Nash Equilibria in Shared Effort Games

Gleb Polevoy
Delft University of Technology
Mekelweg 4, 2628 CD
Delft, The Netherlands
G.Polevoy@tudeflt.nl

Stojan Trajanovski
Delft University of Technology
Mekelweg 4, 2628 CD
Delft, The Netherlands
S.Trajanovski@tudelft.nl

Mathijs de Weerdt
Delft University of Technology
Mekelweg 4, 2628 CD
Delft, The Netherlands
M.M.deWeerdt@tudelft.nl

ABSTRACT

Shared effort games model people's contribution to projects and sharing the obtained profits. Those games generalize both public projects like writing for Wikipedia, where everybody shares the resulting benefits, and all-pay auctions such as contests and political campaigns, where only the winner obtains a profit. In θ-equal sharing (effort) games, a threshold for effort defines which contributors win and then receive their (equal) share. (For public projects $\theta = 0$ and for all-pay auctions $\theta = 1$.) Thresholds between 0 and 1 can model games such as paper co-authorship and shared homework assignments. First, we fully characterize the conditions for the existence of a pure-strategy Nash equilibrium for two-player shared effort games with close budgets and project value functions that are linear on the received contribution and prove some efficiency results. Second, since the theory does not work for more players, fictitious play simulations are used to show when such an equilibrium exists and what its efficiency is. The results about existence and efficiency of these equilibria provide the likely strategy profiles and the socially preferred strategies to use in real life situations of contribution to public projects.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Social and Behavioral Sciences**]: Economics; K.4.4 [**Computers and Society**]: Electronic Commerce

General Terms

Economics, Theory

Keywords

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1. INTRODUCTION

Many real-world situations include a set of people investing resources in several projects. The revenues from these projects are typically divided based on the individual investments. Examples of such situations include contributions to online communities [6], Wikipedia [5], political campaigns [14], paper co-authorship [7], or social exchange networks [8]. Another example is a worker in a company who is investing her time and obtaining some share from the total revenue of the company. We next look into some important special cases of these widespread interactions between resource investors.

The division of the obtained revenue may differ from one setting to another, such as sharing profits equally, proportionally, or in some other way. For instance, in several real life cases, not all contributors receive a (positive) share. Sharing with a minimum contribution threshold means that only the ones who contribute at least this threshold, get a share. This threshold is often present. This threshold might be fixed, but it may also depend on the investments in the project. Assigning bonus points to students from homework exercises is an interesting example, where one needs to achieve at least some percentage of the best grade, to obtain the homework's credits.[1] In this example, participants' utilities (i.e., their gain) are typically equal for anyone who is above the threshold. Another related example a Colonel Blotto game (see e.g. [13]). This example is "highly thresholded" because only the player whose effort per project is maximum collects the complete revenue. Another example is a company where an employee obtains decision power or revenue (e.g., an annual bonus) if she has invested sufficient effort [11]. In conclusion, the interactions where people invest resources in several projects and share the obtained revenues are common. Therefore, understanding them is important in order to know how to behave, such that the total revenue is maximized.

A shared effort game consists of a set of players and a set of projects. Each player has a budget to invest across a predefined subset of projects in any manner she desires. The utility of each player is the sum of the utilities that she obtains from each project, where the latter depends on the project's value. A project's value is a function of the investments of the players in a given project. In particular, we consider a θ-*sharing game* where a player receives utility from a project only if her investment is above a threshold defined by a θ fraction of the maximum investment. In this paper, we assume the project's value is always equally shared among players that invest above this threshold.

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[1]E.g.: `http://www.physics.umd.edu/courses/Phys121/Griffin/grading.html`

We now provide some detailed examples of such interactions. We use them throughout the paper for illustration.

Example 1. *Consider two collaborating scientists in a narrow field. They can work on some papers alone or together. When they collaborate, the second author has to contribute comparably with the first one, in order to be considered as an author.*

Example 2. *Consider a group of people, each having a budget of free time. The possible ways to spend this time are: writing for Wikipedia, coding for Linux, or playing volleyball. Each project yields some utility to its contributors. Specifically, writing to Wikipedia gives emotional satisfaction and respect; Linux provides all the rewards that Wikipedia does and also a plus for a CV; while playing volleyball yields emotional satisfaction and improves health. We assume that the total contribution of a project can be expressed by a single positive real number.*

Contributing to projects where only the maximum contributor to a project obtains the project's value has been considered in all-pay auctions. They can model lobbying, single-winner contests, political campaigns, striving for a job promotion (see e.g. [14]) and to Colonel Blotto games with two players [13]. One of the measures of efficiency is the price of anarchy, which is the total utility of the players in an equilibrium with least total utility relatively to the optimum total utility (see [9]). The price of anarchy was bounded in [2], but assuming some very specific conditions (such as N-approximate Vickrey conditions). However, there is no analysis of the existence of Nash Equilibria (NE) and their efficiency in the general effort sharing interaction, where several people invest in projects and share the values of these projects. As we have seen, these interactions are of large importance. Therefore, the existence and the efficiency of equilibria should be analyzed to recommend socially optimum behavior to the contributors.

Our paper aims to fill this gap. We do the following.

1. Characterize the existence of NE in a partial case.
2. Theoretically study its efficiency, when it exists. By efficiency we mean the optimum total utility divided by the total utility of a NE.
3. Generalize fictitious play[2] to shared effort (infinite) games.
4. Provide a best response algorithm.
5. Simulate the fictitious play to find Nash Equilibria and when found - their efficiency.

A shared effort game is infinite[3] (even non-countable), the set of pure strategies being all the possible splits of each player's budget to the projects where she may contribute. We consider only pure equilibria throughout the paper, even when we do not mention this explicitly.

Our research is organized as follows. After formally defining shared effort games, we treat the existence and efficiency of NE theoretically for a subset of games. To investigate existence and efficiency of NE in some other cases, we develop and employ fictitious play simulations. That is, we define a sequence of plays we call Infinite-Strategy Fictitious Play and simulate it till and if it converges. Eventually, we check whether this is a NE and if that is the case, what its efficiency is. The results improve the understanding of the possible situations that are likely to arise in real life and suggest strategies to play, in order to maximize the social welfare.

[2]The original fictitious play was proposed by Brown [3].
[3]I.e. the number of possible strategies is infinite.

2. MODEL

To model the situation we have described, we now define *shared effort games*, that also appeared in [2]. There are n players $N = \{1, \ldots, n\}$ and a set of projects Ω. Each player $i \in N$ can contribute to projects in Ω_i, where $\emptyset \subsetneq \Omega_i \subseteq \Omega$; the contribution of player i to project $\omega \in \Omega_i$ is denoted by $x^i_\omega \in \mathbb{R}_+$. Each player i has a budget $B_i > 0$, and the strategy space of player i (i.e., the set of her possible actions) is $\left\{x^i = (x^i_\omega)_{\omega \in \Omega_i} \in \mathbb{R}_+^{|\Omega_i|} | \sum_{\omega \in \Omega_i} x^i_\omega \leq B_i\right\}$. The strategies of all the players except i is denoted x^{-i}.

To define the utilities, each project $\omega \in \Omega$ is associated with its *project function*, which determines its *value*, based on the total contribution vector $x_\omega = (x^i_\omega)_{i \in N}$ that it receives; formally, $P_\omega(x_\omega)\colon \mathbb{R}^n_+ \to \mathbb{R}_+$. We assume that every P_ω is increasing and differentiable, in every parameter. Sometimes we give an example of a project function that is a function of a single parameter, like $P_\omega(x) = 2x$. In those cases, we assume that project functions P_ω depend only on the $\sum(x^i_\omega)_{i \in N}$, which can also be denoted by x_ω when it is clear from the context. However, in general $P_\omega(x_\omega)\colon \mathbb{R}^n_+ \to \mathbb{R}_+$, unless otherwise stated. The project's value is distributed among the players in $N_\omega \triangleq \{i \in N | \omega \in \Omega_i\}$ according to the following rule. From each $\omega \in \Omega_i$, each player i gets a share $\phi^i_\omega(x_\omega)\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ with free disposal:

$$\forall \omega \in \Omega : \sum_{i \in N_\omega} \phi^i_\omega(x_\omega) \leq P_\omega(x_\omega). \quad (1)$$

We assume that the sharing functions are non-decreasing.

We denote the vector of all the contributions by $x = (x^i_\omega)^{i \in N}_{\omega \in \Omega}$. The utility of a player $i \in N$ is defined to be

$$u^i(x) \triangleq \sum_{\omega \in \Omega_i} \phi^i_\omega(x_\omega).$$

We now define a specific variant of a shared effort game, called a *θ-sharing mechanism.* This variant is both relevant to many applications and used is the theoretical part. Define, $\forall \theta \in [0, 1]$, the players who get a share

$$N^\theta_\omega \triangleq \left\{i \in N_\omega | x^i_\omega \geq \theta \cdot \max_{j \in N_\omega} x^j_\omega\right\},$$

that is those who bid at least θ fraction of the maximum bid size to ω. We then define the θ-equal sharing mechanism, where the project's value is equally divided between all the users, who contribute at least θ of the maximum bid to the project.

Definition 1. *The* θ-equal sharing mechanism*, denoted by* M^θ_{eq}*, is*

$$\phi^i_\omega(x_\omega) \triangleq \begin{cases} \frac{P_\omega(x_\omega)}{|N^\theta_\omega|} & \text{if } i \in N^\theta_\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Reconsider the example from Section 1 in this model. In example 1, the scientists are the players and papers are the projects. Assume that a paper's total contribution to the reputation of its authors is proportional to the total investment in the paper. That is, the project's functions are linear. In order to be considered an author, a minimum threshold of the maximum contribution is required, and a paper's total contribution to the authors' reputation is equally divided between all its authors. This is a shared effort game with a threshold between 0 and 1 and equal sharing.

In example 2, the people are the players and Wikipedia, Linux and volleyball are the projects. To actually receive these contributions, a person has to invest at least some minimum contribution. - For instance, one cannot efficiently contribute to Wikipedia without initially learning the writing style and the platform. That is, the threshold are positive. The utilities obtained from each project are not equally divided, but in a more complicated manner. This is a shared effort game with a positive threshold and some complicated sharing functions.

3. THEORY OF NASH EQUILIBRIUM

We study the existence of NE, and when it exists we consider its price of anarchy and stability. First, we consider some simple sufficiency results. Then, we provide a characterization of the existence of NE for a special case and some efficiency results. In the next section, simulations help analyzing some cases that are not treated theoretically. Recall that we consider only pure equilibria throughout the paper.

First, we obtain the following.

Theorem 1. *Suppose the strategy sets are non-empty, compact and convex. Then, if each ϕ^i_ω is continuous and concave, then a pure NE exists. If we additionally suppose that the strategy sets are equal to all the payers (in particular, all Ω_is are the same) and that the utility functions are symmetric, then we also conclude that a symmetric NE exists above.*

Proof. Immediate from Proposition 20.3 in [12] (and of Theorem 3 in [4], for the symmetric case). □

For the non-symmetric case of M^0_{eq}, a stronger result can be proven.

Theorem 2. *The game M^0_{eq} admits a potential. If the functions P_ω are continuous and the strategy spaces are compact, then a pure NE exists.*

Proof. Define $P\colon Y \to \mathbb{R}$ by $P(x_\omega) \triangleq \sum_{\omega \in \Omega} \frac{P_\omega(x_\omega)}{|N_\omega|}$. This is a potential function. the game possesses a pure NE, whenever the potential function admits the maximum. In our case, as these functions are continuous and the spaces are compact, they always achieve the maximum (see Lemma 4.3 in [10]). □

3.1 Characterization for Two Players with Close Budgets

We now consider the simple case of two players, where all the project functions are linear with coefficients $\alpha_m \geq \alpha_{m-1} \geq \ldots \geq \alpha_1$. We denote the number of projects with the largest coefficient project functions by $k \in \mathbb{N}$, i.e. $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$. We call those projects *steep*. Assume w.l.o.g. that $B_2 \geq B_1$.

Our goal is to characterize the existence of a NE. We shall need some definitions. We call a project that receives no contribution in a given profile a *vacant* project.

Definition 2. *A player is* dominated *at a project ω, if it belongs to the set $D_\omega \triangleq N_\omega \setminus N^\theta_\omega$. A player is* suppressed *at a project ω, if it belongs to the set $S_\omega \triangleq \left\{i \in N_\omega : x^i_\omega > 0\right\} \setminus N^\theta_\omega$. That is, a player who is contributing to a project but is dominated there.*

In a NE a player is suppressed at a project if and only if it is suppressed at any project where it contributes. This is true since if a player is suppressed at project p but it also contributes to project $q \neq p$ and is not suppressed there, then it would like to move its contribution to p to project q.

We introduce several lemmas, before formulating and proving the characterization. The lemmas describe what must happen in any NE.

Lemma 1. *Consider an equal θ-sharing game with two players with budgets B_1, B_2, w.l.o.g., $B_2 \geq B_1$. Assume $0 < \theta < 1$, and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).*

Then the following hold in any NE.

1. *At least one player contributes to a steep project.*
2. *Suppose that a non-suppressed player, contributing to a steep project, contributes to a non-steep project as well. Then, it contributes either alone or precisely the least amount it should contribute to achieve a portion in the project's value.*
3. *Suppose we have a NE where both players contribute to steep projects and are not suppressed. Then, they never contribute to the same non-steep project together.*

Proof. At least one of the players contributes to a steep project, for the following reasons. If only the non-steep projects receive a contribution, then take any such project p. If a single player contributes there, then this player would benefit from moving to contribute to a vacant steep project. If both players contribute to p, then if one is suppressed, it would like to deviate to any project where it would not be suppressed, and if no-one is suppressed, then a player who contributes not less would like to contribute to a vacant steep project instead.

We prove part 2 now. Let $i \in N$ be any non-suppressed player among those who contribute to a steep project, w.l.o.g. – to project m. Assume first that player $j \neq i$ is not suppressed. Then, for any non-steep project where i contributes, it contributes either alone or precisely the least amount it should contribute to achieve a portion in the project's value, because otherwise i would like to increase its contribution to m on the expense of decreasing its contribution to the considered non-steep project.

Now, consider the case where j is suppressed. Then, even if j contributes to a non-steep project where i contributes (and is suppressed there), i still will prefer to move some budget from this project to m, since i receives the whole value of m as well. Thus, this cannot be a NE.

To prove part 3, notice that part 2 implies that if both such players contribute to the same non-steep project, each contributes precisely the least amount it should contribute to achieve a portion in the project's value. This contradicts the assumption that $\theta < 1$. □

LEMMA 2. *Consider an equal θ-sharing game with two players with budgets B_1, B_2. W.l.o.g., $B_2 \geq B_1$. Assume $0 < \theta < 1$, and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).*

If $B_1 \geq \theta B_2$, then the following hold in any NE.

1. *Each player contributes to every steep project.*
2. *A non-steep project receives the contribution of at most one player.*

Proof. Since $B_1 \geq \theta B_2$, no player is suppressed, because a suppressed player would prefer to contribute to a project where it would not be suppressed, and at any project, a player which concentrates all its value there is not suppressed.

Every steep project receives a positive contribution from each player, since otherwise, the player who does not contribute to some steep project, will profit from contributing there exactly the threshold value, while leaving at least the threshold values at all the projects where it contributed. There is always a sufficient surplus to reach the threshold because $B_1 \geq \theta B_2$.

We next prove the second part of the lemma. Since both players are non-suppressed contributors to steep projects, then, according to part 2 in Lemma 1, we conclude that there exist no non-steep projects where j and i contribute together. □

We shall need another definition.

DEFINITION 3. *A* 2-steep *project is a project that is the most profitable among the non-steep ones.*

We are now ready to characterize the existence of a NE in shared effort games with two players.

THEOREM 3. *Consider an equal θ-sharing game with two players with budgets B_1, B_2. W.l.o.g., $B_2 \geq B_1$. Assume $0 < \theta < 1$, and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g). For $B_1 \geq \theta B_2$, this game has a pure strategy NE if and only if the following both hold.*[4]

1. $\frac{1}{2}\alpha_{m-k+1} \geq \alpha_{m-k}$,
2. $B_1 \geq k\theta B_2$;

The idea of the proof is as follows. To show existence of an equilibrium under the assumptions of the theorem, we just give a strategy profile and prove that no unilateral deviation is profitable. We show the other direction by assuming that a given profile is a NE and deriving the asserted conditions. To do this, we first use the lemmas we have just presented in order to limit the possibilities for an equilibrium profile.

We prepend the following technical lemma.

LEMMA 3. *Consider an equal θ-sharing game with two players with budgets B_1, B_2. W.l.o.g., $B_2 \geq B_1$. Assume $0 < \theta < 1$, and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g). Assume that no player is suppressed anywhere, and player j does not contribute to a non-steep project p. Consider player $i \neq j$.*

Then, the following hold.

1. *If $\frac{1}{2}\alpha_m \geq \alpha_p$, then it is not profitable for i to move any budget $\delta > 0$ from any subset of the steep projects to p (or to a set of such non-steep projects).*
2. *If $\frac{1}{2}\alpha_m > \alpha_p$, then it is (strictly) profitable for i to move any budget $\delta > 0$ from p to any subset of the steep projects. If j is suppressed after such a move, then requiring $\frac{1}{2}\alpha_m \geq \alpha_p$ is enough.*
3. *If $\frac{1}{2}\alpha_m < \alpha_p$ and it is possible to move $\delta > 0$ from any subset of the steep projects to p, such that i received and still receives half of the value of these steep projects, then it is (strictly) profitable for i.*

Proof. Before moving, player i obtains in total $\sum_{q \in \Omega} (\frac{1}{2}\text{or } 1)\alpha_m \cdot (x_q^1 + x_q^2)$.

We begin by proving part 1. Assume $\frac{1}{2}\alpha_m \geq \alpha_p$. If i moves $\delta > 0$ from the steep projects to p, then its utility from the steep projects decreases by at least $0.5\alpha_m\delta$, and its utility from p increases by $\alpha_p\delta$. The total change is $(-0.5\alpha_m + \alpha_p)\delta$, and since $\frac{1}{2}\alpha_m \geq \alpha_p$, this is non-positive.

We prove part 2 now. Moving δ from p to a subset of the steep projects decreases the utility of i by $\alpha_p\delta$ and increases it by at least $0.5\alpha_m\delta$, and since $\frac{1}{2}\alpha_m > \alpha_p$, the sum of these is (strictly positive). If j is suppressed by such this move, then the increase is more than $0.5\alpha_m\delta$, thus requiring $\frac{1}{2}\alpha_m \geq \alpha_p$ is enough.

To prove part 3, assume that $\frac{1}{2}\alpha_m < \alpha_p$, and let $\delta > 0$ be an amount that is possible to take from some of the steep projects where i receives half of the value so as to keep receiving a half of the new value. Then, moving this δ to p decreases i's utility from the steep projects by $0.5\alpha_m\delta$ and its utility from p increases by $\alpha_p\delta$. The total change is $(-0.5\alpha_m + \alpha_p)\delta$, and since $\frac{1}{2}\alpha_m < \alpha_p$, this is (strictly) positive. □

We are now set to prove the theorem.

Proof. ($\Rightarrow$) We prove the existence of NE under the conditions of the theorem. Suppose that $B_1 \geq k\theta B_2$ and $\frac{1}{2}\alpha_{m-k+1} \geq \alpha_{m-k}$. Let both players allocate $1/k$th of their respective budgets to each of the steep projects. We prove now that this is a NE. With this profile, each player receives $k \cdot \frac{1}{2}\alpha_m \cdot \frac{B_1+B_2}{k} = \frac{1}{2}\alpha_m \cdot (B_1 + B_2)$. For player i, moving $\delta > 0$ to some non-steep projects is not profitable, according to part 1 of Lemma 3. Another possible deviation is reallocating budget among the steep projects. Since $B_1 \geq k\theta B_2$, we conclude that $B_2 \leq \frac{B_1}{k\theta}$, so 2 is not able to suppress 1 (and the other way around is impossible, even more so) and therefore - only reallocating among the steep projects will not increase the profit. The only deviation that remains to be considered is a simultaneous allocating $\delta > 0$ to some non-steep projects and reallocating the rest of the budget among the steep ones. Any such potentially profitable deviation can be looked at as two consecutive deviations: first allocating $\delta > 0$ to some non-steep projects, and then reallocating the rest of the budget among the steep ones. Part 1 of Lemma 3 shows that bringing back all $\delta > 0$ from non-steep projects to the steep ones, without getting suppressed anywhere (which is possible since $B_1 \geq \theta B_2$) will always bear a

[4]If α_{m-k} does not exist, consider the containing condition to be vacuously true.

non-negative profit. Therefore - we can ignore the last form of deviations. Therefore, this is a NE.

($\Leftarrow$) We show the other direction now. We now assume the existence of a NE and derive the conditions of the theorem. Assume that a given profile is a NE.

Since $B_1 \geq \theta B_2$, then according to Lemma 2, each player contributes to every steep project. Suppose to the contrary that $\frac{1}{2}\alpha_{m-k+1} < \alpha_{m-k}$. Let i be a player that contributes to m more than its threshold there, and let j be the other player. Then, by part 3 of Lemma 3, all non-steep projects with coefficients larger than $0.5\alpha_m$ must get positive contribution from j, for otherwise i would profit by transferring there part of its budget from m. Therefore, the non-steep projects with coefficients larger than $0.5\alpha_m$ receive no contribution from i, according to Lemma 2.

Therefore, for all the steep projects - player j contributes exactly its threshold value, while i contributes above it. Also, i contributes nothing to any non-steep project: we have shown this for the non-steep projects with coefficients larger than $0.5\alpha_m$, now we show for the rest. If i contributed to a non-steep project with coefficient at most $0.5\alpha_m$, it would be profitable for him to deviate to a steep one, according to part 2 of Lemma 3 (when the coefficient is exactly $0.5\alpha_m$, we use the fact that j would be suppressed by such a deviation).

We assume that $B_1 \geq \theta B_2$, and thus, for any $i \neq j$ we have

$$\theta B_j \leq B_i \iff B_j - \theta B_i \leq \frac{B_i}{\theta} - \theta B_i$$
$$\iff B_j - \theta B_i \leq \frac{B_i - \theta^2 B_i}{\theta}.$$

Thus, a non-steep project with coefficients larger than $0.5\alpha_m$ receives from j at most $\frac{B_i - \theta^2 B_i}{\theta}$, and since i can transfer to that project $B_i - \theta^2 B_i$ without losing a share at the steep projects, i can transfer exactly θ-share of j's contribution there and profit thereby. This profitable deviation contradicts our assumption and we conclude that $\frac{1}{2}\alpha_{m-k+1} \geq \alpha_{m-k}$.

It is left to prove that $B_1 \geq k\theta B_2$. According to part 2 of Lemma 3, there are no contributions to non-steep projects, since they would render the deviation to the steep projects profitable, unless $\frac{1}{2}\alpha_{m-k+1} = \alpha_{m-k}$, in which case a 2-steep project can get a positive investment from one player. Thus, the players' utility is at most the same as when each steep project obtains contributions from both players, and other projects receive nothing. Thus, each player's utility is at most $k \cdot (\alpha_m/2)(\frac{B_1+B_2}{k}) = (\alpha_m/2)(B_1+B_2)$. If 2 could deviate to contribute all B_2 to a steep project while suppressing 1 there, player 2 would obtain $\alpha_m(B_2+y)$, for some $y > 0$. This is always profitable, since

$$B_2 \geq B_1 \Rightarrow B_2 + 2y > B_1$$
$$\iff \alpha_m(B_2+y) > (\alpha_m/2)(B_1+B_2).$$

Thus, since we are in a NE, 2 may not be able to suppress i and therefore $B_2 \leq \frac{B_1}{k}\frac{1}{\theta} \Rightarrow B_1 \geq k\theta B_2$. And we have proved that the conditions of the theorem hold. $\square$

The proof of the necessity of the conditions of this theorem relays on the lemmas that describe structure of a NE, that are not easily generalized for $n > 2$. However, some of the sufficiency conditions can be proven analogously for a general n.

THEOREM 4. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$, $0 < \theta < 1$ (the order is w.l.o.g.), and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).*

This game has a pure strategy NE if $B_{n-1} \geq \theta B_n$ and the following both hold.[5]

1. $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$,
2. $B_1 \geq k\theta B_n$;

Proof. It is analogous to the proof for $n = 2$. All the players equally divide their budgets among all the steep projects. $\square$

Since there may be various Nash Equilibria, it is important to analyze their efficiency. We consider the price of anarchy (PoA) that is the ratio of the worst NE's efficiency to the optimum possible one, and the price of stability (PoS) that is the ratio of the best NE's efficiency to the optimum possible one.

THEOREM 5. *Consider an equal θ-sharing game with two players with budgets B_1, B_2. W.l.o.g., $B_2 \geq B_1$. Assume $0 < \theta < 1$, and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).*[6]

Assume that $B_1 \geq \theta B_2$ and the following both hold.

1. $\frac{1}{2}\alpha_{m-k+1} > \alpha_{m-k}$,
2. $B_1 \geq k\theta B_2$;

Then, there exists a pure strategy NE and there holds: PoA = PoS = 1.

Proof. According to the proof of Theorem 3, equally dividing all the budgets between the steep projects is a NE. Therefore, PoS = 1.

Consider any NE. By Lemma 2, each player contributes to all steep projects and if it contributes to a non-steep project, then it is the only contributor there. Then, according to part 2 of Lemma 3, there can be no contribution to a non-steep project, since a deviation to a steep project would be profitable. Therefore, PoA = 1. We have fully proven the theorem. $\square$

We now generalize some of the efficiency results for a general $n \geq 2$.

THEOREM 6. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$, $0 < \theta < 1$ (the order is w.l.o.g.), and linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).*[7]

Assume that $B_{n-1} \geq \theta B_n$ and the following both hold.

1. $\frac{1}{n}\alpha_{m-k+1} > \alpha_{m-k}$,
2. $B_1 \geq k\theta B_n$;

[5]If α_{m-k} does not exist, consider the containing condition to be vacuously true.

[6]If α_{m-k} does not exist, consider the containing condition to be vacuously true.

[7]If α_{m-k} does not exist, consider the containing condition to be vacuously true.

Then, there exists a pure strategy NE and there holds: $\mathrm{PoS} = 1$.

Proof. According to proof of Theorem 4, equally dividing all the budgets between the steep projects is a NE. Therefore, $\mathrm{PoS} = 1$. □

4. SIMULATION

For a wide subset of shared effort games, the question when a game possesses at least one NE is still unanswered. To study this, we generalize the fictitious play, originally suggested by Brown [3] for mixed extensions of finite games, and simulate it to find a NE.

In order to be able to simulate this, we must be able to find a best response to a given opponents' profile, if it exists. A best response of a player is a best strategy for her, given all the other players' strategies. The set of all the best responses of player i to the profile x^{-i} of other players is denoted $\mathrm{BR}(x^{-i})$.

If best responses exists long enough throughout the simulation, such that a candidate for a limit of the empirical distributions has been found, we check whether this candidate is a NE. Thereby, our simulations may find a NE for some games but never assert that no NE exists.

4.1 Infinite-Strategy Fictitious Play for Shared Effort Games

Since the classical fictitious play is defined for mixed extensions of finite games, and we are dealing with pure infinite games, a generalization is in order. The intuition of our generalization is to consider a strategy as a combination of several contributions of the whole budget, each to a single project.

Definition 4. *Given a shared effort game with players N, budget-defined strategies $A_i = \left\{x^i = (x^i_\omega)_{\omega \in \Omega_i} \in \mathbb{R}_+^{|\Omega_i|} | \sum_{\omega \in \Omega_i} x^i_\omega \leq B_i\right\}$ and utilities $u^i(x) \triangleq \sum_{\omega \in \Omega_i} \phi^i_\omega(x_\omega)$, define a* Infinite-Strategy Fictitious Play (ISFP) *as the following set of sequences. Consider a (pure) strategy in this game, i.e. $((x^i(1))^{i \in N} = ((x^i(1)_\omega)_{\omega \in \Omega_i})^{i \in N}$, and define recursively, for each $i \in N$ and $t \geq 0$:*

$$x^i(t+1) \triangleq \frac{tx^i(t) + \mathrm{BR}(x^{-i}(t))}{t+1}, \tag{2}$$

where for each strategy in $\mathrm{BR}(x^{-i}(t))$ *we obtain another sequence in the ISFP.*

We say that an ISFP converges *to $x^* \in {\mathbb{R}_+}^n$, if at least one of its sequences converges to x^* in every coordinate.*

Since $\mathrm{BR}(x^{-i}(t))$ is a set, there may be multiple ISFP sequences. For an ISFP to be defined, we need that $\mathrm{BR}(x^{-i}(t)) \neq \emptyset$, that is the utility functions attain a maximum. Since the functions are, generally speaking, not upper semi-continuous, they may sometimes not attend a maximum, rendering the ISFP undefined.

In ISFP, all the plays obtains equal weights. In the other extreme, a player just best-responds to the previous strategy profile of other players, thereby attributing the last play with the weight of 1 and all the other plays with 0. In general, we define an $\alpha -$ *ISFP* play as in Definition 4, but with the following formula instead

$$x^i(t+1) \triangleq \frac{\alpha t x^i(t) + \mathrm{BR}(x^{-i}(t))}{\alpha t + 1}, \tag{3}$$

Now, we turn to solve the algorithmic problem of finding whether a best response exists, and if yes, what it is.

4.2 Best Response in a 2-project Game with Partially Convex and Weakly Monotone Share Functions

Let the projects be $\Omega = \{\psi, \omega\}$. For a player $i \in N$, given all the other players' strategies $a^{-i} \in A^{-i}$, we would like to find a best response. From the weak monotonicity of the share functions, we may assume w.l.o.g. that the best responding player contributes all her budget. Then, a strategy is uniquely determined by the contribution to project ψ and we shall write x^i for x^i_ψ, meaning that $x^i_\omega = B_i - x^i$.

In the following theorem, consider M^θ_{eq} sharing functions that define what player i obtains from each project, given other players' strategies a^{-i}. Let $D^i_0 < D^i_1 < \ldots < D^i_m$ and $W^i_0 < W^i_1 < \ldots < W^i_l$ be the jumps of ϕ^i_ψ and ϕ^i_ω respectively. (The first points in each list are the minimum contributions to projects ψ, ω, respectively, required for i to obtain a share. The other points are the points at which another player becomes suppressed at the respective project.) The possible discontinuity points of the total utility of i are thus $D^i_0 < D^i_1 < \ldots < D^i_m$ and $B_i - W^i_l < \ldots < B_i - W^i_1 < B_i - W^i_0$. Denote the distinct points of these lists merged in the increasing order by L.

Theorem 7. *Let all the contributions of the players in $N \setminus \{i\}$ be fixed as they are given, and consider x^i as the only variable. Assume M^θ_{eq} sharing and a convex project value functions. Let L_{B_i} denote the points of the list L that are on $[0, B_i]$, together with 0 and B_i, and let M_{B_i} be L_{B_i} with an arbitrary point added between each two consecutive points. Then, the maximum of the one-sided limits at the points of L_{B_i} and of the values at the points of M_{B_i} yields the utility supremum of the responses of player i. This supremum is a maximum (that is a best response exists) if and only if it is achieved at a point of M_{B_i}.*

Proof. The utility of i is $u^i(x^i) = \phi^i_\psi(x^i) + \phi^i_\omega(B_i - x^i)$. Consider the open intervals between the consecutive points of L_{B_i}. On each of these segments, the function $\phi^i_\psi(x^i)$ is convex, being proportional to the convex project value function, and $\phi^i_\omega(B_i - x^i)$ is convex because the function $B_i - x$ is convex and concave and ϕ^i_ω is convex and weakly monotone. Therefore, the utility is also convex, as the sum of convex functions.

Therefore, the utility's supremum of each convexity interval is achieved as the one-sided limit of at least on of its edge points. This supremum can be a maximum if and only if it is not larger than the maximum of the utility at an interval edge point or at an internal point of an interval (in the last case, the utility is constant on this interval, from the convexity). □

4.3 The Simulation Method

For each of the considered shared effort games, $\alpha -$ ISFP are run, for several αs. Theorem 7 shows how to find whether a best response exists, and if yes, what it is. If at least once in the process of simulating an $\alpha-$ISFP, there exists no best response, then we either stop this attempt or approximate the supremum and continue. While running an $\alpha-$ ISFP, we stop at a convergence or after a predefined number of iterations. In any case, after stopping, we check whether the final distribution is a NE, employing Theorem 7 to find a best response for each player; a profile is a NE if and only if each player has a best response and reacts by a best response. If the final distribution is not a NE, then this attempt does not solve this game; otherwise, a NE has been found.

To conclude, we never state here that a game does not possess a NE. We only may assert the existence of a NE.

4.4 Simulation Settings

We consider some cases that are not covered by the theoretical analysis. That is, where $\theta \neq 0$ and also when theorem 3 does not hold. We consider only the θ-equal 2-project case, since in this case we know how to calculate the best response. We work with several cases of 2 and 3 players and with some setting of players' budgets and the threshold value. For each such number of players, we consider several linear utility functions for each of the projects, with the coefficients $0.1, 0.2, \ldots, 2.0$. For each such game, we generate 30 fictitious plays by randomly and independently picking original history for each player, uniformly from the possible histories. While simulating, when there are multiple best responses, we choose one which is closest to the current belief state of the fictitious play. If, in at most 100 iterations, we reach an iteration, where each of the players' history changes less than $\epsilon = 0.001$ from its previous value, then we consider the process to have converged. In any case, we check whether profile we have arrived at is a NE. For each found NE, we calculate its efficiency by dividing its total profit by the optimum possible total profit.

We plot whether a NE has been found for various combinations of the parameters. For a found NE, its efficiency is shown as well.

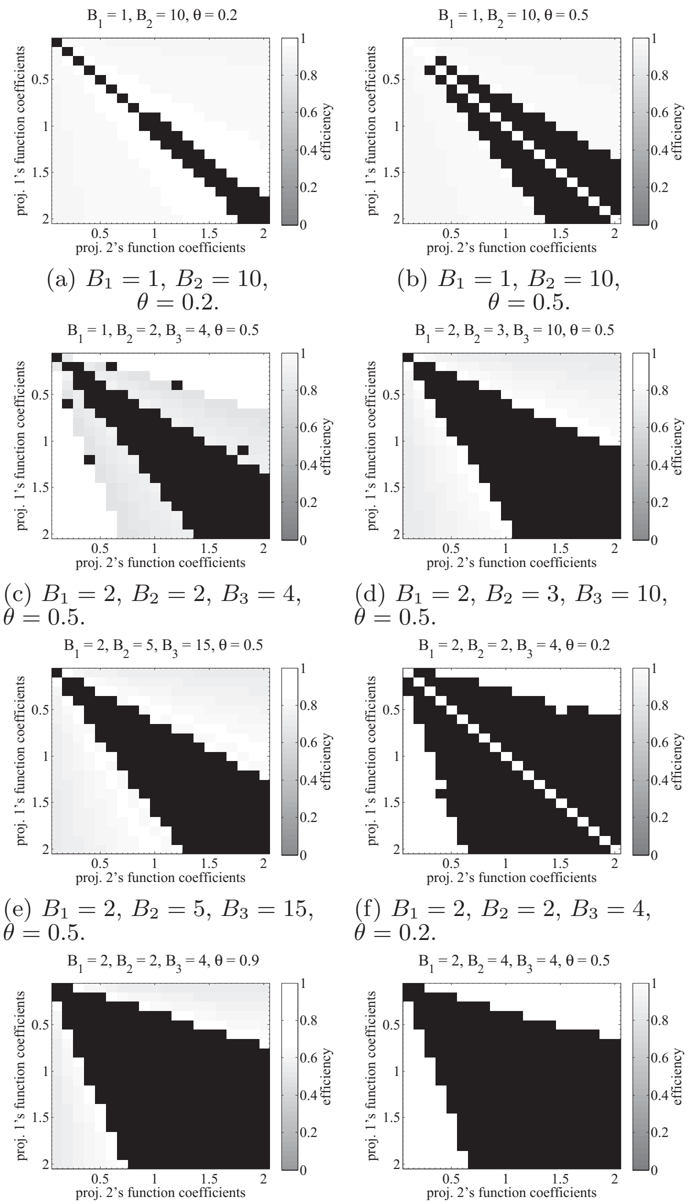
4.5 Results and Conclusions

The results appear in Fig. 1. For two players with far away budgets ($B_1 < \theta B_2$), a NE exists when the project function coefficients are not too close to one another. For $\theta = 0.5$, an equilibrium exists also when the project functions are exactly the same, since player 2 can just dominate player 1 everywhere. When a NE exists, the efficiency is quite high, it begins with 9.14 and increases are the project functions become closer to each other.

For three players, the following behavior takes place. A NE exists besides a cone in the graph, that is except when the project functions are quite close to each other. Only in once case do NE exist also when the project functions are exactly the same. When the budgets are relatively close (within the factor of θ), and a NE exists, the efficiency is close to optimum. When the budgets are far away and a NE exists, its efficiency drops from somewhere in the interval $[0.8, 0.6]$ when the project function coefficients are quite close, to somewhere in the interval $[0.68, 0.53]$ when the project functions differ the most.

5. RELATED WORK

The price of anarchy is the lowest total utility of a NE divided by the optimum total utility of any strategy profile. Under very specific, but somewhat artificial conditions



(a) $B_1 = 1$, $B_2 = 10$, $\theta = 0.2$.

(b) $B_1 = 1$, $B_2 = 10$, $\theta = 0.5$.

(c) $B_1 = 2$, $B_2 = 2$, $B_3 = 4$, $\theta = 0.5$.

(d) $B_1 = 2$, $B_2 = 3$, $B_3 = 10$, $\theta = 0.5$.

(e) $B_1 = 2$, $B_2 = 5$, $B_3 = 15$, $\theta = 0.5$.

(f) $B_1 = 2$, $B_2 = 2$, $B_3 = 4$, $\theta = 0.2$.

(g) $B_1 = 2$, $B_2 = 2$, $B_3 = 4$, $\theta = 0.9$.

(h) $B_1 = 2$, $B_2 = 4$, $B_3 = 4$, $\theta = 0.5$.

Figure 1: The simulation results for various parameters. Bold black color means that Nash Equilibrium has not been found. For all the other cases, the efficiency, a value in $[0, 1]$, is shown in the appropriate color, according to the color maps.

(N-approximate Vickrey conditions and $\theta = 0$), Bachrach et al. [2] have shown that the price of anarchy (PoA) is at most the number of players. They also show upper bounds for convex project functions, where each player receives at least a constant share of its marginal contribution to the project's value. In this paper, we use study more general $\theta \in [0, 1]$ sharing mechanisms without these conditions, an we provide precise conditions for existence and efficiency results. Anshelevich and Hoefer [1] considered an undirected graph model, where the nodes are the players and each player splits

its budget between its adjacent edges in minimum effort games (where the edges are the projects), each of which equally rewards both sides by measure of the project's success (i.e., duplication instead of division). They proved the existence and the complexity of funding a NE and found that the PoA is at most 2. A related setting of *multi-party computation games* appeared in [15]. There, the players are interested in reducing their own cost. This differs from our work, since they consider utility minimization and different strategical setting, that is "computing the correct value or free-riding".

To conclude, there has been no research of our problem's NE in the general case.

6. CONCLUSIONS AND FUTURE WORK

This paper considered shared effort games where players may contribute to some given projects, and subsequently share the profits of these projects, conditionally on the allocated effort. The paper studied existence and efficiency of the NE of these games, arriving at the following results. In these games, a NE exists if utility functions are continuous and concave, and the strategy sets are non-empty, compact and convex. We also characterized the existence of NE in a subcase of shared effort games and considered the price of anarchy and stability. In some cases when a NE existed and the budgets were close, the price of anarchy and price of stability was found to be 1. That is, all the NE in those cases are socially optimum. For the shared effort games that have not been analyzed theoretically, we simulated fictitious play, to estimate the existence of NE. To this end, we generalized the fictitious play and described some of the best responses of a player to the other players' strategies. In the cases where we found a NE, we also estimated its efficiency. For two players, a NE is usually almost of the optimum efficiency. For three players, the efficiency of a NE can be suboptimal, as we saw in Section 4.5. In general, we found some cases where we characterized the existence of NE and found its efficiency, when existed.

The current approach has its limitations, providing a number of interesting directions for future work. First, we would like to extend our theoretical characterization of the existence of NE for far away budgets, and also for more than two players and to non-linear project functions and to find the price of anarchy and stability in the cases where a NE exists. Next, extending simulations for more than two projects and finding precise new methods to locate NE would improve our understanding of the various NE. Considering existence and efficiency of mixed Nash equilibria is also interesting.

The theoretical analysis of efficiency implies that in the analyzed cases for two players with close budgets, regulation is not needed, since the price of anarchy is 1. The price of anarchy is very high also for two players with far away budgets. For three or more players, some regulation may improve the total utility, though it does not go below 0.52 in all the considered cases. To conclude, we have analyzed the existence of NE and provided some practical insights as to when regulation should be used to improve efficiency.

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8. REFERENCES

[1] E. Anshelevich and M. Hoefer. Contribution games in social networks. pages 158–169, 2011.

[2] Y. Bachrach, V. Syrgkanis, and M. Vojnović. Efficiency and the Redistribution of Welfare. Technical report, Microsoft Research, October 2011.

[3] G. W. Brown. Iterative solution of games by fictitious play. *Activity analysis of production and allocation*, 13(1):374–376, 1951.

[4] S.-f. Cheng, D. M. Reeves, Y. Vorobeychik, and M. P. Wellman. Notes on Equilibria in Symmetric Games. In *Proceedings of the 6th International Workshop On Game Theoretic And Decision Theoretic Agents*, pages 1–5, 2004.

[5] A. Forte and A. Bruckman. Why do people write for wikipedia? incentives to contribute to open-content publishing. In *GROUP Workshop: Sustaining Community: The Role and Design of Incentive Mechanisms in Online Systems. Sanibel Island, FL*, pages 6–9, 2005.

[6] F. M. Harper, S. X. Li, Y. Chen, and J. A. Konstan. Social comparisons to motivate contributions to an online community. In *Proceedings of the 2nd international conference on Persuasive technology*, pages 148–159, Berlin, Heidelberg, 2007. Springer-Verlag.

[7] J. Kleinberg and S. Oren. Mechanisms for (mis)allocating scientific credit. In *Proceedings of the 43rd annual ACM symposium on Theory of computing*, pages 529–538, New York, NY, USA, 2011. ACM.

[8] J. Kleinberg and E. Tardos. Balanced outcomes in social exchange networks. In *Proceedings of the 40th annual ACM symposium on Theory of computing*, pages 295–304, New York, NY, USA, 2008. ACM.

[9] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. *Computer Science Review*, 3(2):65–69, 2009.

[10] D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behavior*, 14(1):124 – 143, 1996.

[11] B. D. Nguyen and K. M. Nielsen. What death can tell: Are executives paid for their contributions to firm value? AFA Denver Meetings Paper, 2011.

[12] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*, volume 1 of *MIT Press Books*. The MIT Press, April 1994.

[13] B. Roberson. The Colonel Blotto game. *Economic Theory*, 29:1–24, 2006.

[14] R. Siegel. All-pay contests. *Econometrica*, 77(1):71–92, 2009.

[15] R. Smorodinsky and M. Tennenholtz. Overcoming free riding in multi-party computations – the anonymous case. *Games and Economic Behavior*, 55(2):385–406, 2006.