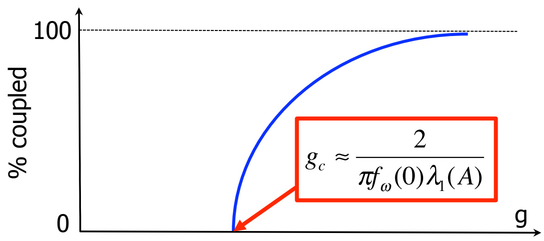



Interaction equals sums of sinus of phase difference of each neighbor:

$$\dot{\theta}_k = \omega_k + g\sum_{j=1}^{N} a_{kj}\sin(\theta_j - \theta_k)$$

coupling strength

natural frequency

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J. G. Restrepo, E. Ott, and B. R. Hunt. Onset of synchronization in large networks of coupled oscillators, Phys. Rev. E, vol. 71, 036151, 2005

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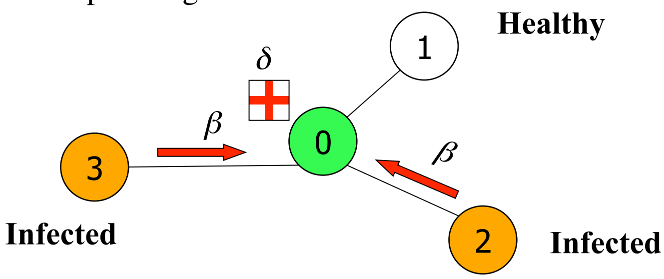
## Simple SIS model

- Homogeneous birth (infection) rate $\beta$ on all edges between infected and susceptible nodes
- Homogeneous death (curing) rate $\delta$ for infected nodes

$\tau = \beta / \delta$ : effective spreading rate



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## Epidemics in networks: modeling & immunization

network protection

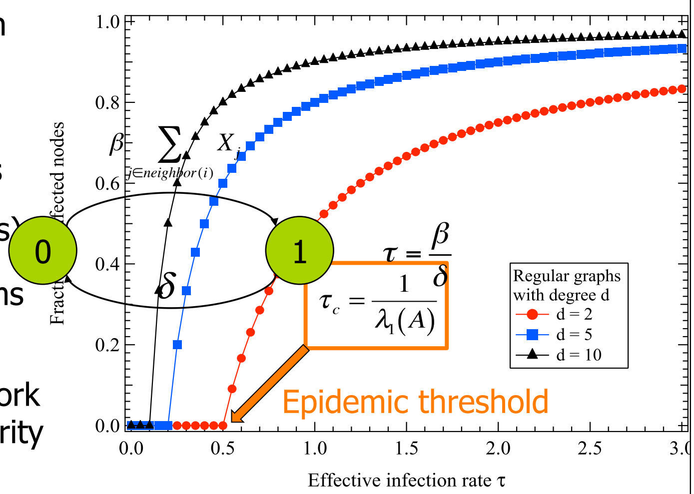
self-replicating objects (worms)

propagation errors

rumors (social nets)

epidemic algorithms (gossiping)

cybercrime : network robustness & security






- Van Mieghem, P., J. S. Omic and R. E. Kooij, 2009, "Virus Spread in Networks", IEEE/ACM Transaction on Networking, Vol. 17, No. 1, February, pp. 1-14.
- Van Mieghem, P., 2011, "The N -Intertwined SIS epidemic network model", Computing (Springer), Vol. 93, Issue 2, p. 147-169.

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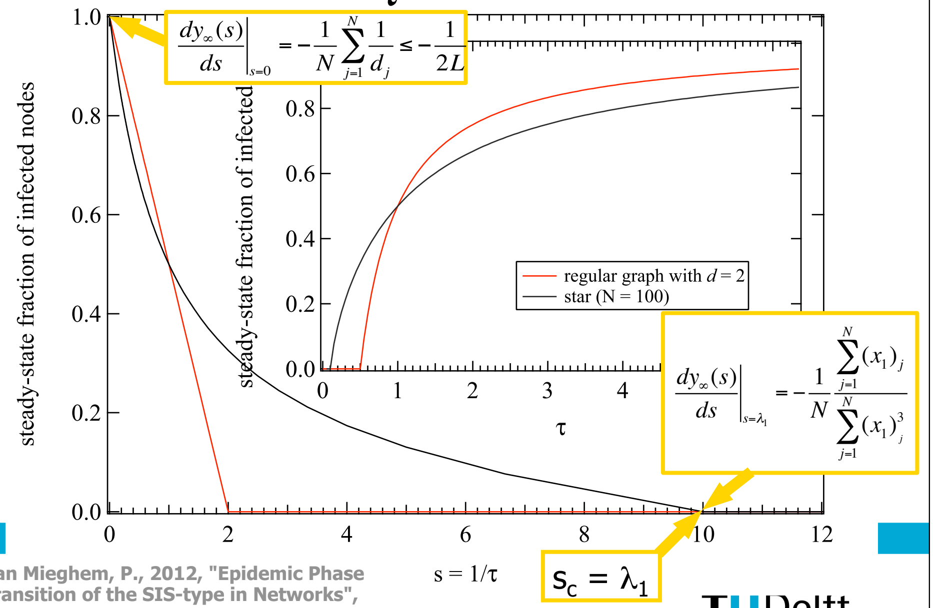
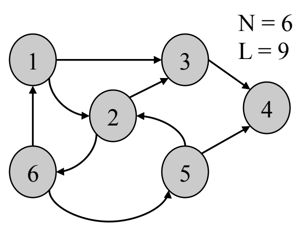
# Transformation $s = \frac{1}{\tau}$ & principal eigenvector




Van Mieghem, P., 2012, "Epidemic Phase Transition of the SIS-type in Networks", Europhysics Letters (EPL), Vol. 97, Februari, p. 48004.

# Algebraic graph theory: notation

Any graph *G* can be represented by an adjacency matrix *A* and an incidence matrix *B*, and a Laplacian *Q*




$$A_{N\times N} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} = A^T$$

$$B_{N\times L} = \begin{bmatrix} 1 & 1 & -1 & \ldots & 0 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & -1 \\ 0 & 0 & 1 & & 1 \end{bmatrix}$$

$$Q = BB^T = \Delta - A$$

$$\Delta = diag(d_1 \quad d_2 \quad \ldots \quad d_N)$$

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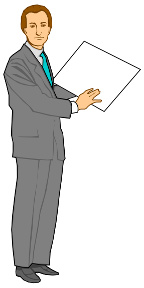

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## Affecting the epidemic threshold

- Degree-preserving rewiring (assortativity of the graph)
  - Van Mieghem, P., H. Wang, X. Ge, S. Tang and F. A. Kuipers, 2010, "Influence of Assortativity and Degree-preserving Rewiring on the Spectra of Networks", The European Physical Journal B, vol. 76, No. 4, pp. 643-652.
  - Van Mieghem, P., X. Ge, P. Schumm, S. Trajanovski and H. Wang, 2010, "Spectral Graph Analysis of Modularity and Assortativity", Physical Review E, Vol. 82, November, p. 056113.
  - Li, C., H. Wang and P. Van Mieghem, 2012, "Degree and Principal Eigenvectors in Complex Networks", IFIP Networking 2012, May 21-25, Prague, Czech Republic.
- Removing links/nodes (optimal way is NP-complete)
  - Van Mieghem, P., D. Stevanovic, F. A. Kuipers, C. Li, R. van de Bovenkamp, D. Liu and H. Wang, 2011,"Decreasing the spectral radius of a graph by link removals", Physical Review E, Vol. 84, No. 1, July, p. 016101.
- Quarantining and network protection
  - Omic, J., J. Martin Hernandez and P. Van Mieghem, 2010, "Network protection against worms and cascading failures using modularity partitioning", 22nd International Teletraffic Congress (ITC 22), September 7-9, Amsterdam, Netherlands.
  - Gourdin, E., J. Omic and P. Van Mieghem, 2011, "Optimization of network protection against virus spread", 8th International Workshop on Design of Reliable Communication Networks (DRCN 2011), October 10-12, Krakow, Poland.

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E.R. van Dam, R.E. Kooij, The minimal spectral radius of graphs with a given diameter, *Linear Algebra and its Applications,* 423, 2007, pp. 408-419.

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## Assortativity

Reformulation of Newman's definition into algebraic graph theory

$$\rho_D = \frac{E\left[D_{l^+}D_{l^-}\right] - E\left[D_{l^+}\right]E\left[D_{l^-}\right]}{\sqrt{Var\left[D_{l^+}\right]Var\left[D_{l^+}\right]}} = \frac{N_1 N_3 - N_2^2}{N_1 \sum_{j=1}^{N} d_j^3 - N_2^2}$$

where $N_k = u^T A^k u$ is the total number of walks with $k$ hops:

$$N_0 = \sum_{j=1}^{N} d_j^0 = N, N_1 = \sum_{j=1}^{N} d_j^1 = 2L, N_2 = d^T d = \sum_{j=1}^{N} d_j^2 \qquad N_k \le \sum_{j=1}^{N} d_j^k$$

A network is (degree) *assortative* if $\rho_D > 0$

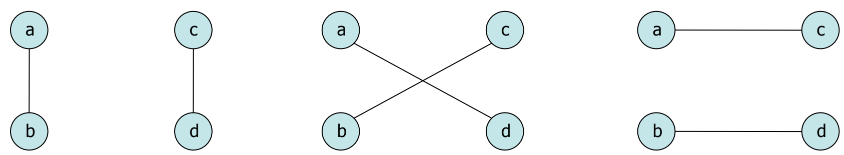
A network is (degree) *disassortative* if $\rho_D < 0$



Van Mieghem, P., H. Wang, X. Ge, S. Tang and F. A. Kuipers, 2010,"Influence of Assortativity and Degree-preserving Rewiring on the Spectra of Networks", The European Physical Journal B, vol. 76, No. 4, pp. 643-652.



## Degree-preserving rewiring




$$\rho_D = 1 - \frac{\sum_{i \sim j} \left(d_i - d_j\right)^2}{\sum_{j=1}^{N} d_j^3 - \frac{1}{2L}\left(\sum_{j=1}^{N} d_j^2\right)^2}$$

→ only two terms change

↓

degree-preserving rewiring algorithm

# Bounds largest eigenvalue adjacency matrix

Classical bounds: $E[D] = \frac{2L}{N} \leq \lambda_1(A) \leq d_{\max}$ $\quad \sqrt{d_{\max}} \leq \lambda_1(A) \leq d_{\max}$

Walks based: $\lambda_1(A) \geq \left(\frac{N_{2k}}{N}\right)^{1/(2k)} \geq \left(\frac{N_k}{N}\right)^{1/k}$

$$k = 2 : \lambda_1(A) \geq \left(\frac{d^T d}{N}\right)^{1/2} = \frac{2L}{N}\sqrt{1 + \frac{Var[D]}{(E[D])^2}}$$

$$k = 3 : \lambda_1^3(A) \geq \frac{N_3}{N} = \frac{1}{N}\left(\rho_D\left(\sum_{j=1}^{N} d_j^3 - \frac{N_2^2}{N_1}\right) + \frac{N_2^2}{N_1}\right)$$

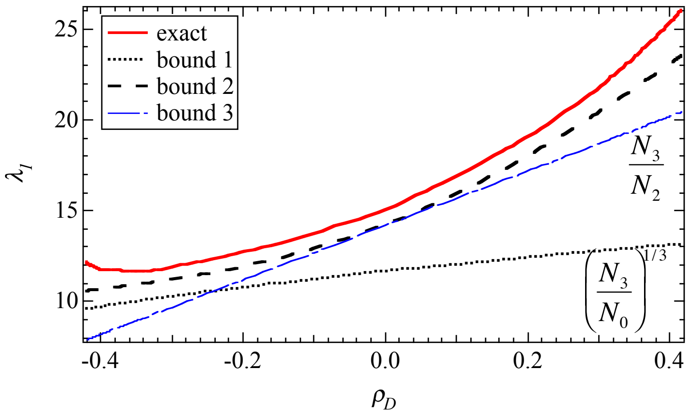
Optimized: $\lambda_1(A) \geq \frac{NN_3 - N_1N_2 + \sqrt{(NN_3)^2 - 6NN_1N_2N_3 + others}}{2(NN_2 - N_1^2)}$



**P. Van Mieghem, Graph Spectra for Complex Networks, Cambridge University Press, 2011**

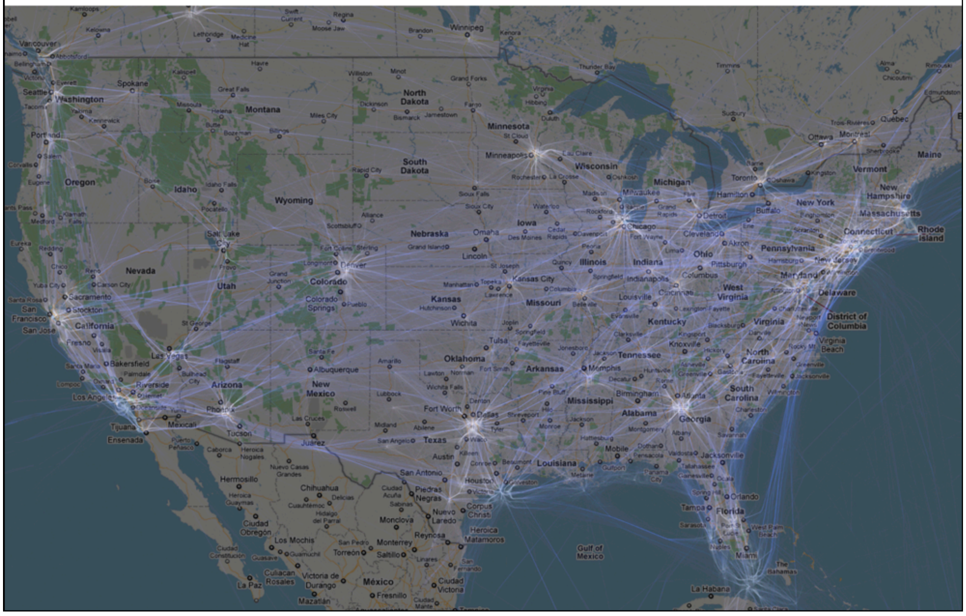


# Comparison

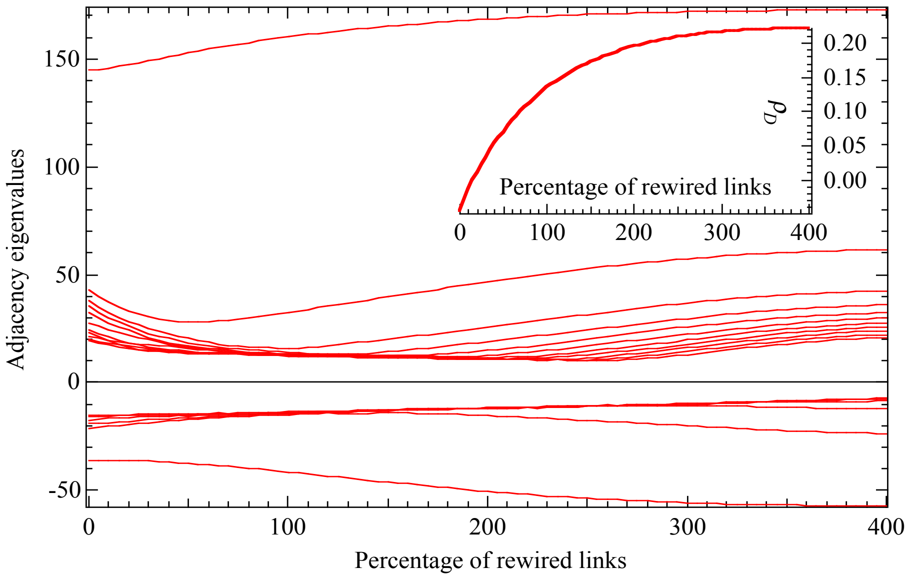



Barabasi-Albert power law graph with $N$ = 500 and $L$=1920

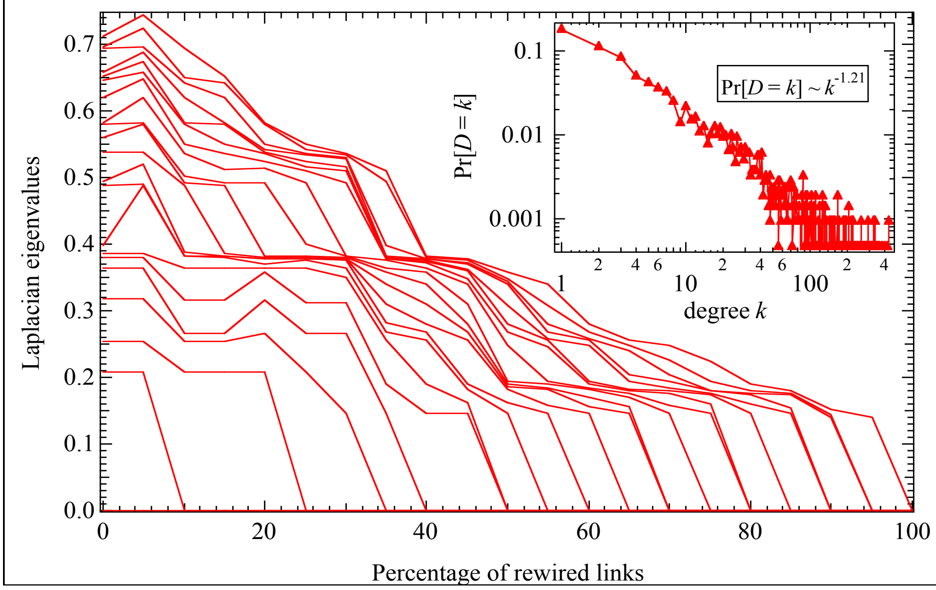
## USA air transportation network



## Degree-preserving rewiring USA air transport network: adjacency eig.



## Degree-preserving rewiring USA air transport network: Laplacian eig.




## Decreasing the spectral radius

- Which set of *m* removed links (or nodes) decreases the spectral radius most? **[NP-complete problem]**
- Scaling law:

$$\lambda_1(A) - \lambda_1(A_m) = \frac{c_G m^\beta}{N}$$

- What is the best heuristic strategy?
  - remove the link *l* with maximum product of the eigenvector components: $x_{l+}\, x_{l-}$

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Van Mieghem, P., D. Stevanovic, F. A. Kuipers, C. Li, R. van de Bovenkamp, D. Liu and H. Wang, 2011,"Decreasing the spectral radius of a graph by link removals", Physical Review E, Vol. 84, No. 1, July, p. 016101.

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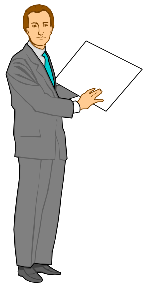

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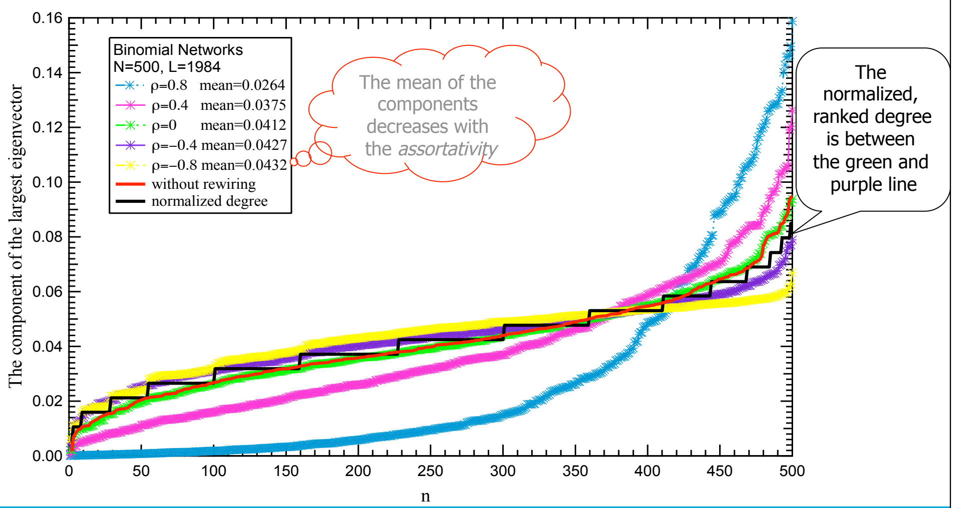
## Binomial networks




Li, C., H. Wang and P. Van Mieghem, 2012, "Degree and Principal Eigenvectors in Complex Networks", IFIP Networking 2012, May 21-25, Prague, Czech Republic.

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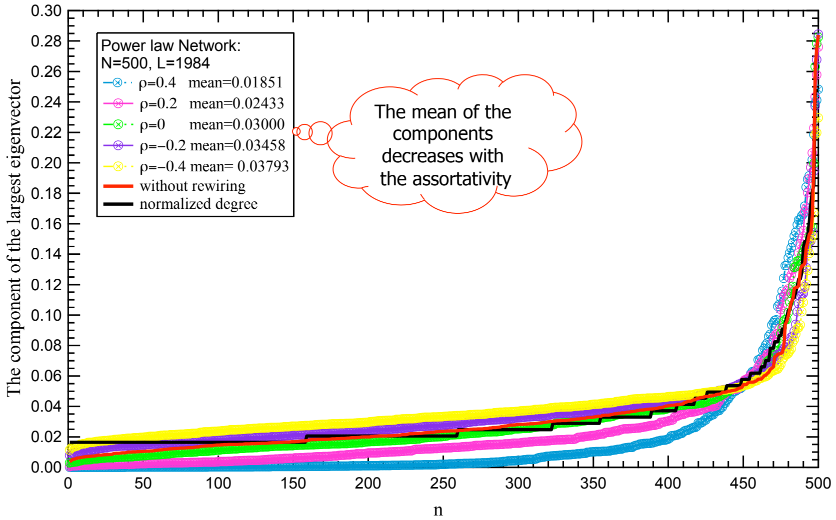
# Power law networks

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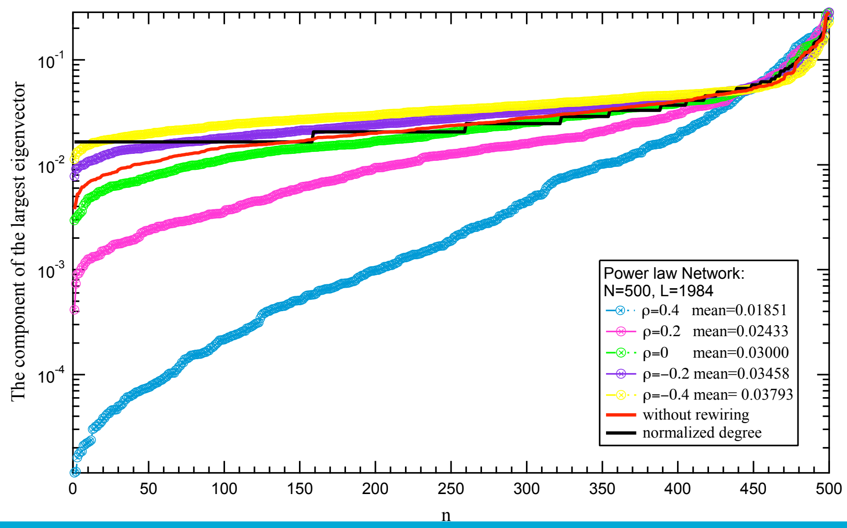
# Power law networks (log-scale)

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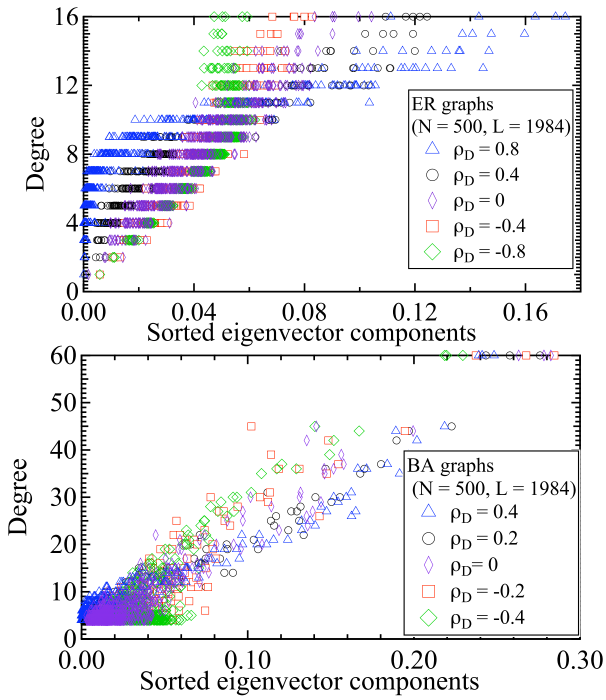


E[D] = 8

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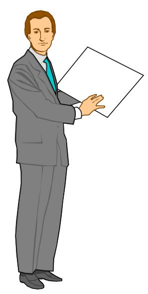

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## Upper bound principal eigenvector of A

Let $x$ denote the principal eigenvector of $A$ belonging to the largest eigenvalue $\lambda_1(A)$.
Let $N_m$ denote the set of nodes that are removed from G

$$\sum_{n\in N_m} x_n^2 \le \frac{1}{2}\left\{1+\frac{1}{\lambda_1(A)}\sum_{j\in N_m}\sum_{i\in N_m} a_{ij}x_i x_j\right\}$$

If $m = N$, equality holds

If $m = 1$, then $x_n \le \frac{\sqrt{2}}{2}$

S.M. Cioaba, A necessary and sufficient eigenvector condition for a connected graph to be bipartite, Electron. J. Linear Algebra (ELA) 20 (2010) 351–353.



Li, C., H. Wang and P. Van Mieghem, 2012, "Bounds for the spectral radius of a graph when nodes are removed", Linear Algebra and its Applications, vol. 437, pp. 319-323.



## Lower bound principal eigenvector of A

Let $\prod_{j=1;j\neq m}^{n}(z-x_j)=\sum_{k=0}^{n-1} b_k(m)z^k$ and all eigenvalues are different

Then

$$x_m x_m^T = \frac{1}{\prod_{k=1;k\neq m}^{N}(\lambda_m-\lambda_k)}\sum_{k=0}^{N-1} b_k(m)A^k = \frac{\prod_{k=1;k\neq m}^{N}(A-\lambda_k I)}{\prod_{k=1;k\neq m}^{N}(\lambda_m-\lambda_k)}$$

This follows after inversion of $A^k=\sum_{j=1}^{N}\lambda_k x_k x_k^T$ using the inversion of the Vandermonde matrix, which can be related to Lagrange's interpolation theorem

Corollary: for any connected graph, it holds that

$$(x_1)_j \ge \max\left(\frac{\sqrt{2}}{\prod_{k=2}^{N}(\lambda_1-\lambda_k)}\max_{1\le j\le N}\sum_{k=H_{ij}}^{N-1} b_k(1)\left(A^k\right)_{ij},\ \sqrt{\frac{1}{e\lambda_1^{N-1}}\sum_{k=H_{ij}}^{N-1} b_k(1)\left(A^k\right)_{jj}}\right) > 0$$

## Spectral identity

For any symmetric matrix A

$$\det\left(A-\lambda I\right)=\frac{\det\left(A_{\backslash\{i\}}-\lambda I\right)\det\left(A_{\backslash\{k\}}-\lambda I\right)-\left(\det\left(A_{\backslash row_i col_k}-\lambda I\right)\right)^2}{\det\left(A_{\backslash\{i,k\}}-\lambda I\right)}$$

*Is this identity new?*

Using this identity, we can deduce

$$(x_k)_j^2=\frac{\det\left(A_{\backslash\{j\}}-\lambda_k I\right)}{\sum_{n=1}^{N}\det\left(A_{\backslash\{n\}}-\lambda_k I\right)}$$



**P. Van Mieghem, Graph Spectra for Complex Networks, Cambridge University Press, *second edition in preparation.***

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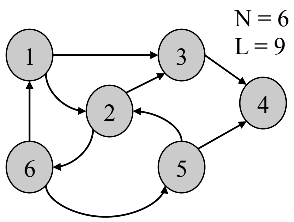
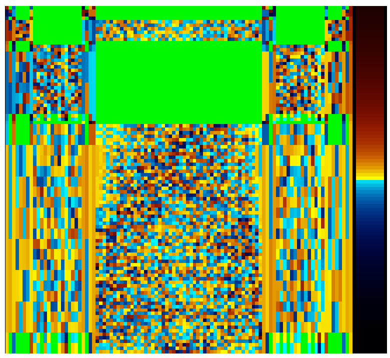
## Topology domain

## Spectral domain

$$A=X\Lambda X^T$$




Most network problems:
- shortest path
- graph metrics
- network algorithms



$X_{3\text{-ary tree}}$ : green = 0



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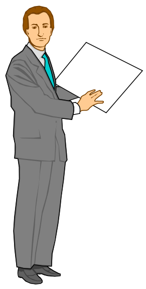
## Spectral invariants

- The number of eigenvector components with positive (negative) sign and those with zero value
- Numerical results (Javier Martin-Hernandez):
  - about 50 % of the eigenvector components are positive
  - invariant? due to orthonormal matrix properties?
  - What does it mean?





## Outline



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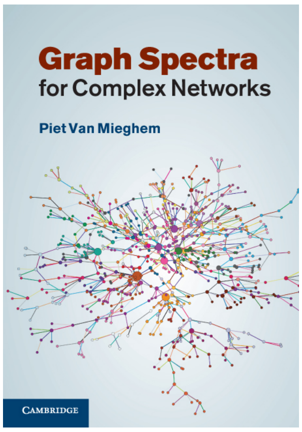
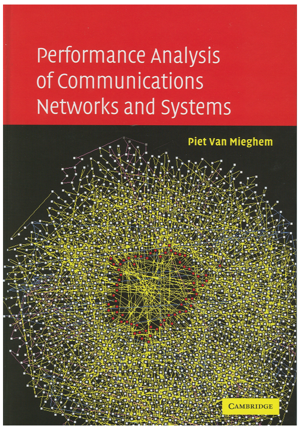
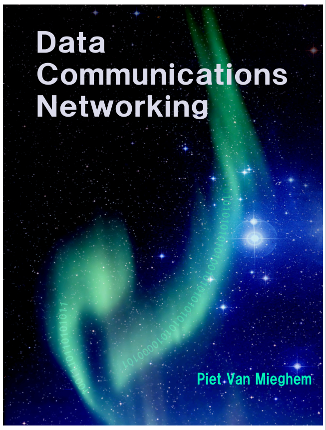
- *Real epidemics*: phase transition at $\tau_c > 1/\lambda_1$, but the **N-Intertwined mean-field approximation** is very useful in network engineering via spectral analysis
- Epidemic threshold engineering:
  - Degree-preserving *assortative* rewiring increases $\lambda_1$, while degree-preserving *disassortative* rewiring decreases $\lambda_1$
  - Removing links/nodes to maximally decrease $\lambda_1$ is NP-hard
- What do eigenvalues and eigenvectors "physically" mean?



## Books

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Thank You
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