Temporal profiles of avalanches on networks

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Science Foundation Ireland

Collaborators, funding, references

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- Yamir Moreno, Zaragoza
- Raquel A Baños, Zaragoza
- Jonathan Ward, Leeds
- William Lee, Portsmouth

- Science Foundation Ireland
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- Gleeson and Durrett, “Temporal profiles of avalanches on networks”, arXiv: 1612.06477

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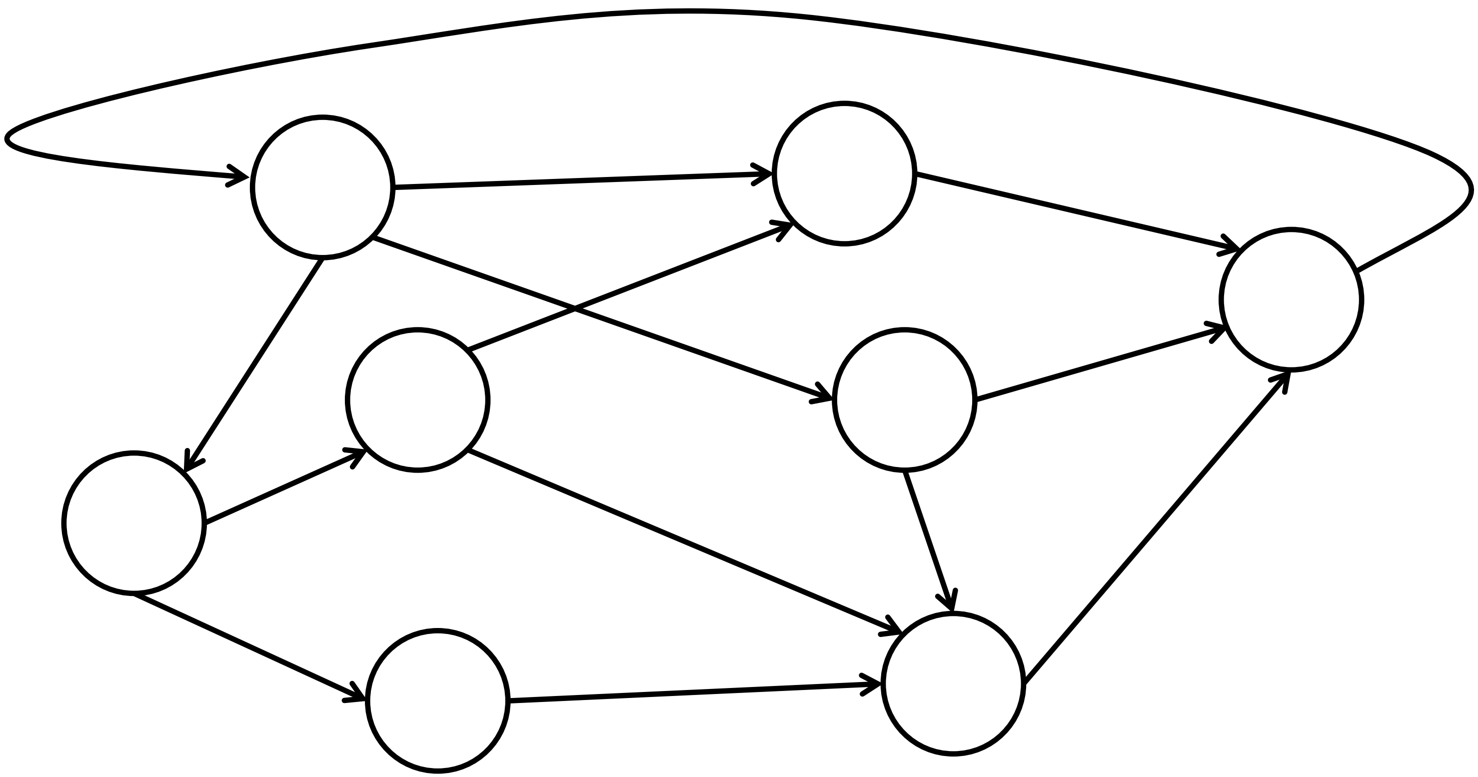
Overview

- Prologue: criticality in a model of meme diffusion on Twitter

1. Average avalanche shape functions (and beyond…)
2. Analytical results
3. Numerical simulations

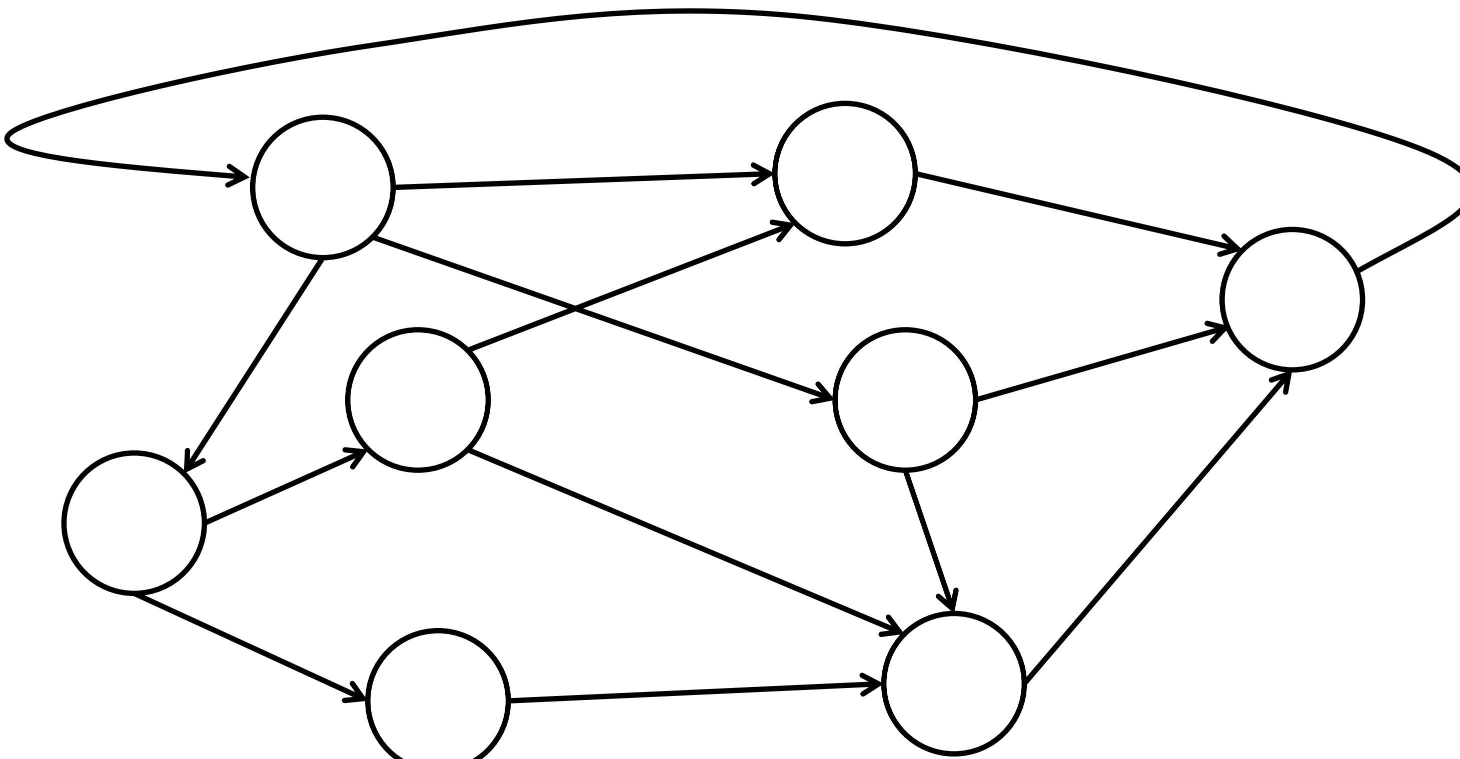
A meme diffusion model

- A simplified version of the model of Weng, Flammini, Vespignani and Menczer, “Competition among memes in a world with limited attention” Scientific Reports 2, 335 (2012)



A meme diffusion model

- Network structure: a node has k followers (out-degree k) with probability p_k (mean degree $z = \sum k\, p_k$)



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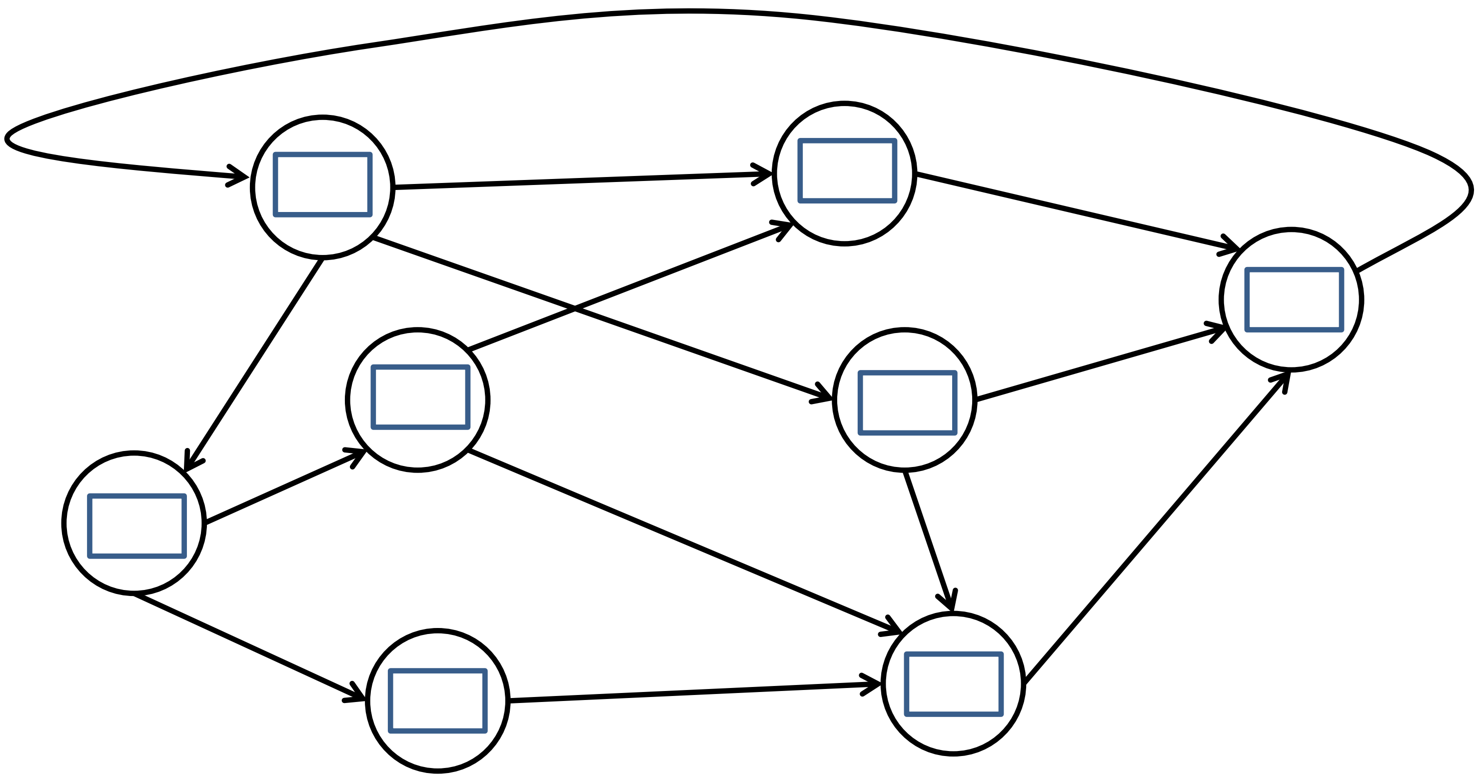
- Each node (of N) has a *memory screen*, which holds the meme of current interest to that node. Each screen has capacity for only one meme.



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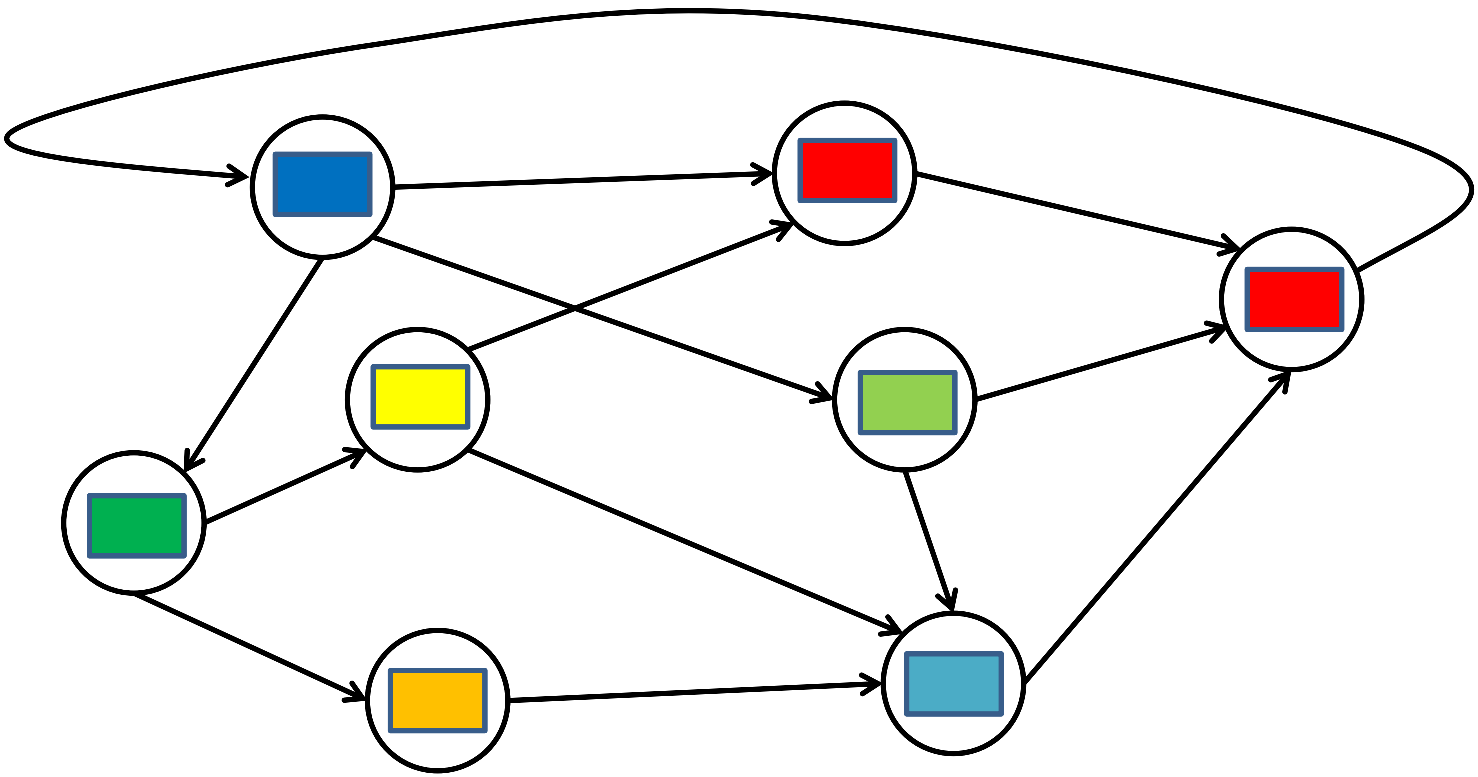
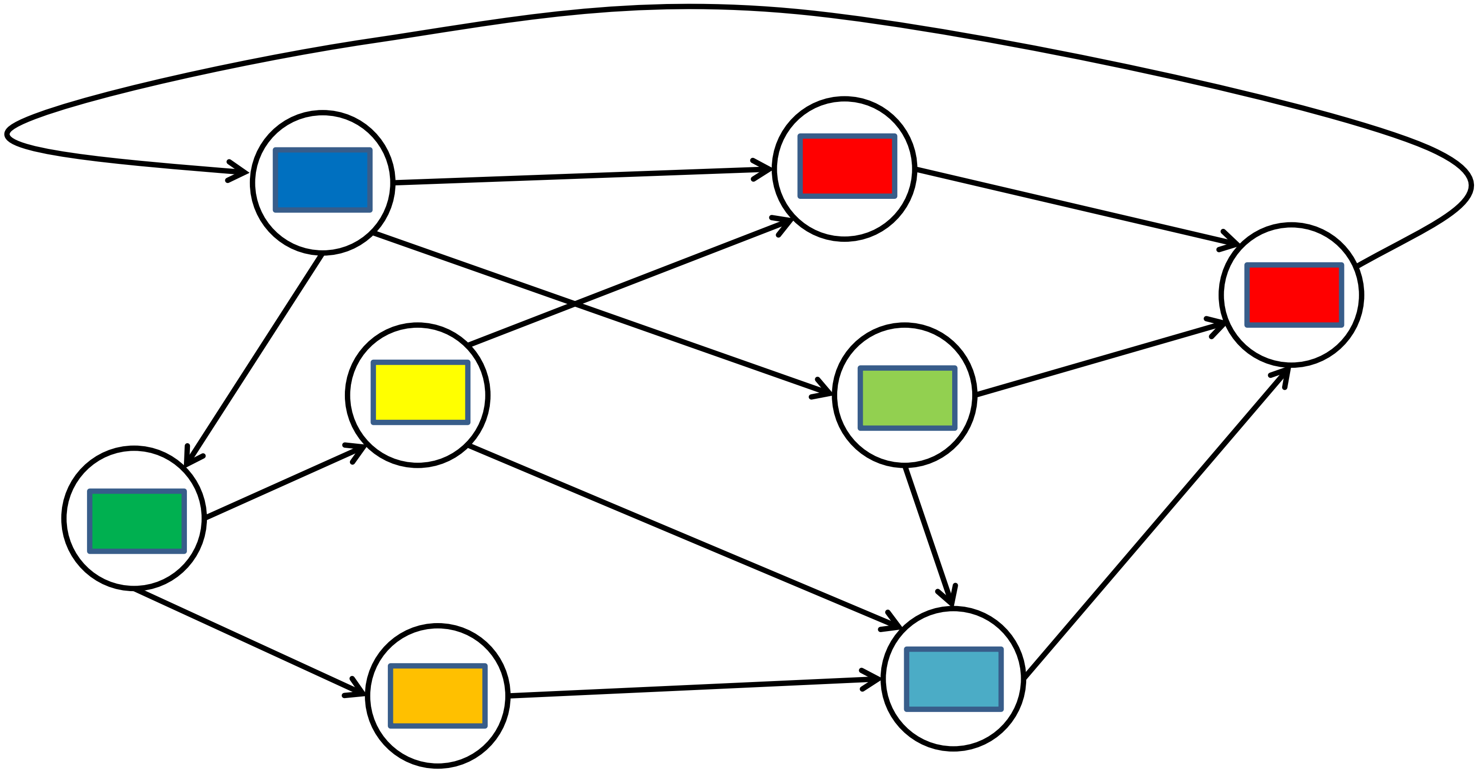


A meme diffusion model

- During each time step (with time increment $\Delta t = 1/N$), one node is chosen at random.
- With probability μ, the selected node *innovates*, i.e., generates a brand-new meme, that appears on its screen, and is tweeted (broadcast) to all the node's followers.



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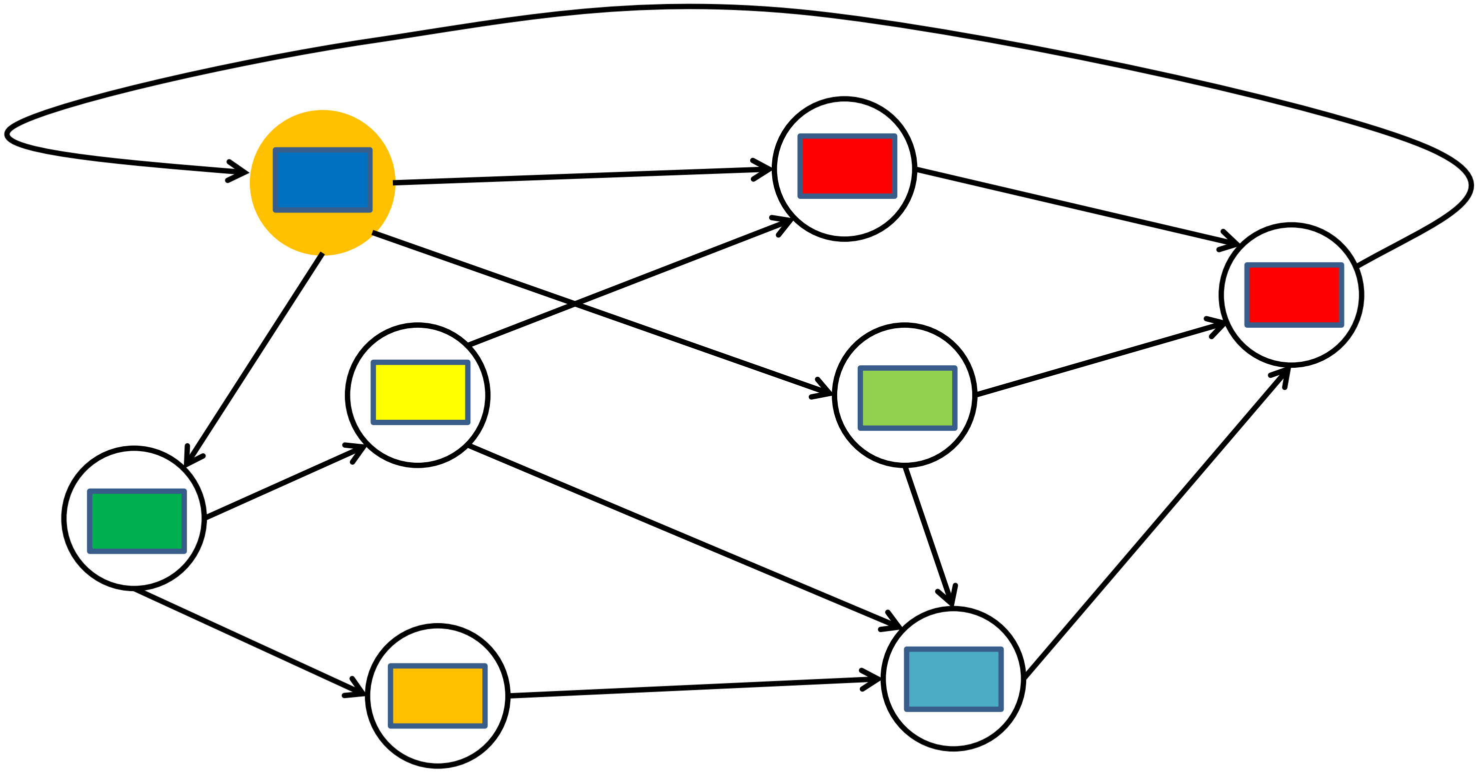

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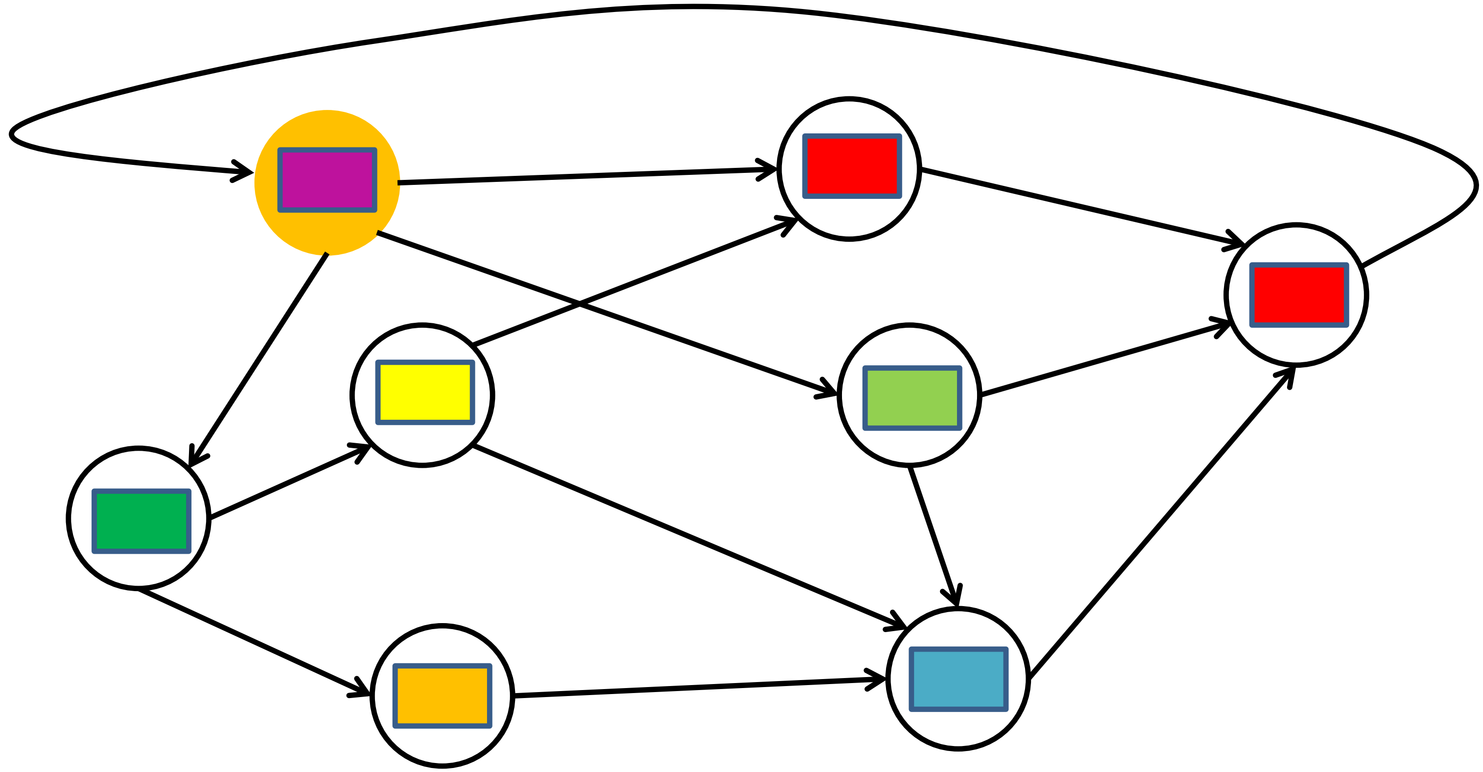

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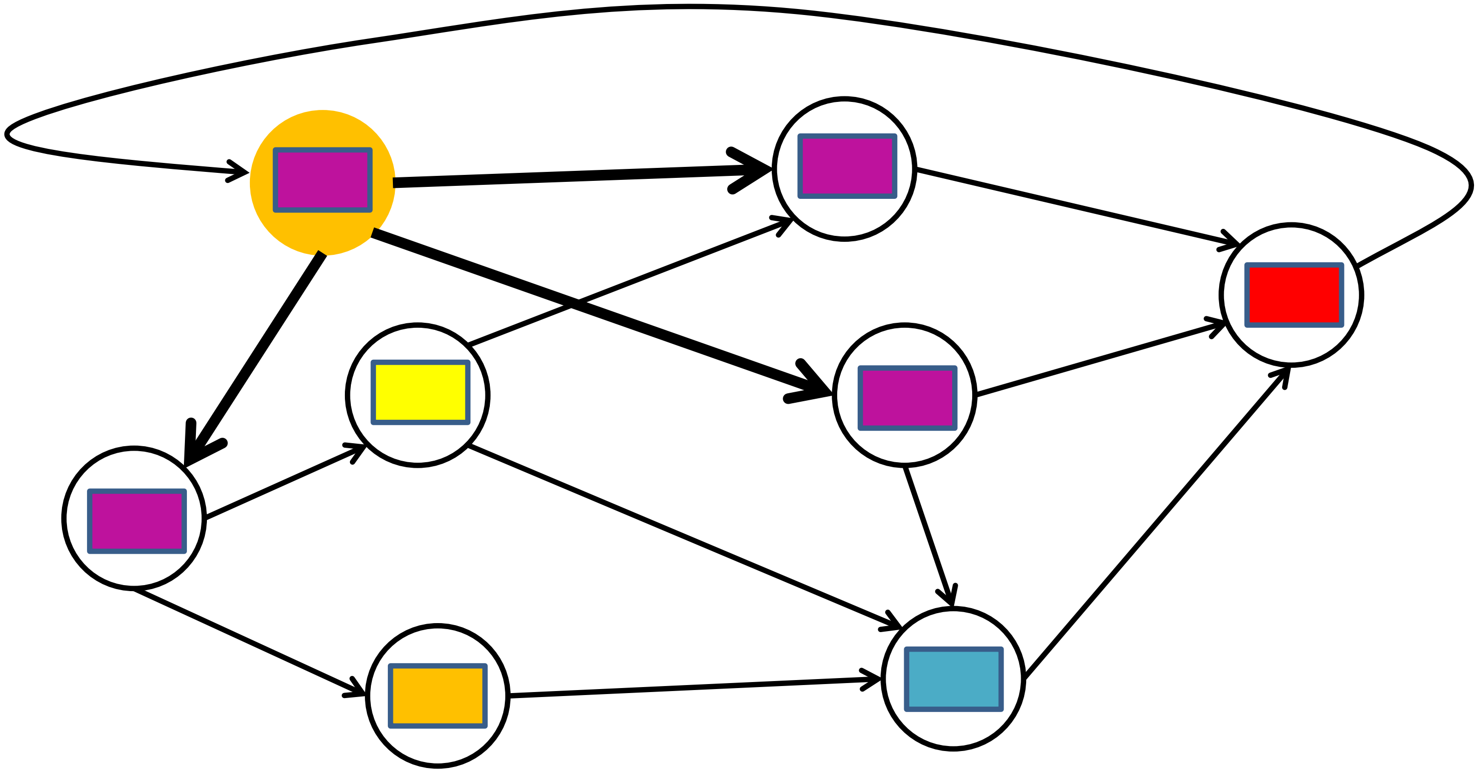

A meme diffusion model

- Otherwise (with probability $1-\mu$), the selected node (re)tweets the meme currently on its screen (if there is one) to all its followers, and the screen is unchanged. If there is no meme on the node's screen, nothing happens.

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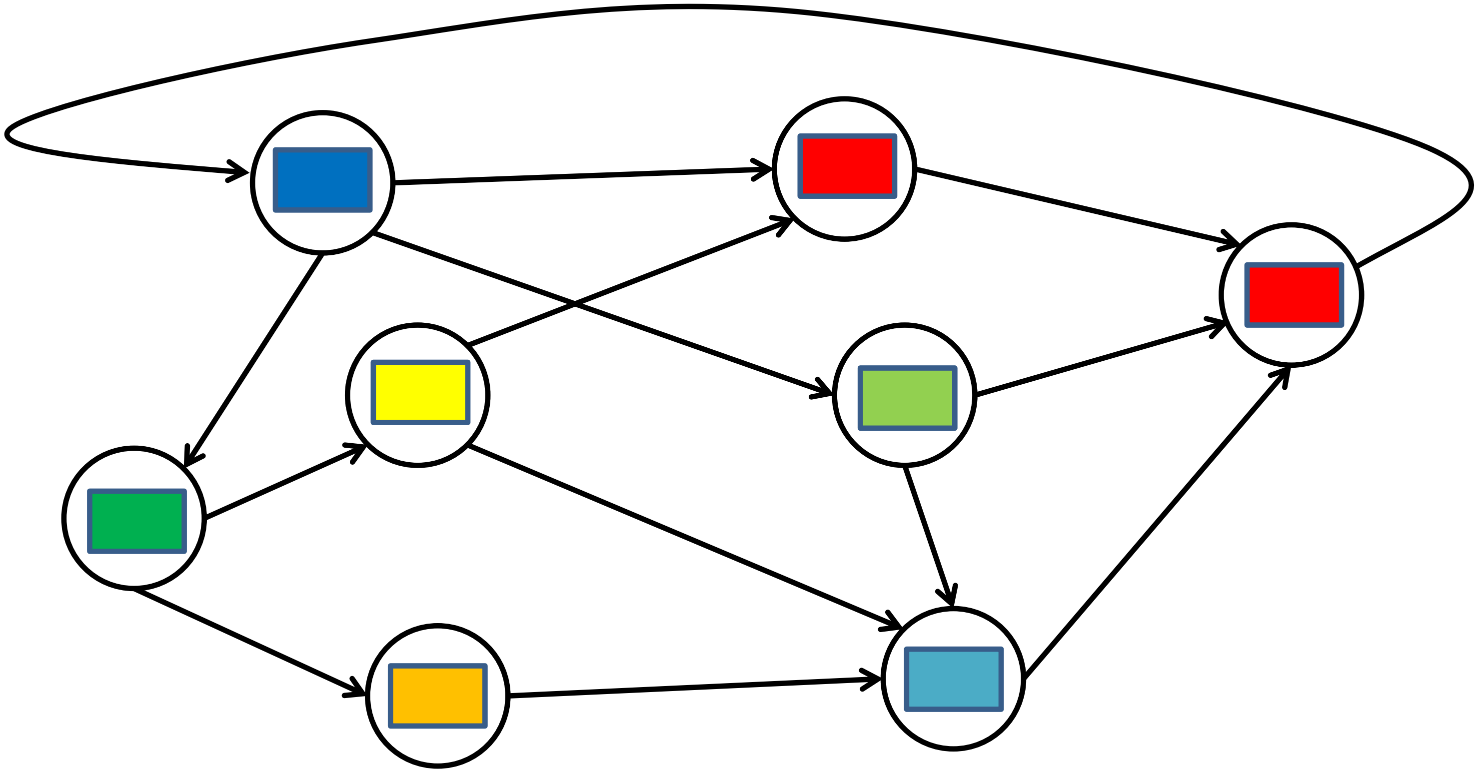

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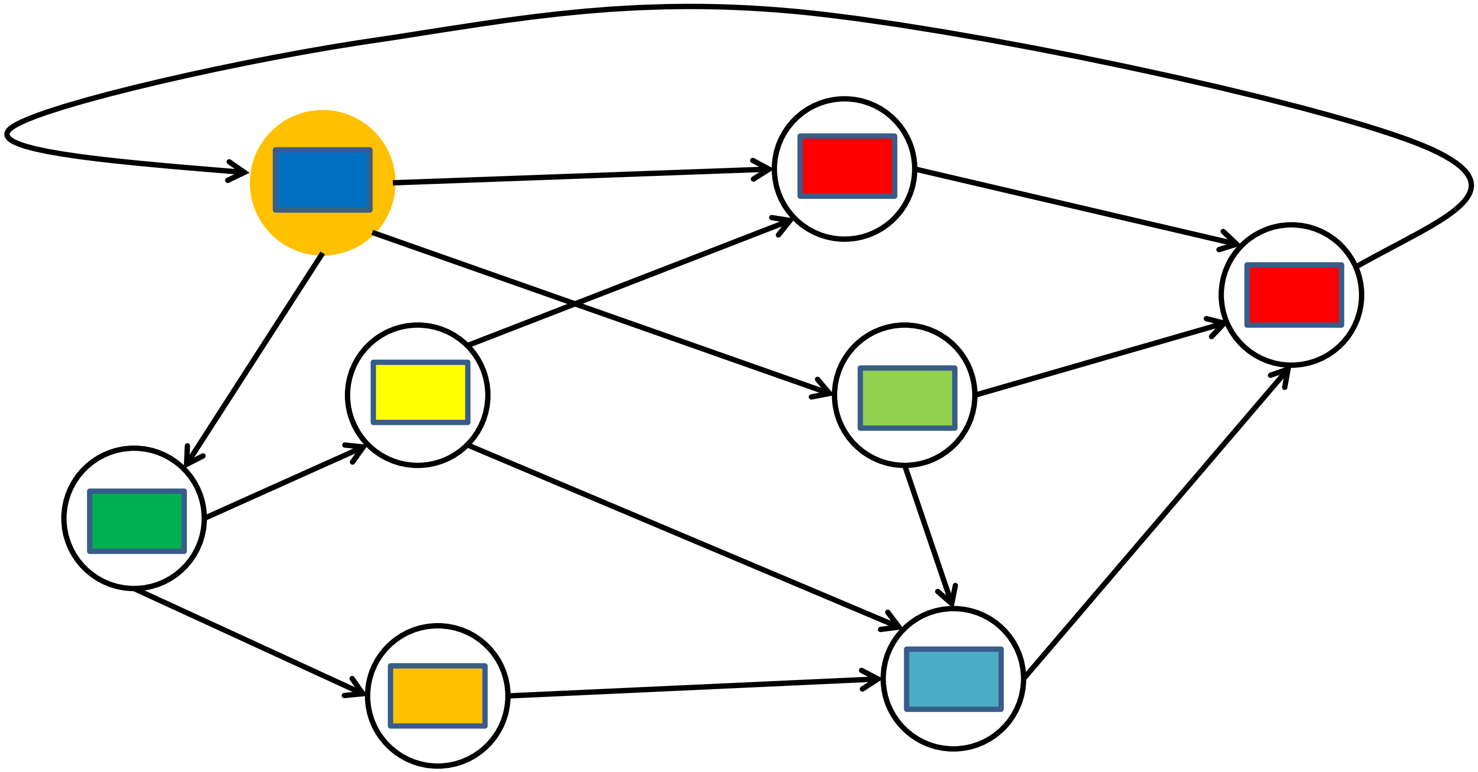

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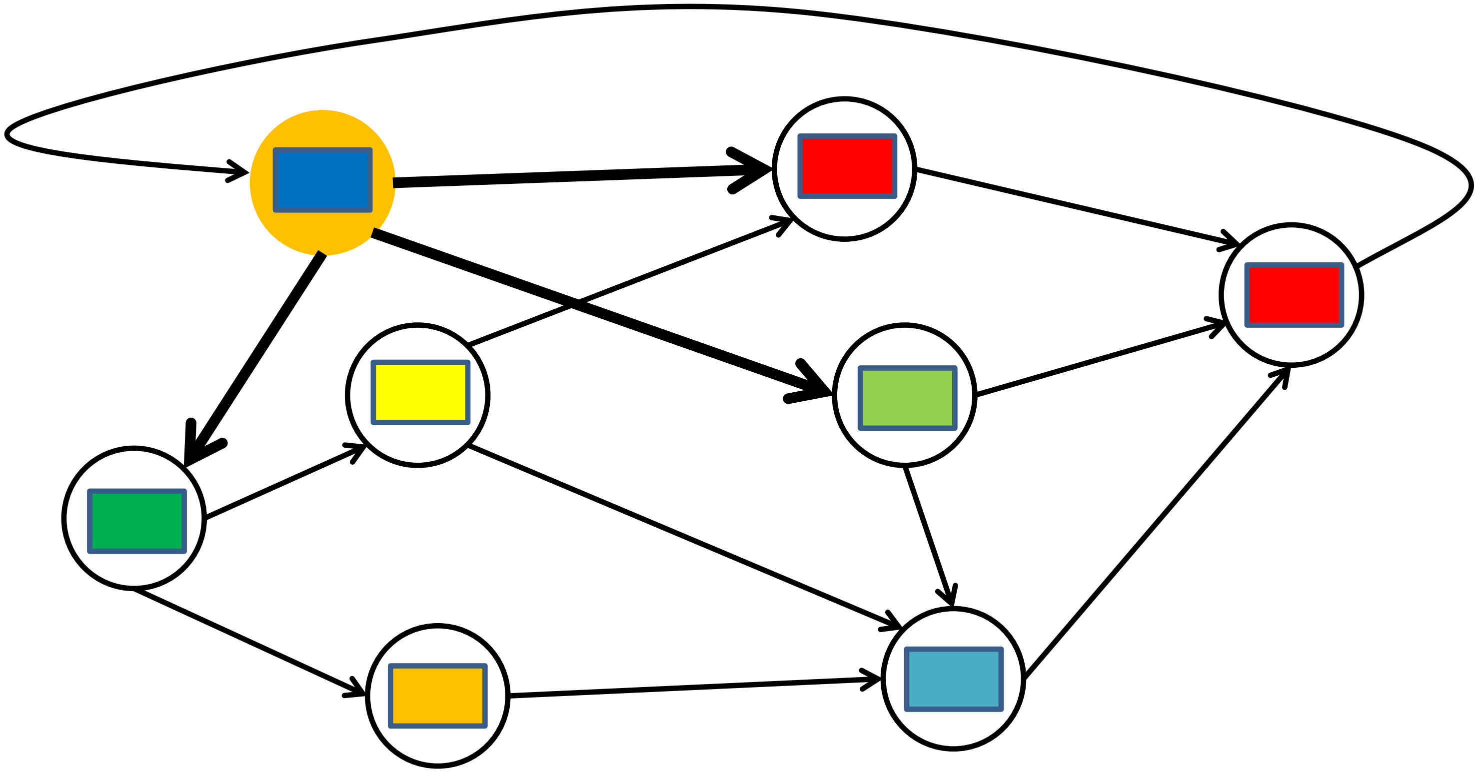

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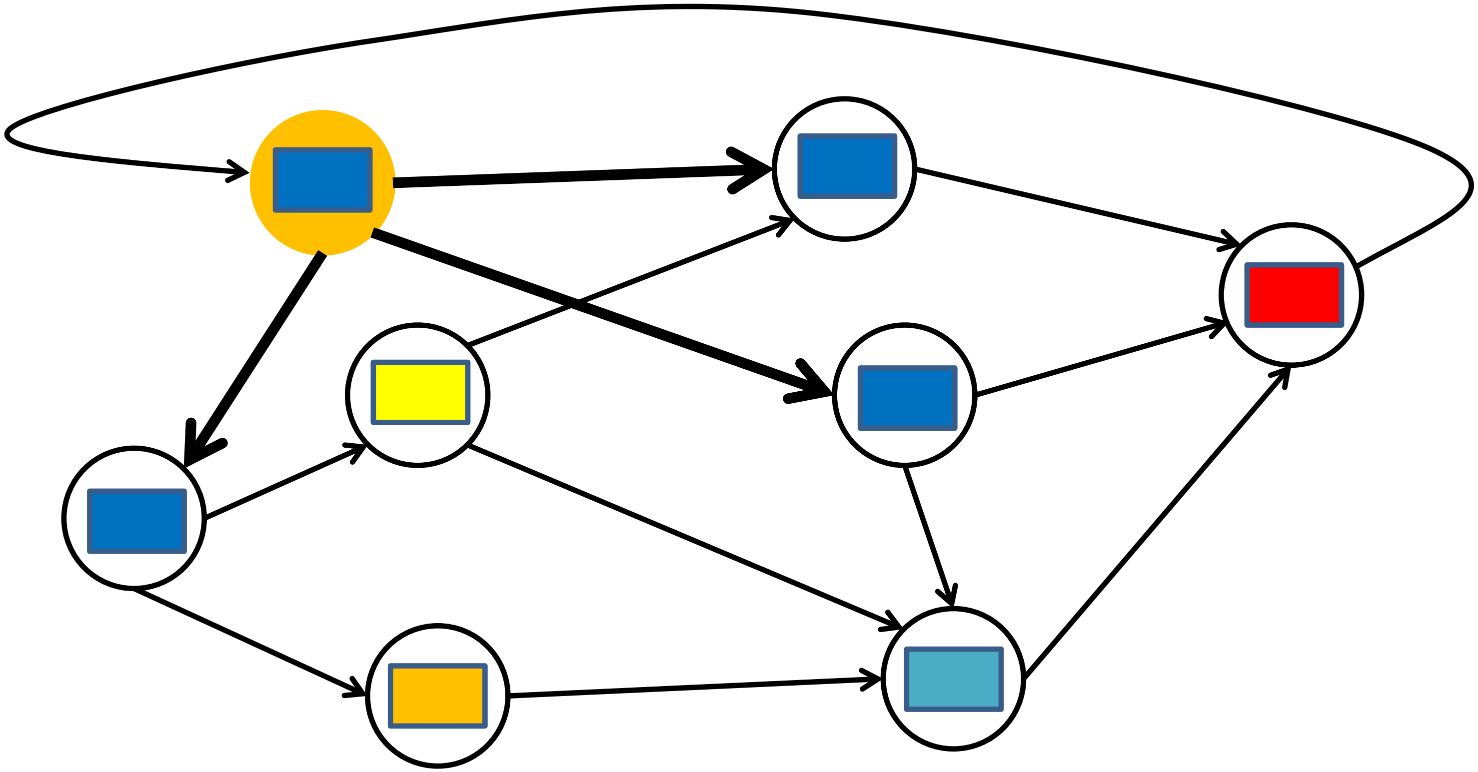
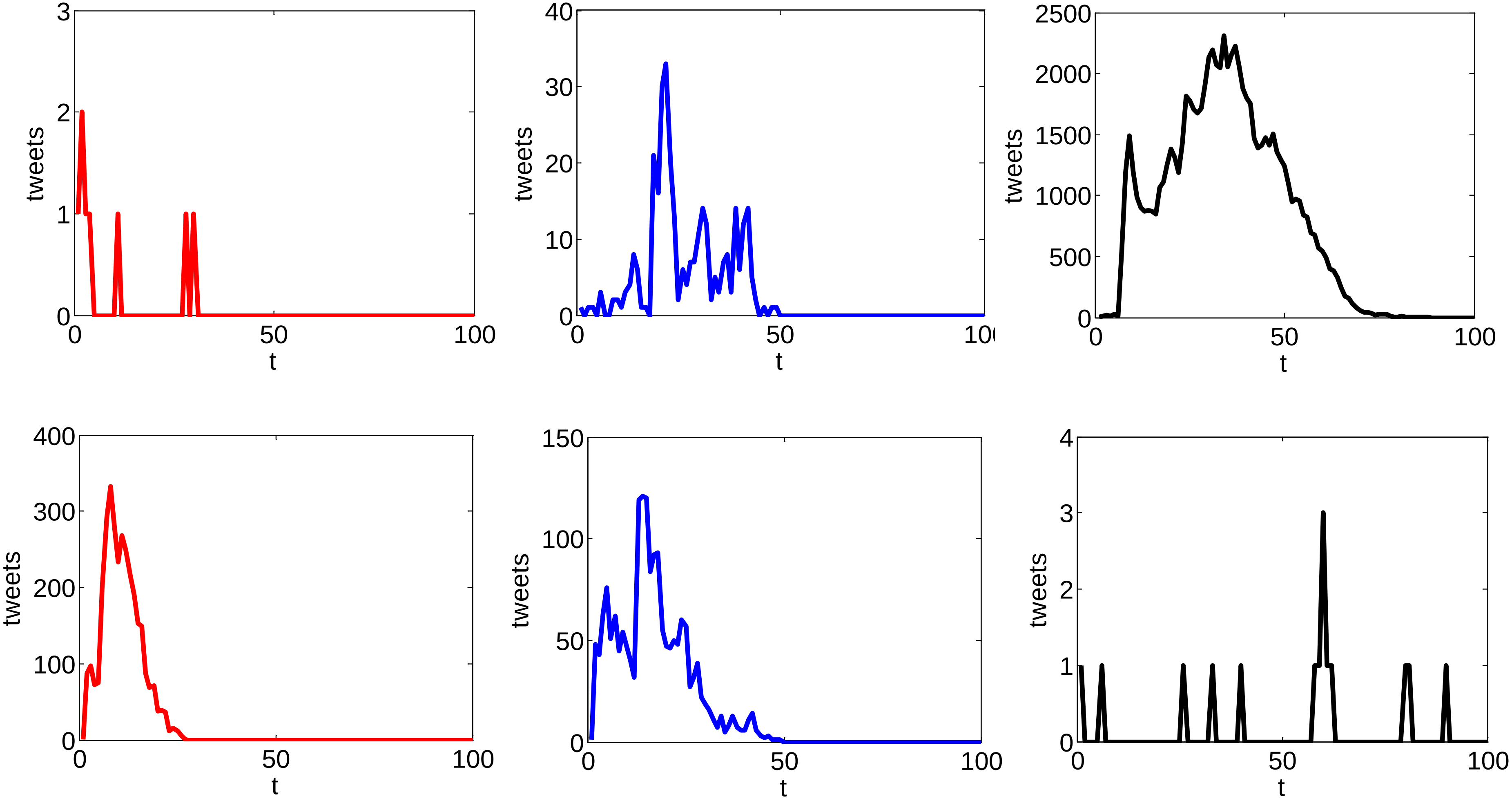
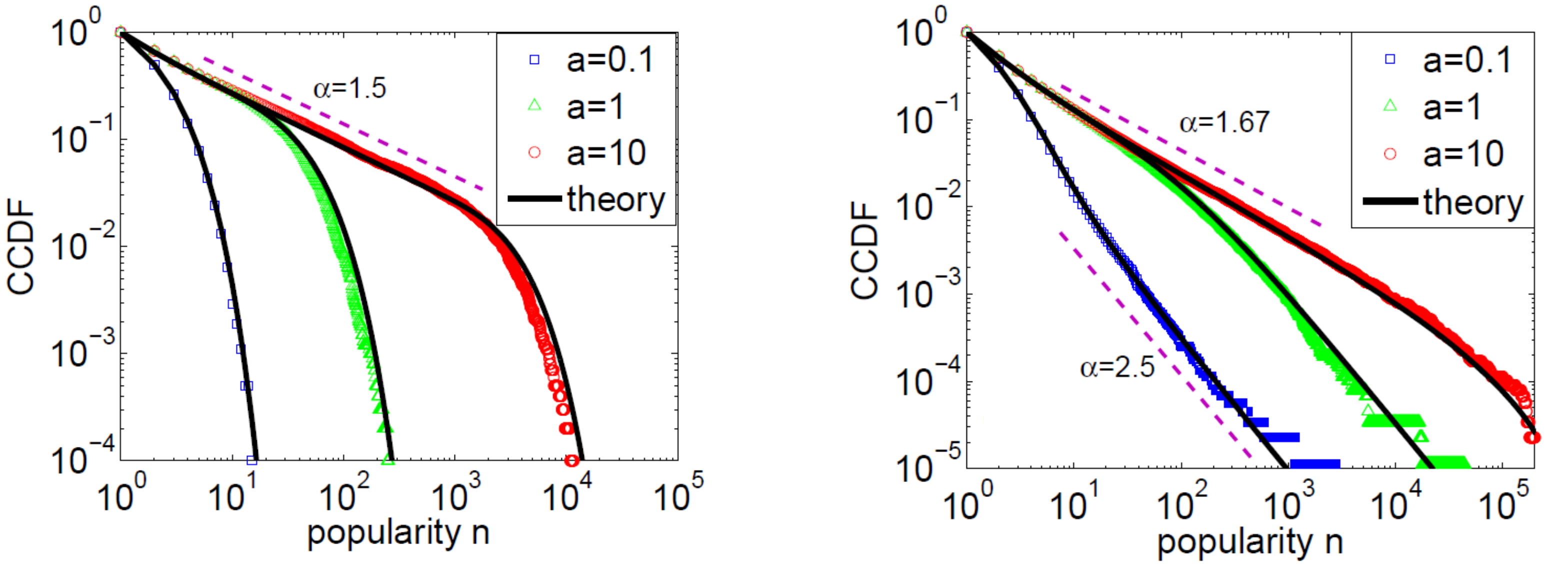


Examples of retweet avalanches from model



Competition-induced criticality in the $\mu \to 0$ limit

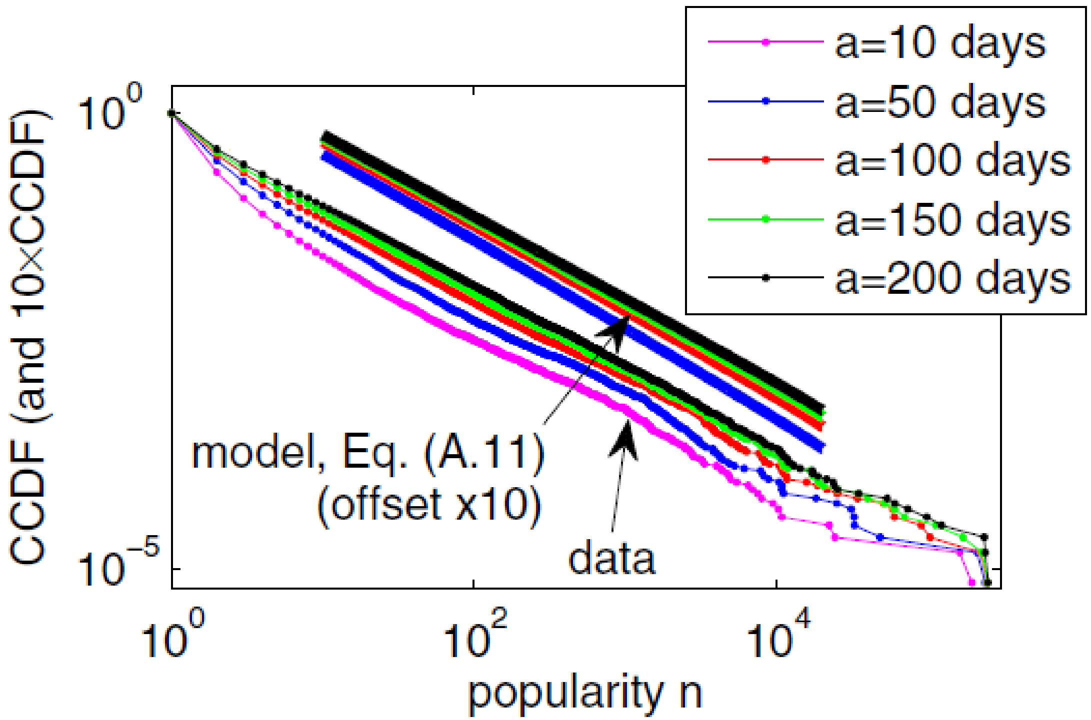
$p_k = \delta_{k,10}; \mu = 0$ $\qquad$ $p_k \propto k^{-\gamma}, \gamma = 2.5; \mu = 0.01$



Competition-induced criticality: competition between memes for the limited resource of user attention induces criticality in the $\mu \to 0$ limit

- Phys. Rev. Lett., 112, 048701 (2014)

Competition-induced criticality: comparison with data



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- Phys. Rev. X., 6, 021019 (2016)

Overview

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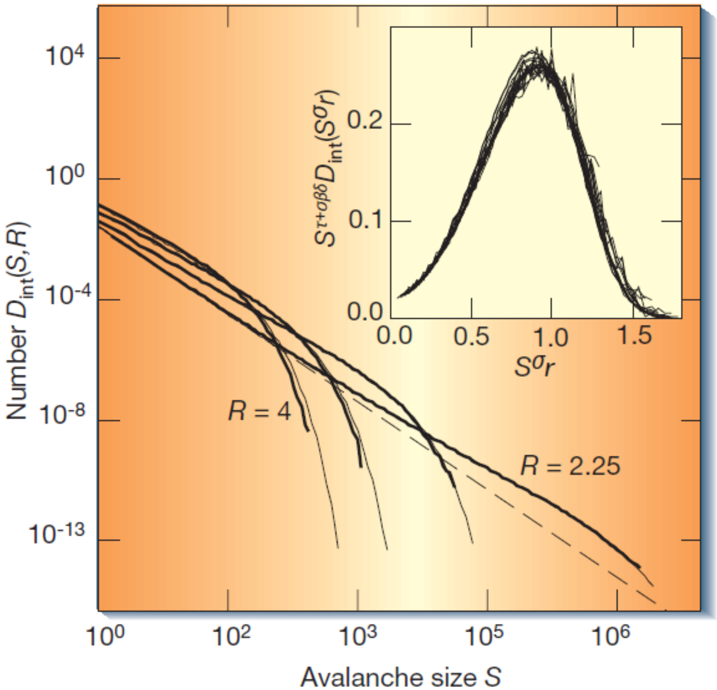
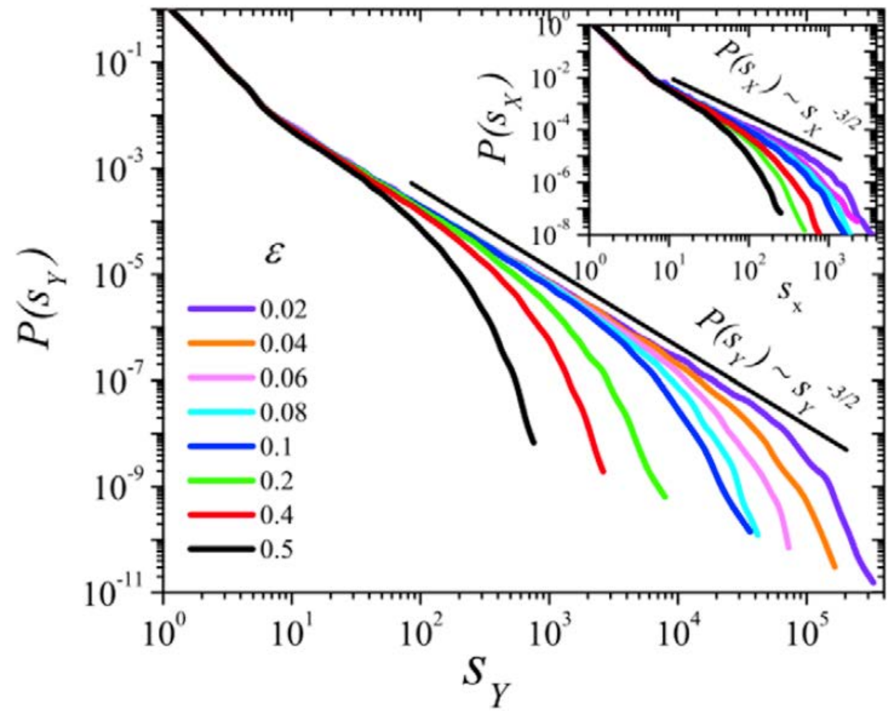
Criticality: going beyond power-law avalanche size distributions

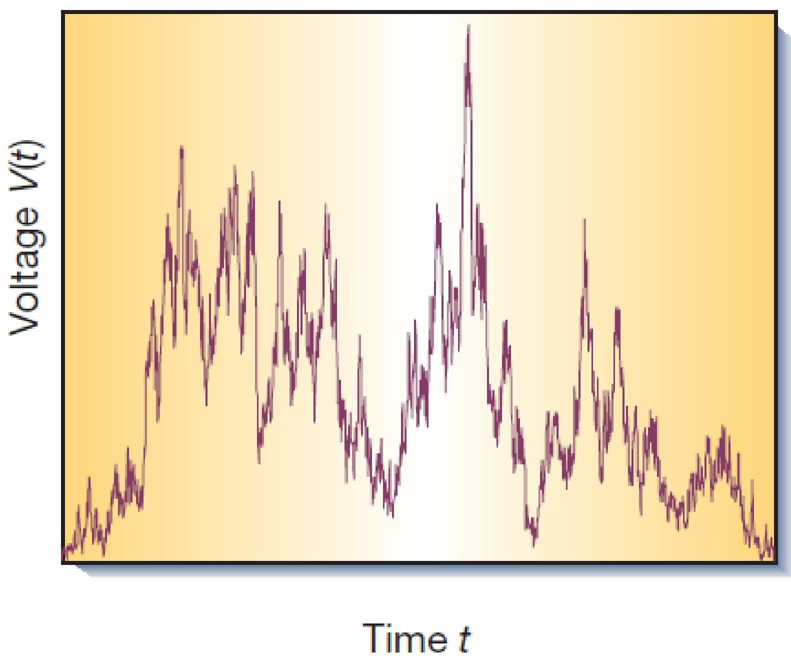


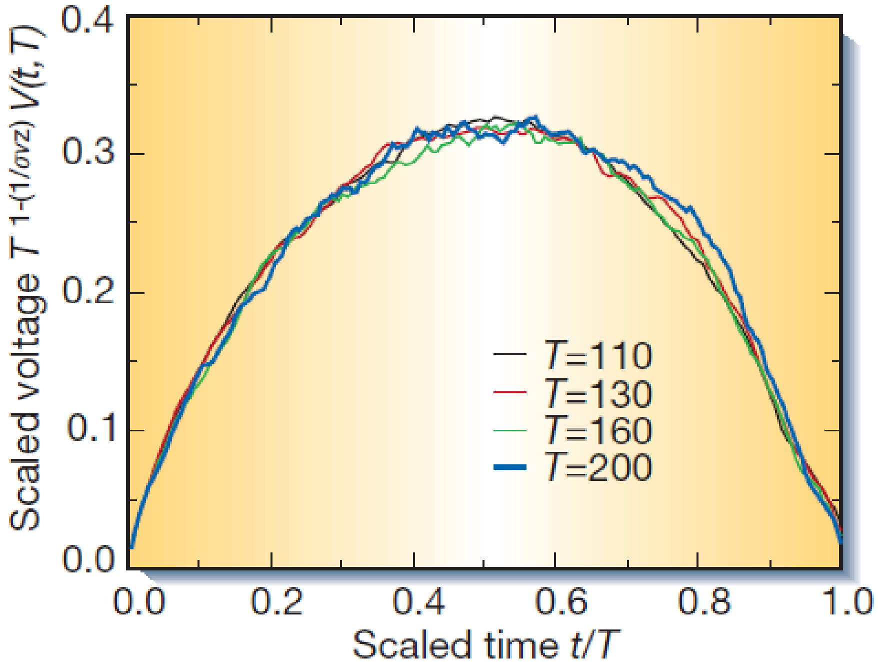


From Sethna et al., 2001 “Crackling noise”, Nature, 410: 242

From Pinto and Muñoz, 2011 “Quasi-neutral theory of epidemic outbreaks”, PLoS ONE, 6:e21946

Average avalanche temporal profiles





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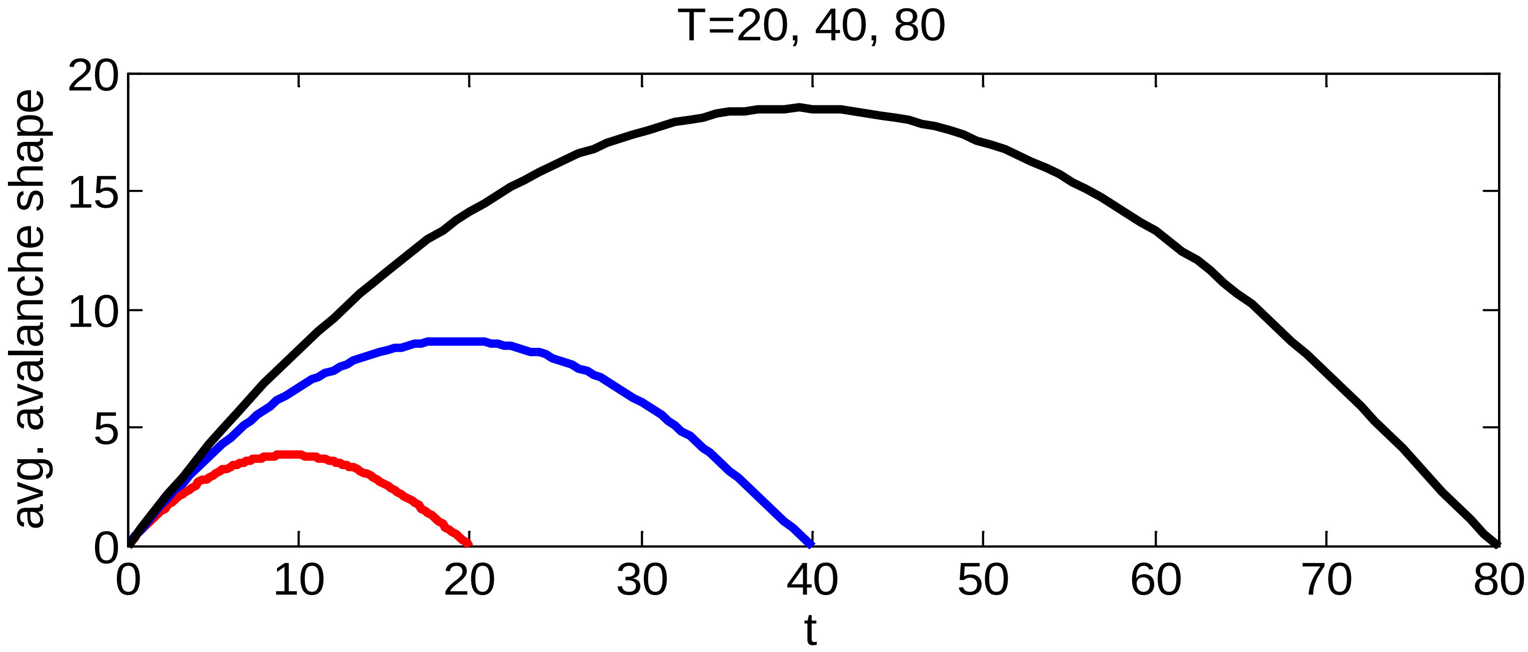
Criticality and the average avalanche shape



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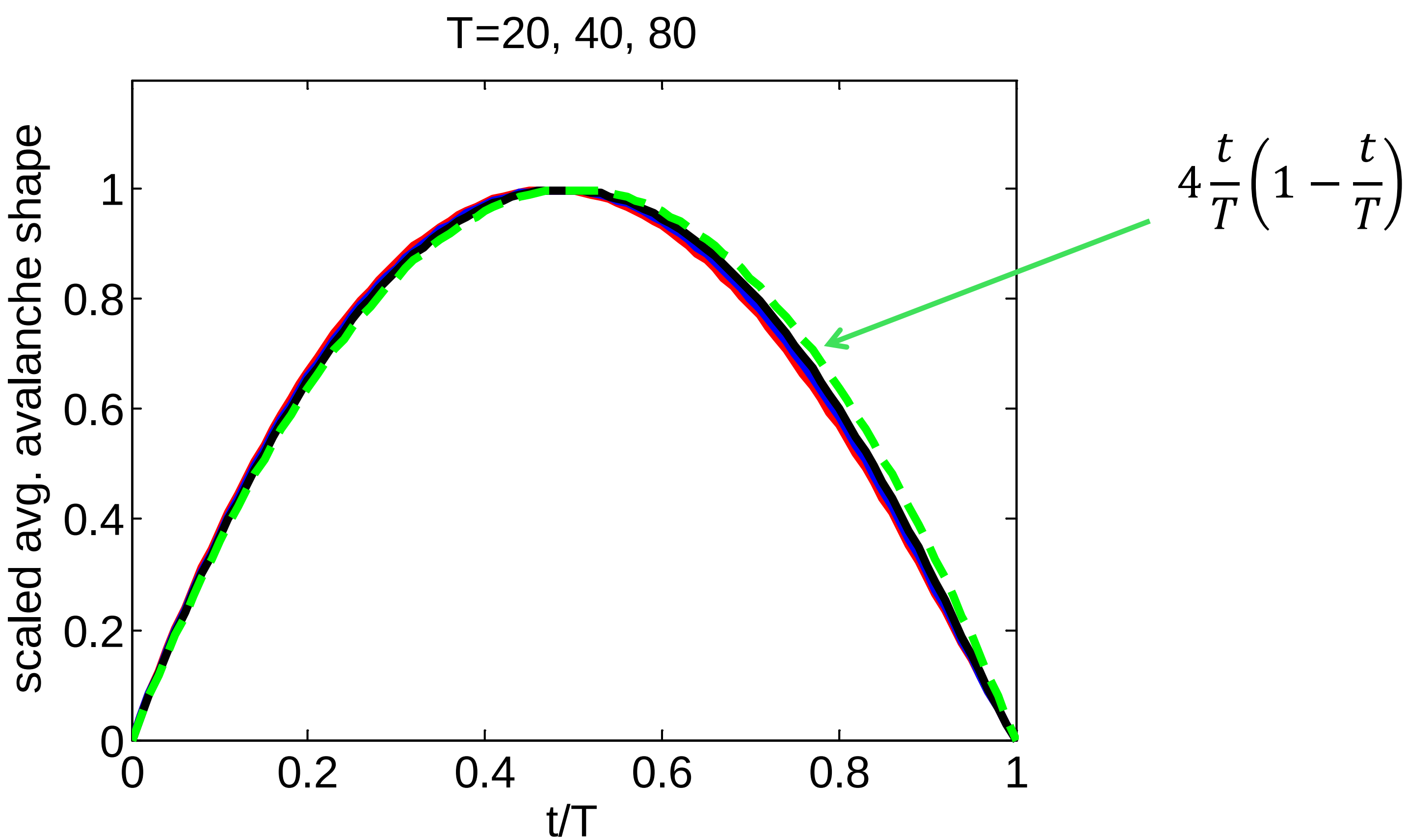
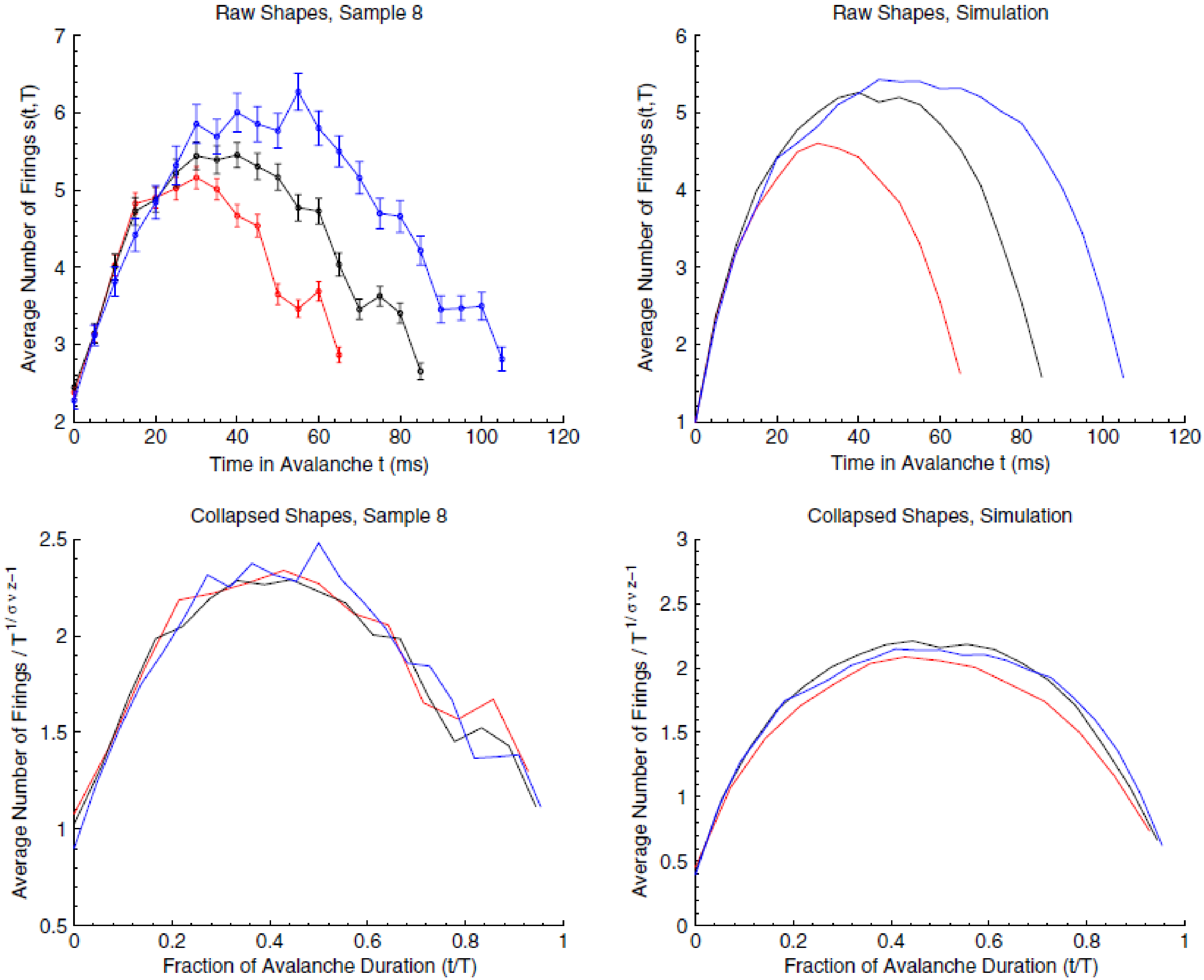
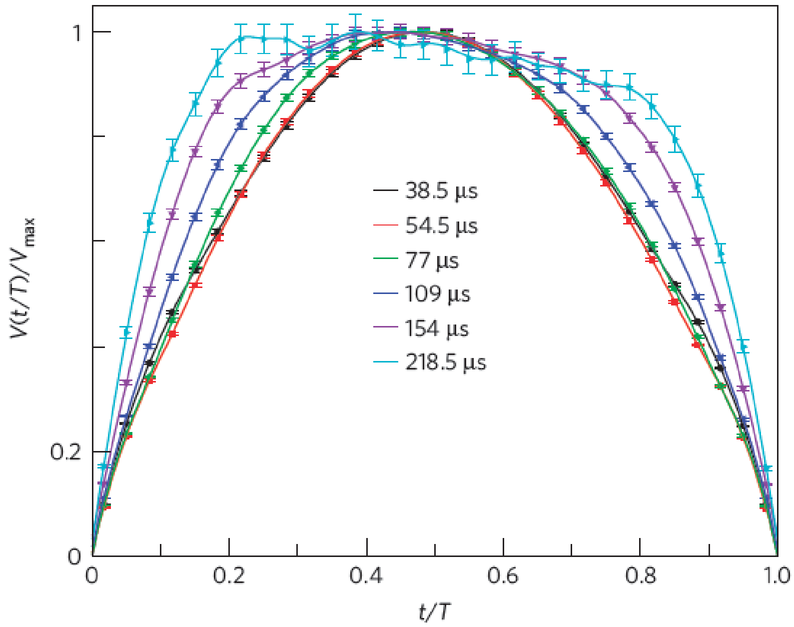


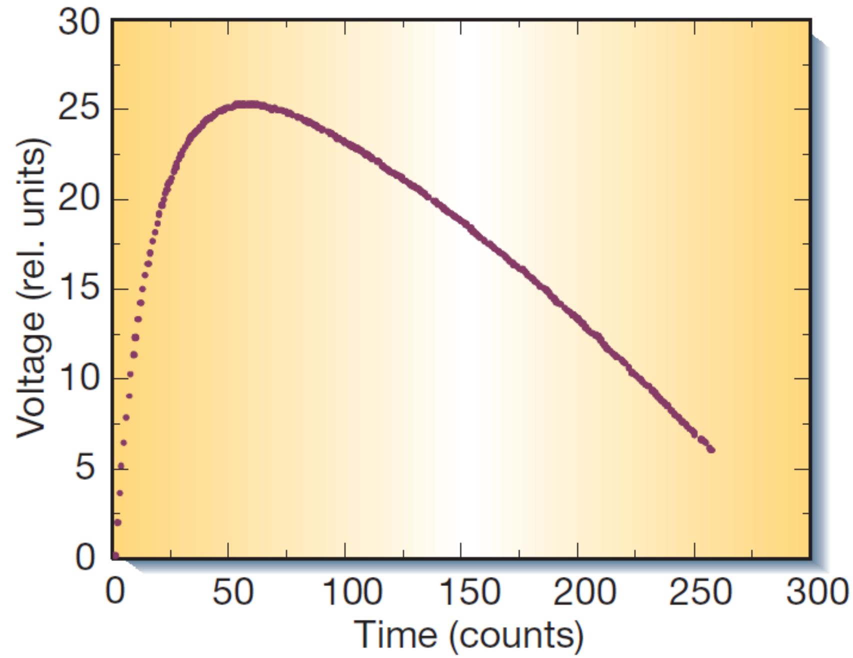
Examples of average avalanche shape analysis



From Friedman et al., 2012 "Universal critical dynamics in high resolution neuronal avalanche data", Phys. Rev. Lett., 108: 208102

Examples of nonsymmetric average avalanche shapes





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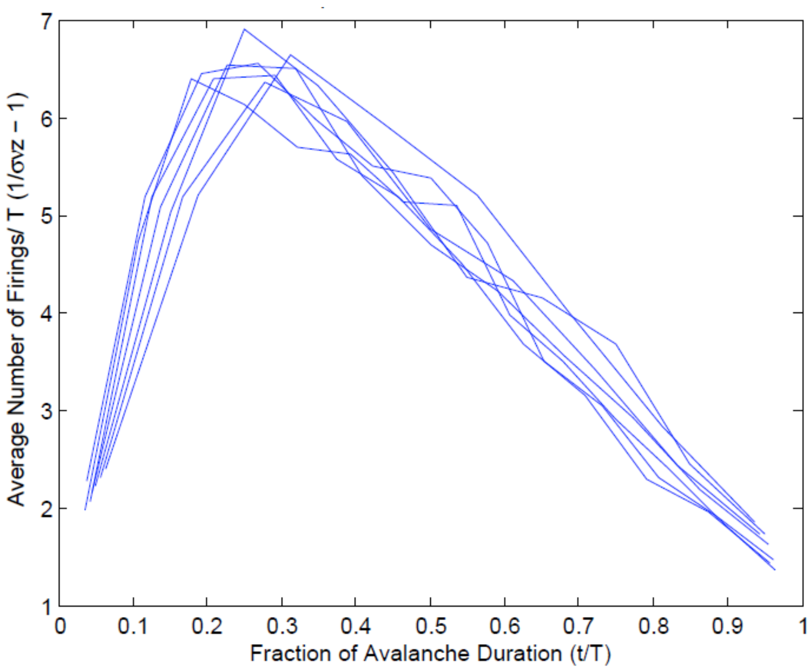

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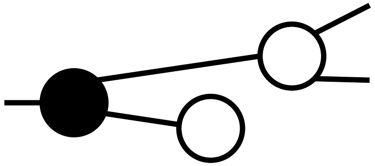
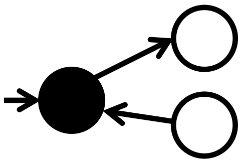
Cascade models: examples

- Threshold model (undirected network)



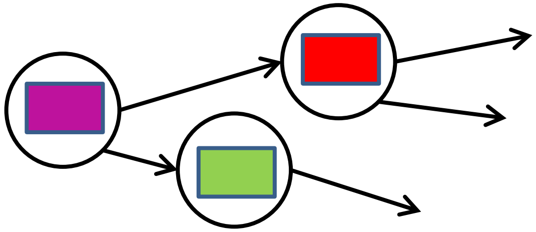
Each node has a threshold r that is assigned randomly from a distribution. All nodes are initially inactive, except for one seed node. When an inactive node is updated, it becomes active if the number m of its active neighbours exceeds its threshold r.

- Neuronal dynamics model of Friedman et al. (directed network)



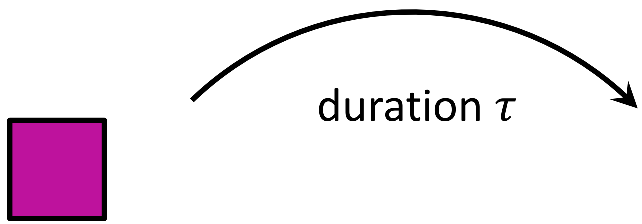
The weight ϕ_{ij} of each directed edge from neuron i to neuron j is assigned randomly from a uniform distribution on $[0, \phi_{max}]$. When neuron i fires (becomes active), it causes neuron j to become active (in the next discrete time step) with probability ϕ_{ij}. After a neuron fires, it returns to the inactive state in the next time step.

- Meme diffusion model (directed network)



Branching process theory and cascades on networks

Particles:



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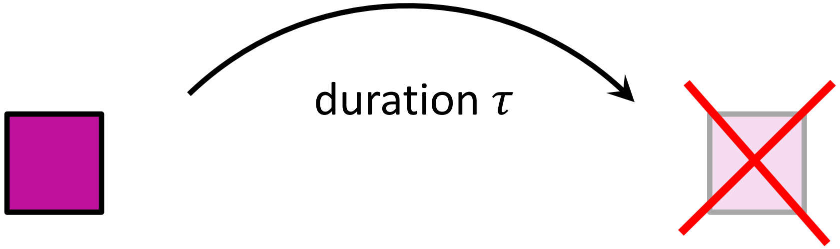

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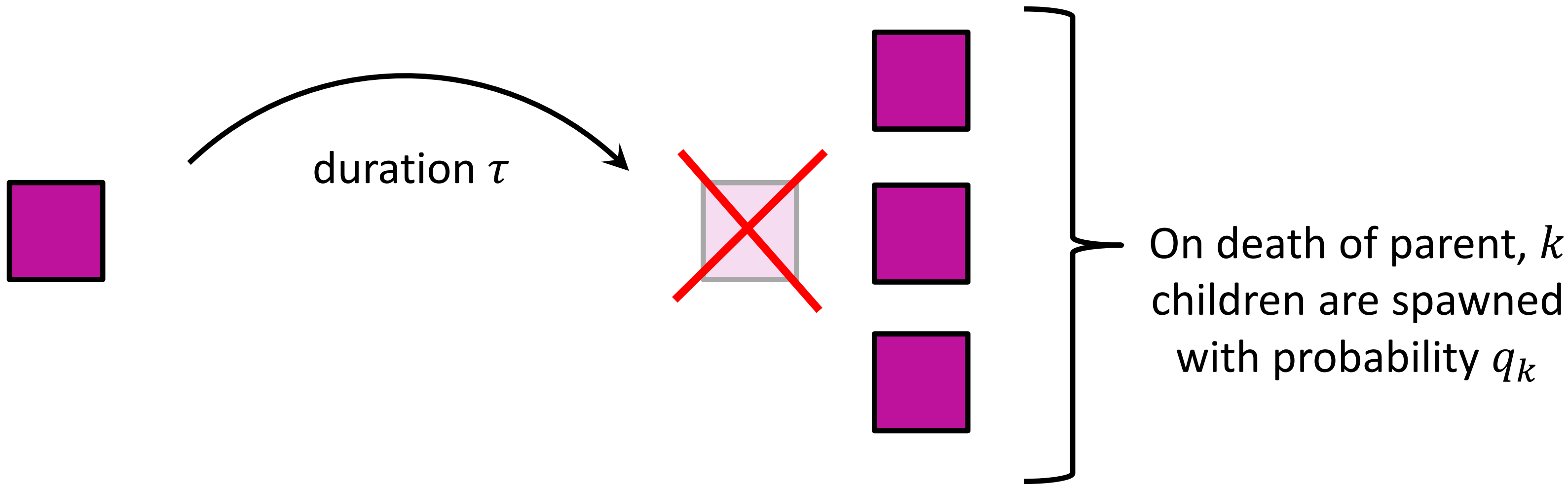

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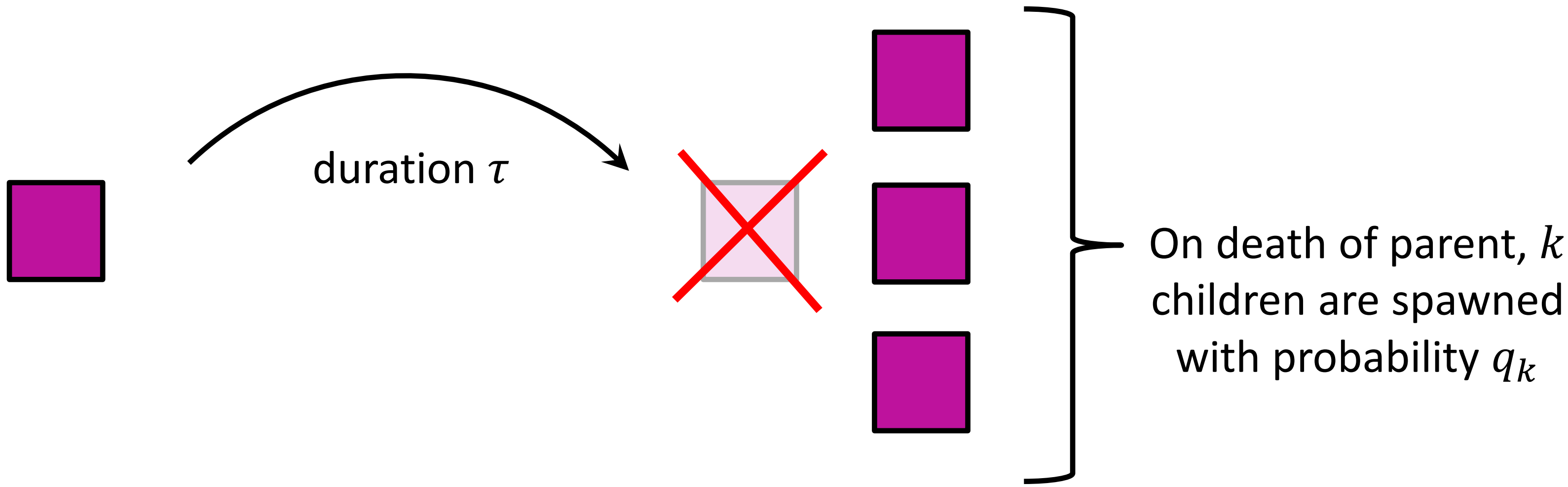
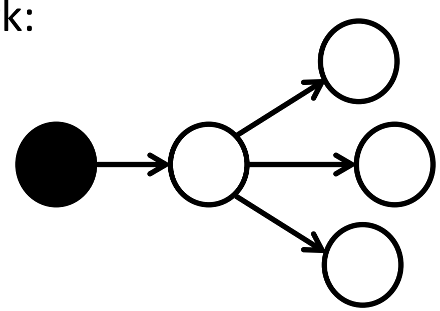


Network:



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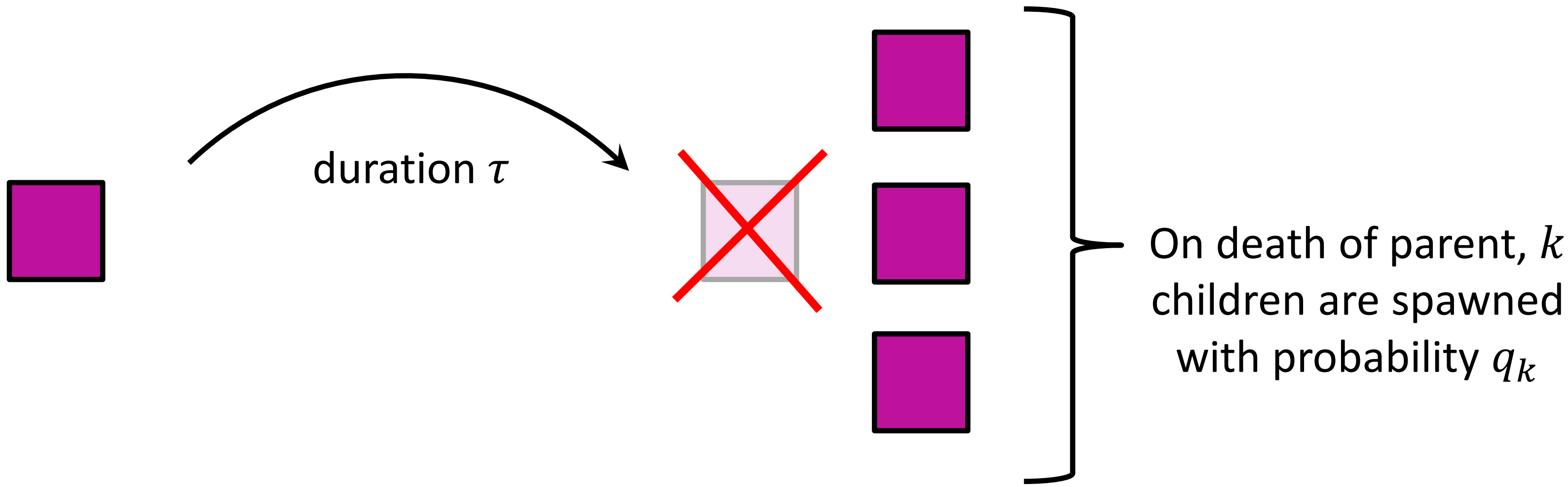

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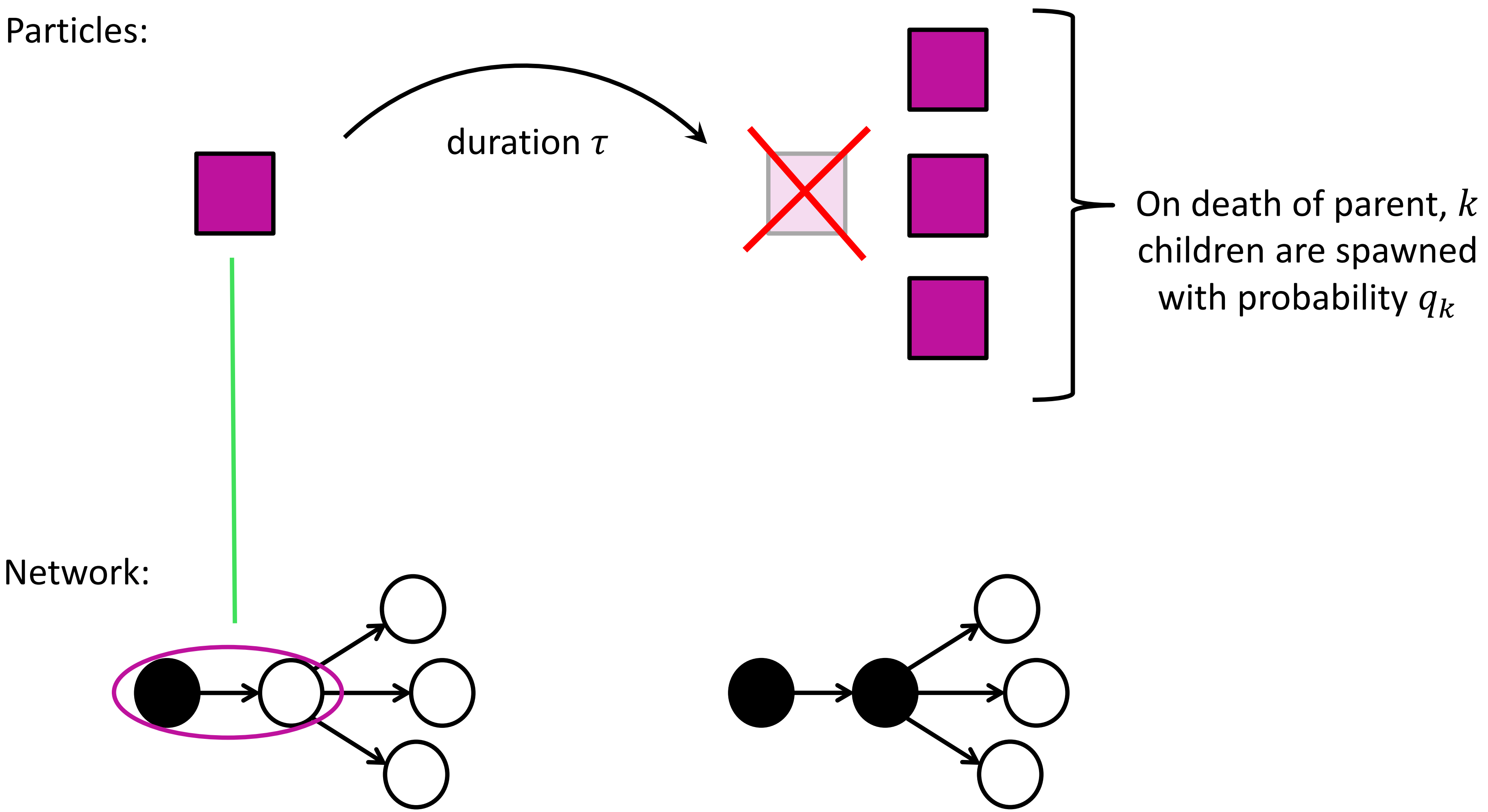

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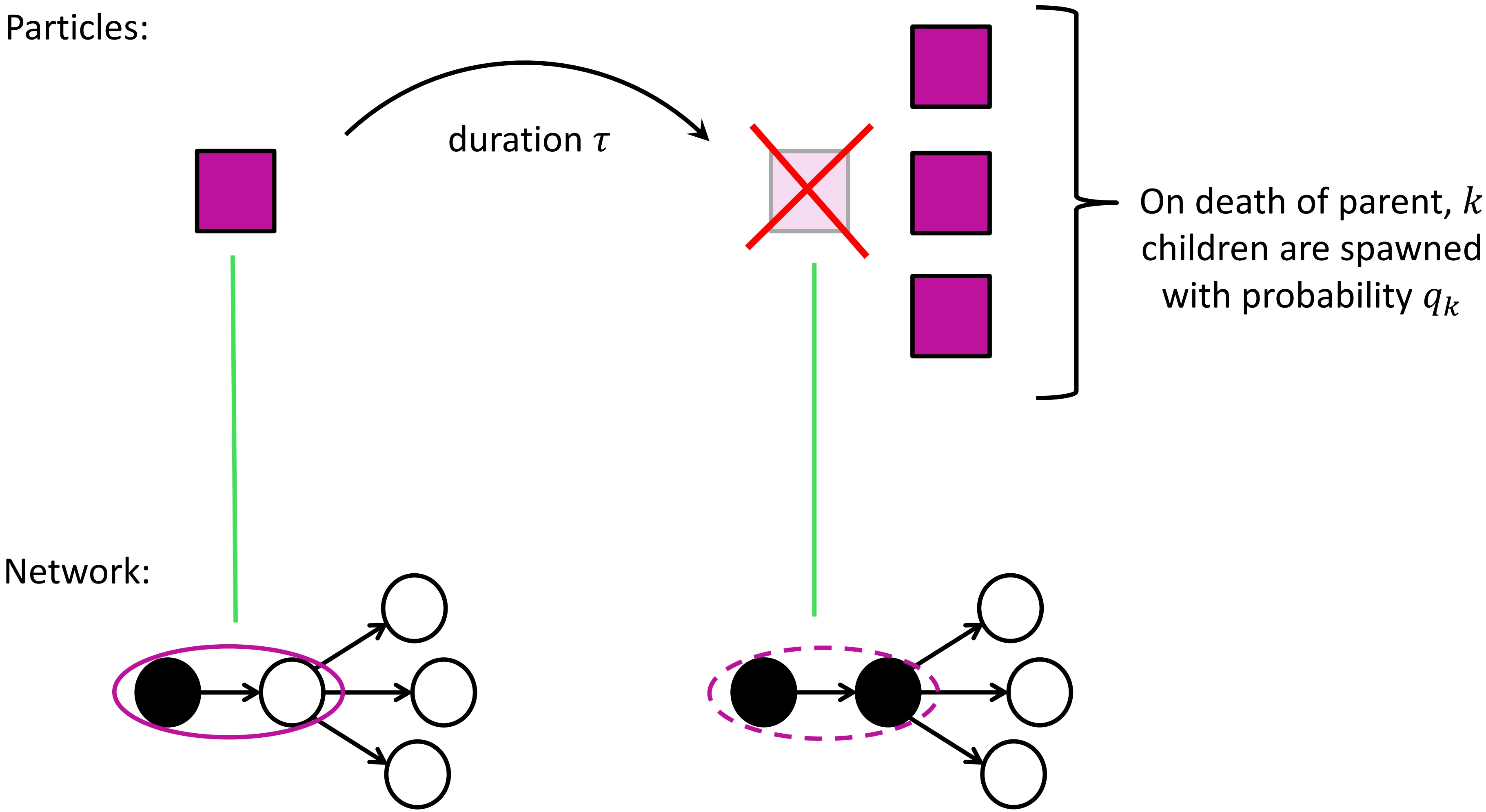

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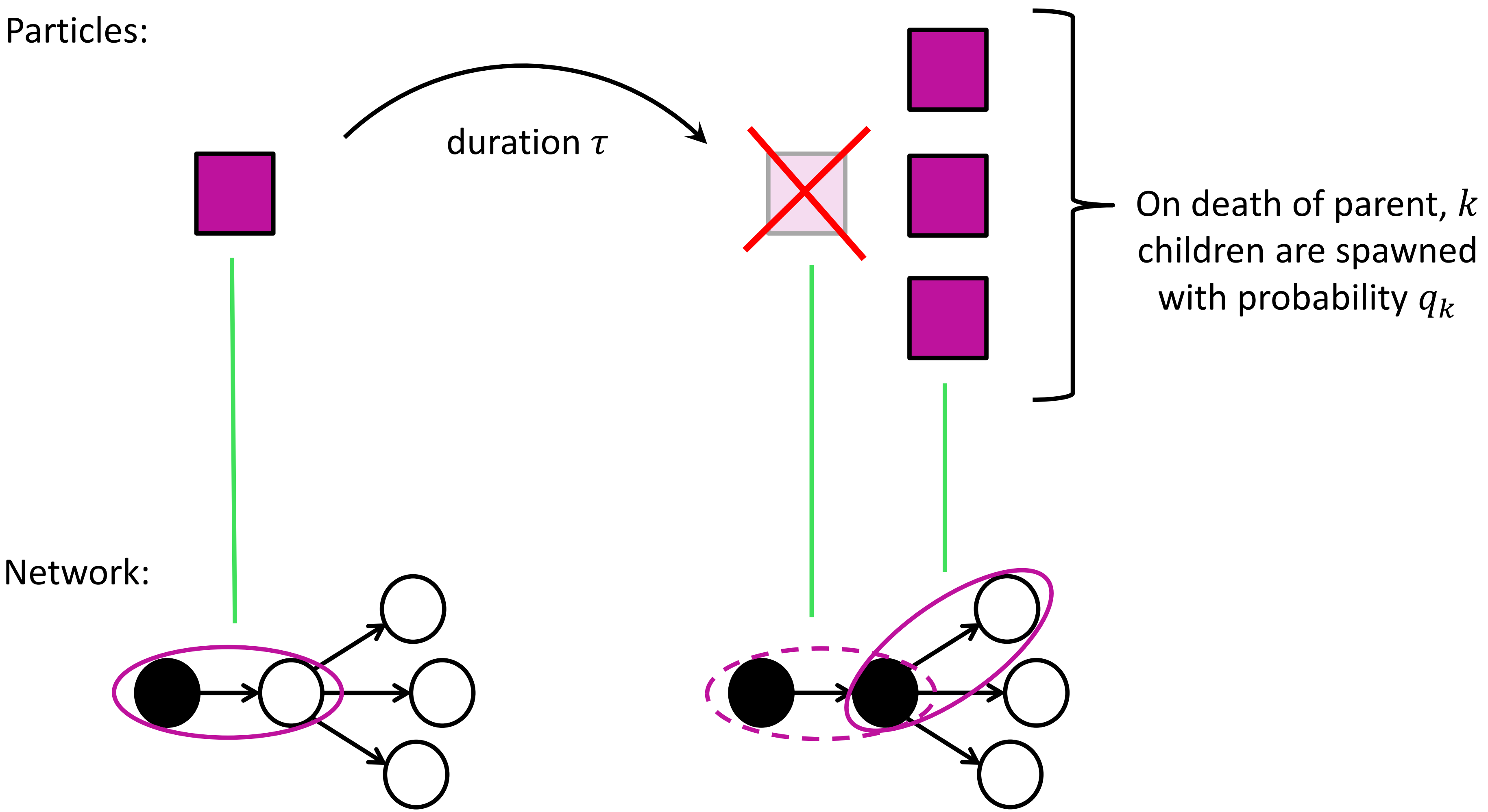

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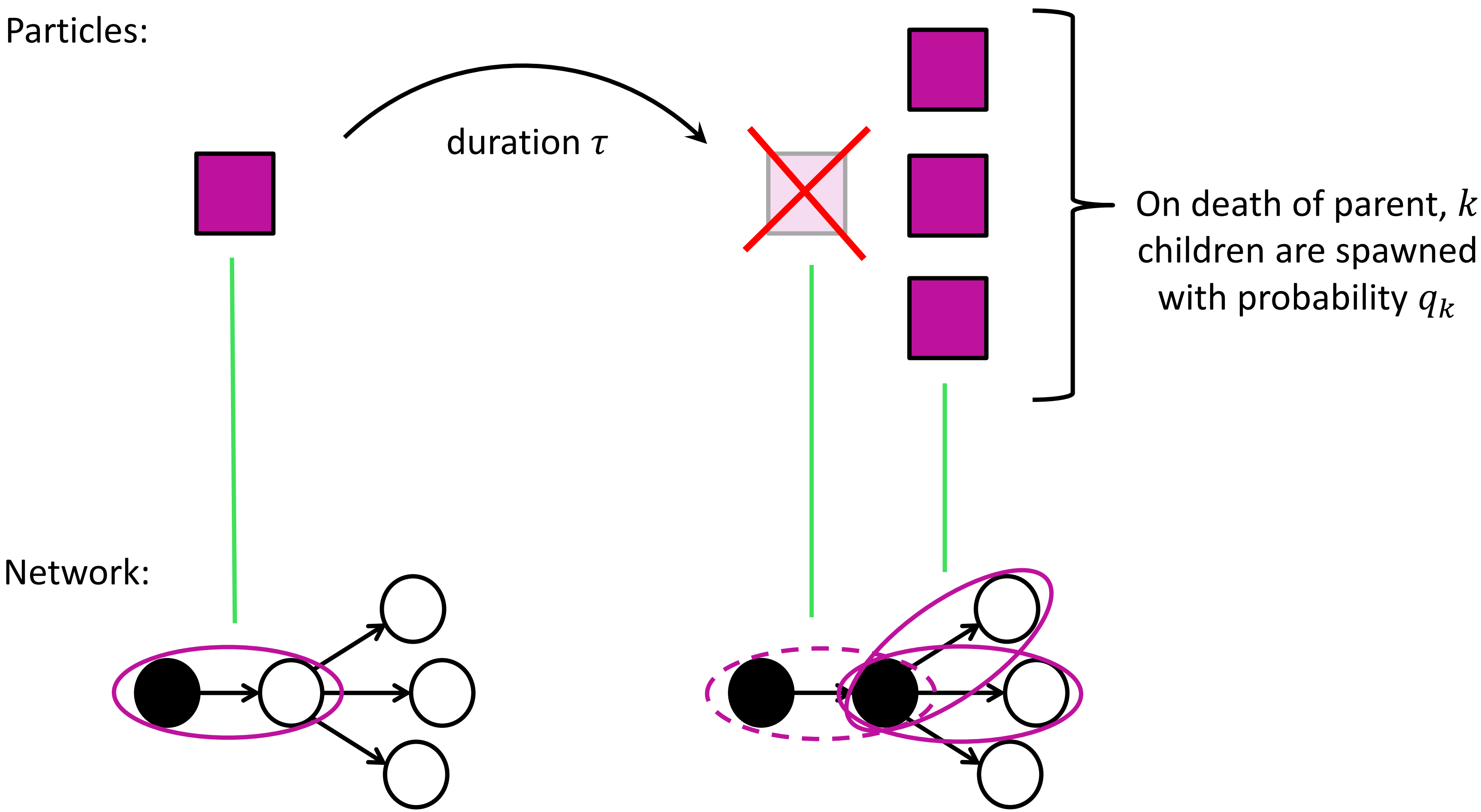

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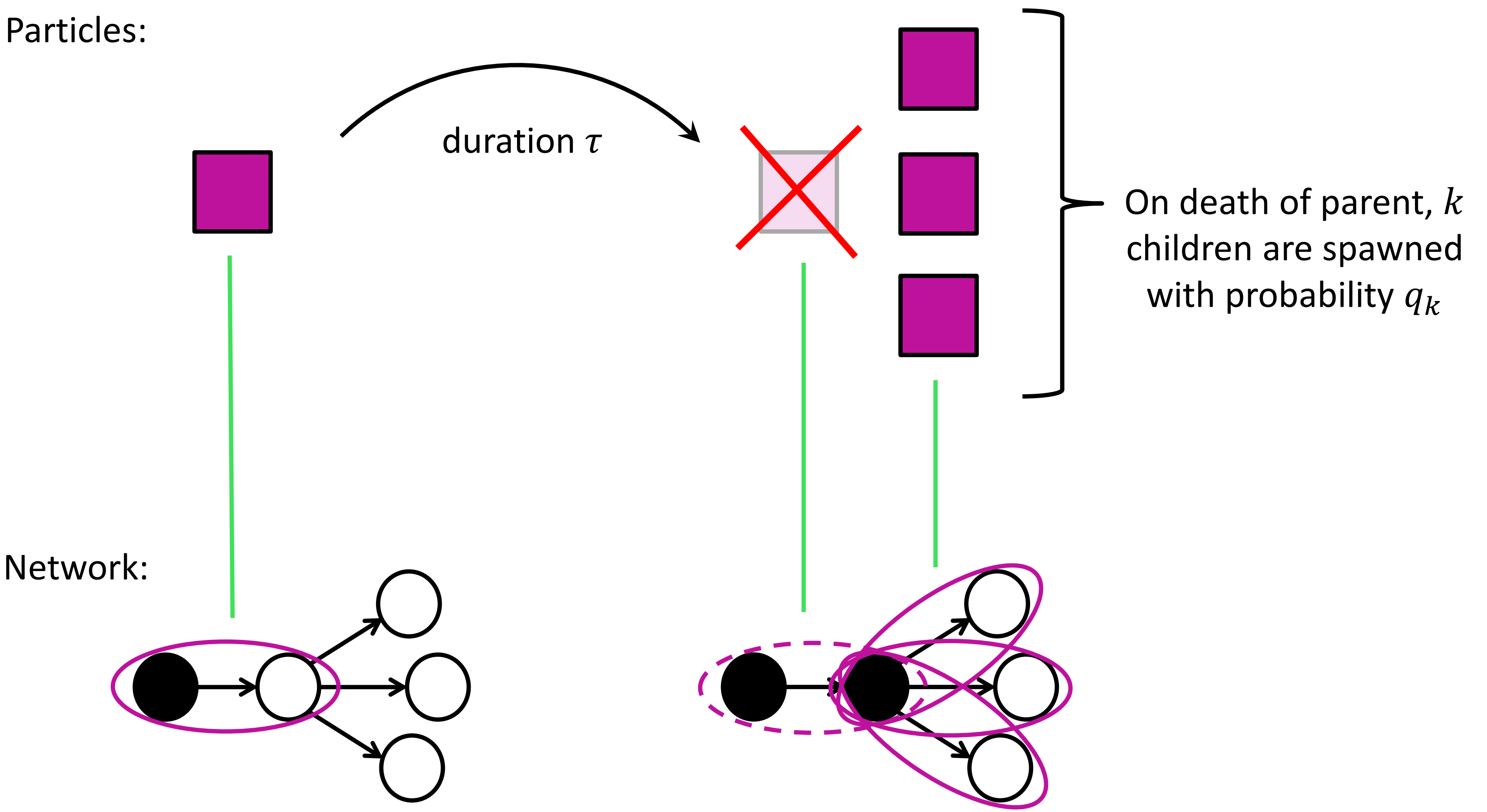

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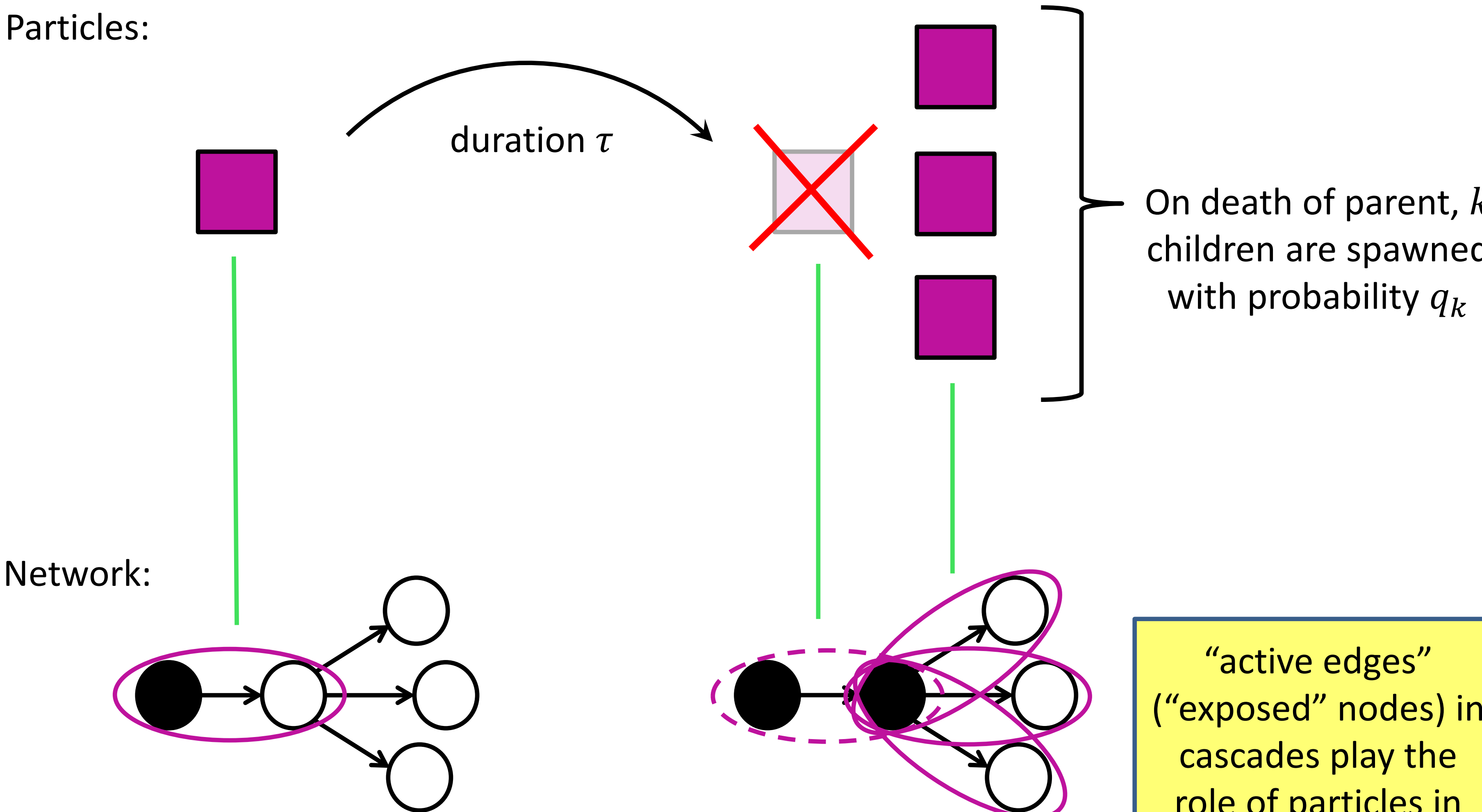

"active edges" ("exposed" nodes) in cascades play the role of particles in branching processes

Branching process theory and cascades on networks

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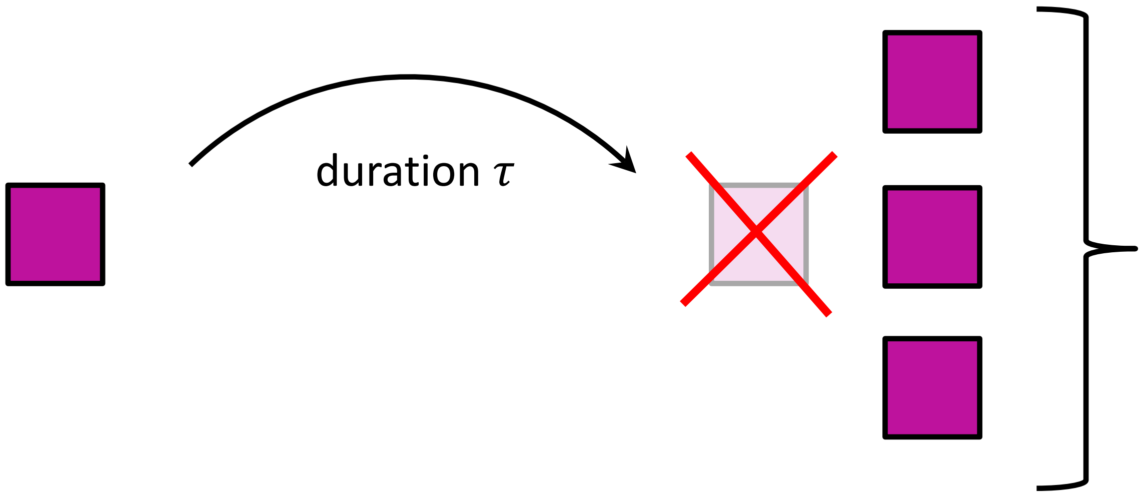
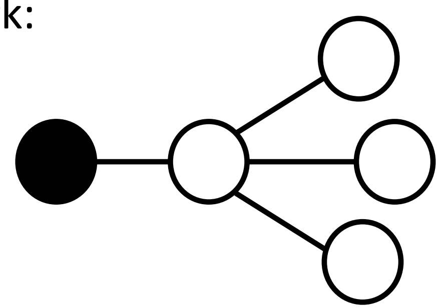


On death of parent, k children are spawned with probability q_k

Network:



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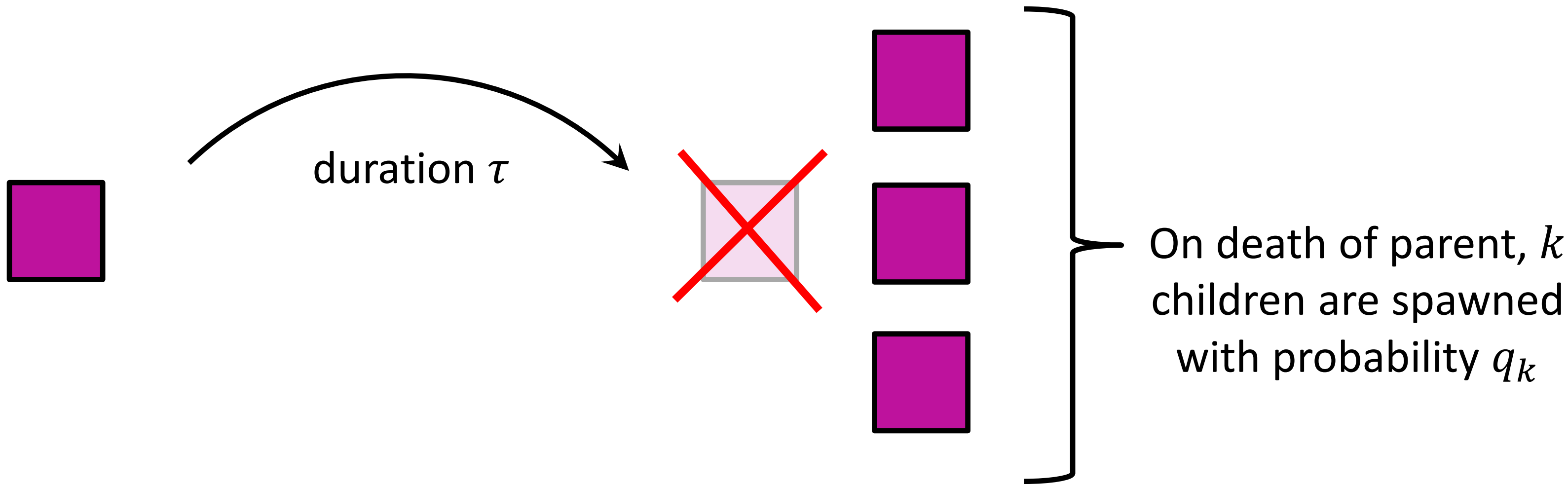

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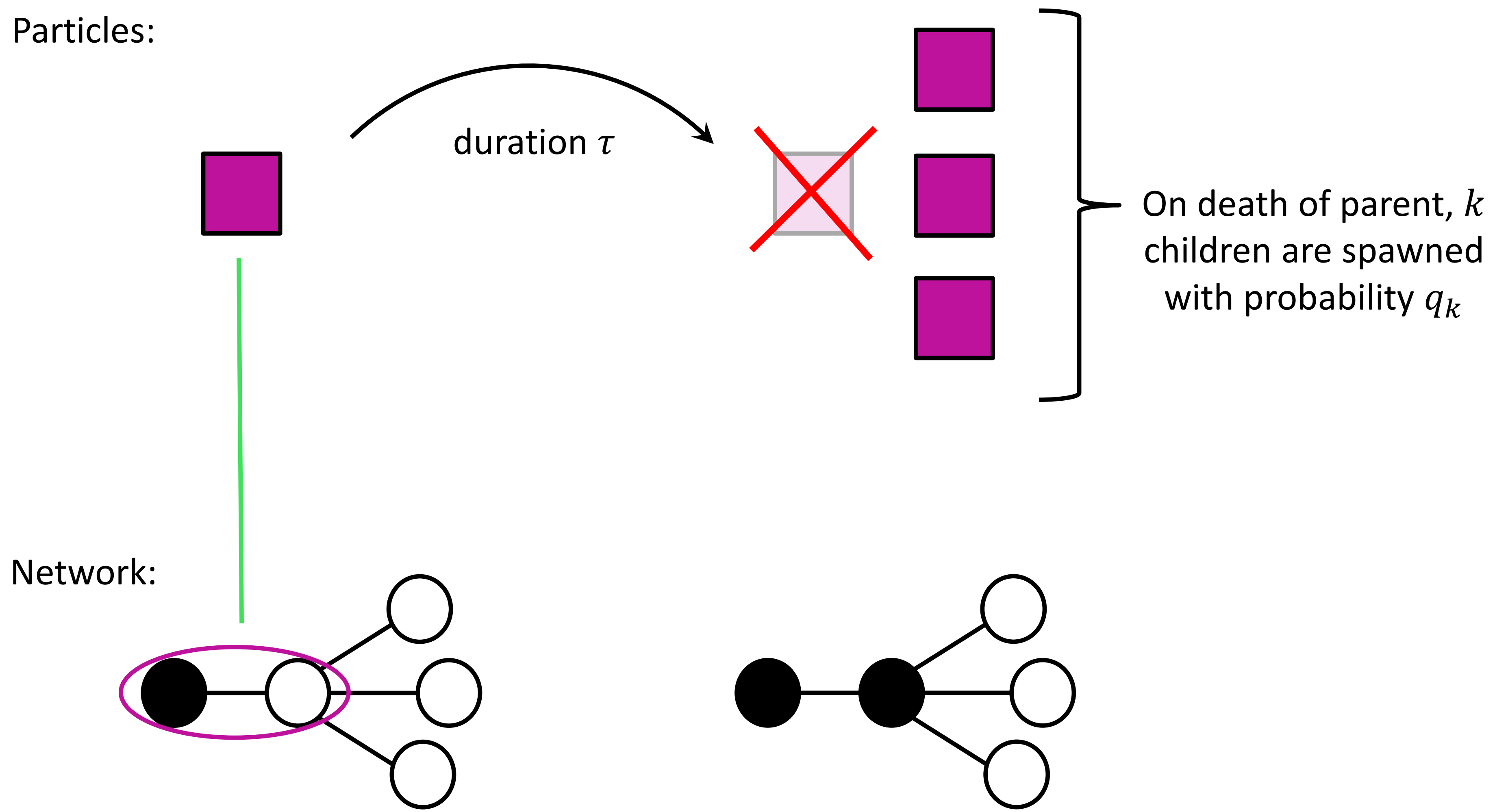

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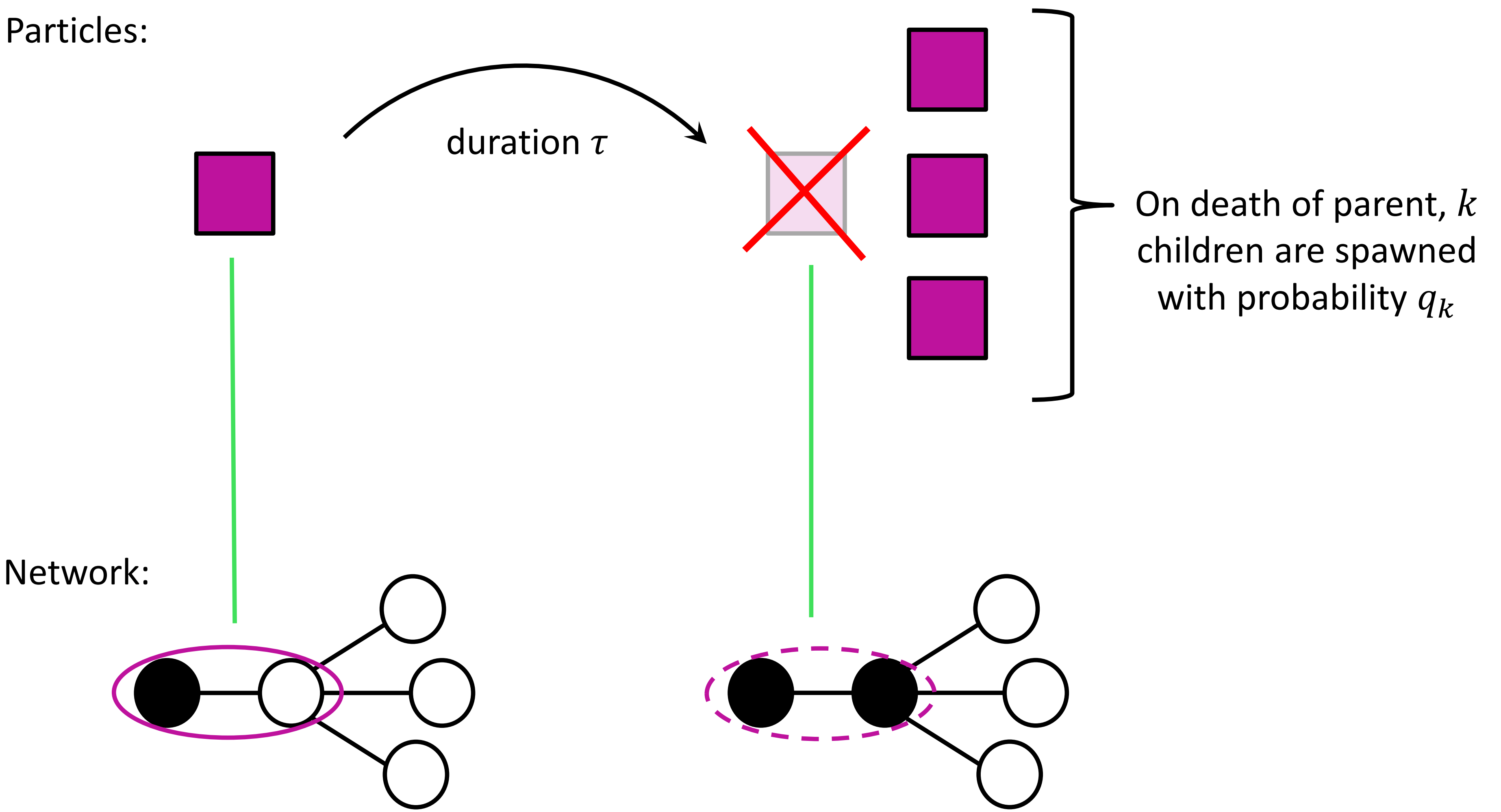

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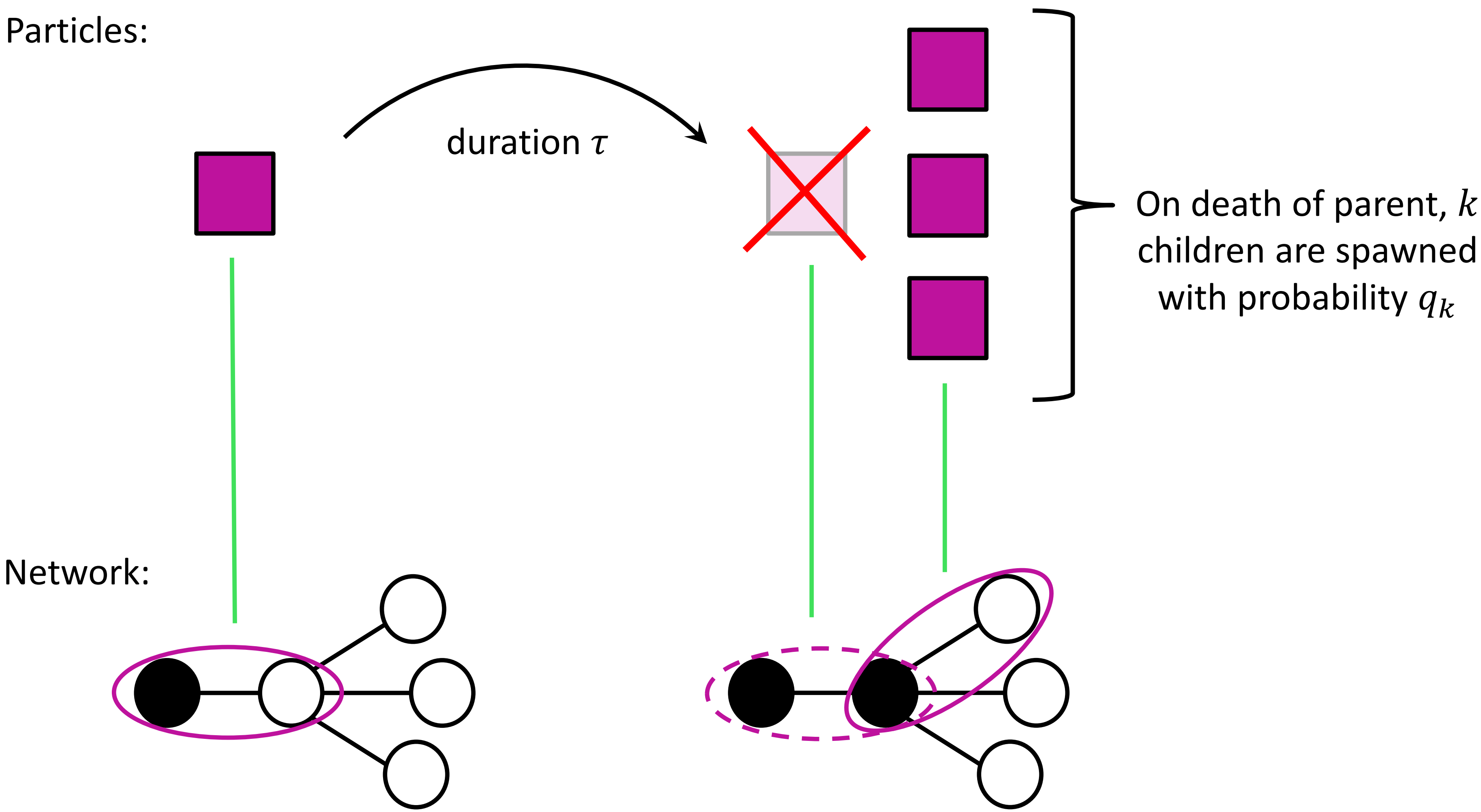

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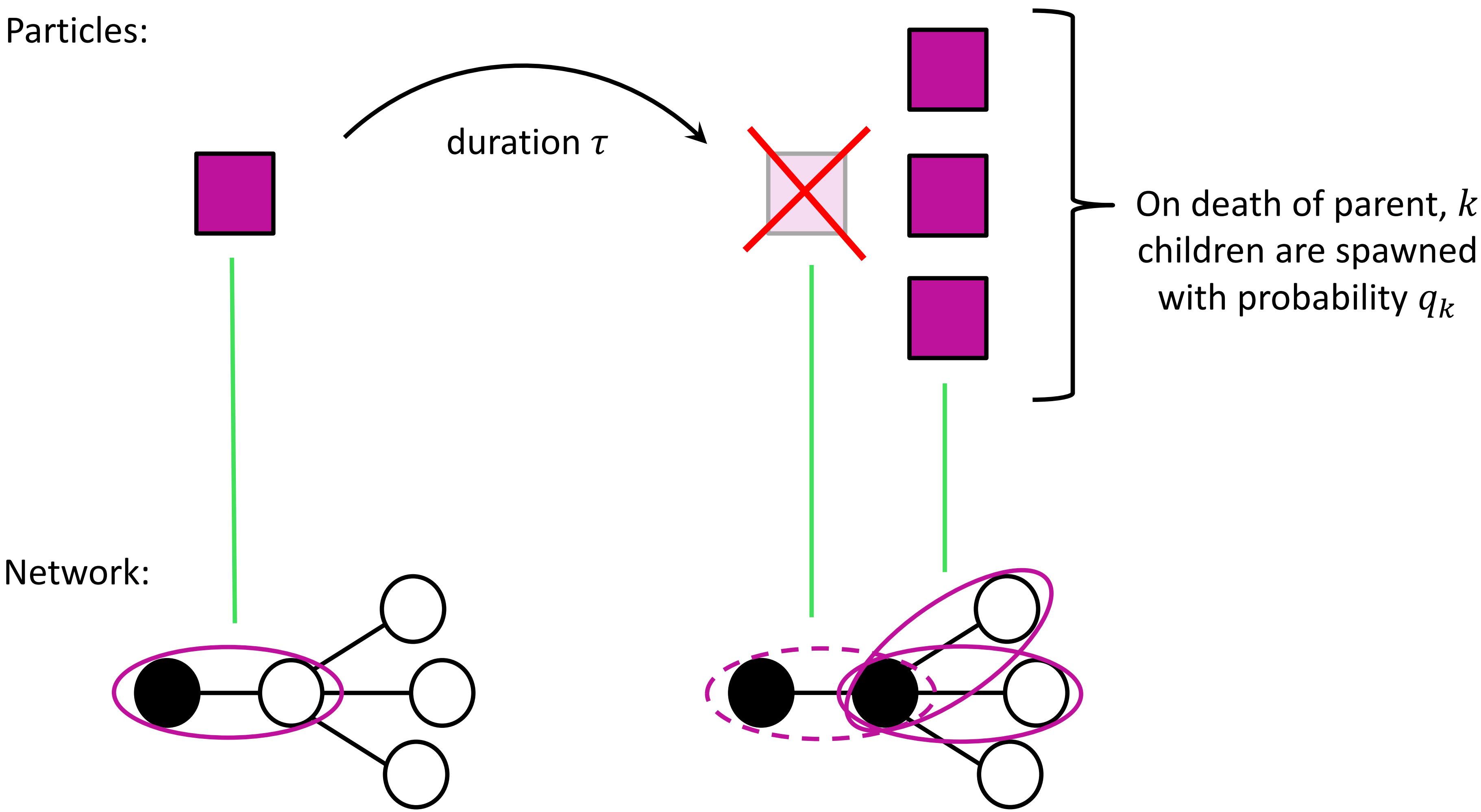

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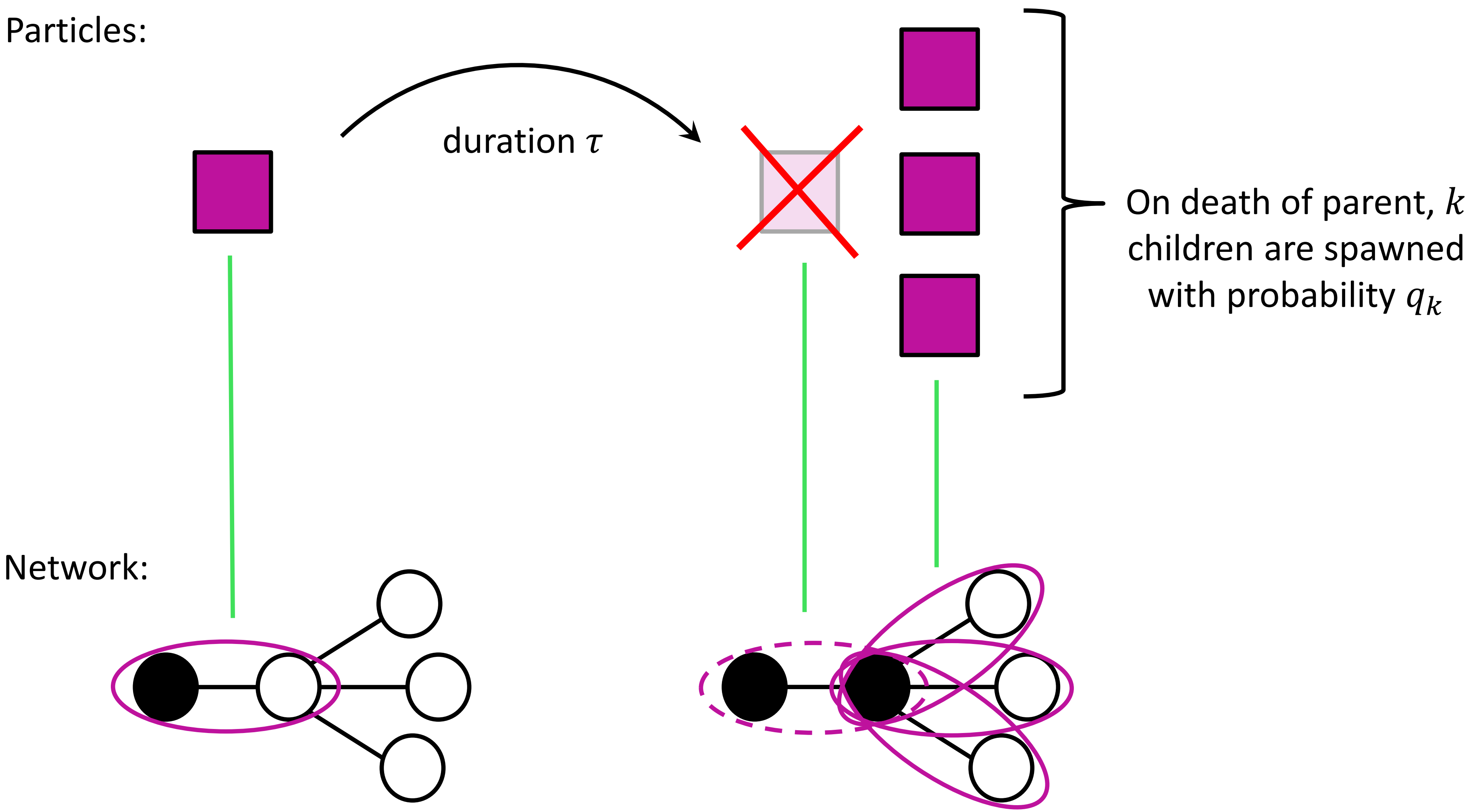

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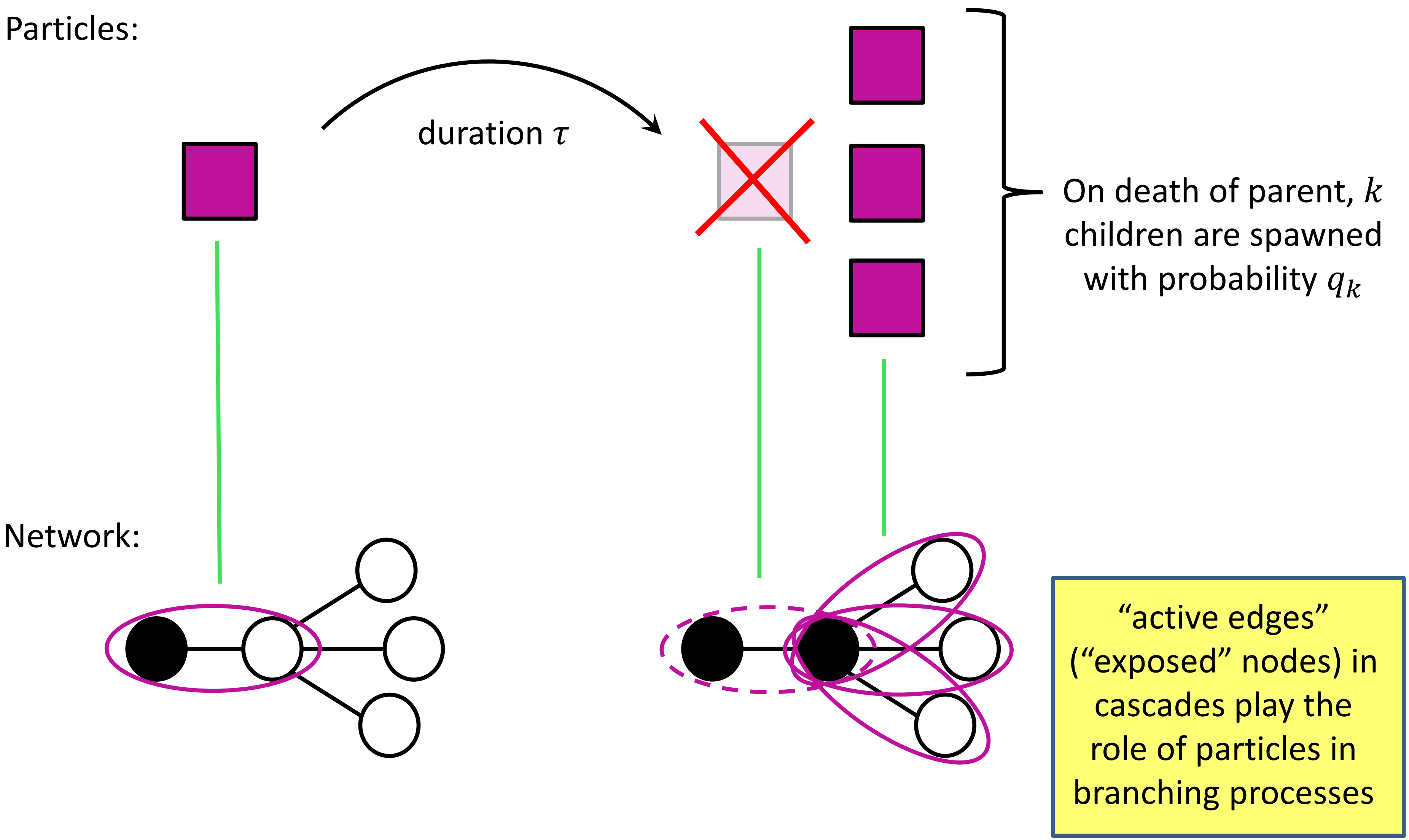
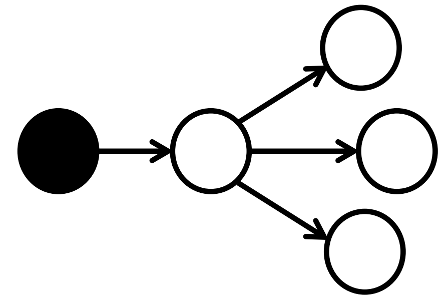


Effective offspring distribution



- Directed network:

$$q_k = \sum_j \frac{j}{z} p_{jk} \, v_{jk}$$

A node with in-degree j and out-degree k is *vulnerable* with probability v_{jk}: This is the probability that the activation of a single in-neighbour (at time t_1) will lead to the activation of the node at some time $t > t_1$, assuming that no other in-neighbour of the node becomes active by time t.

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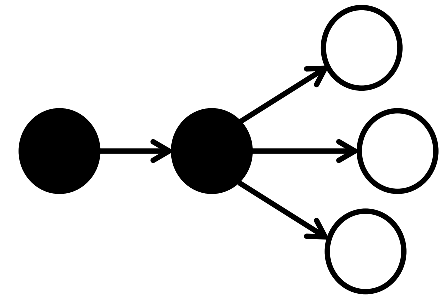

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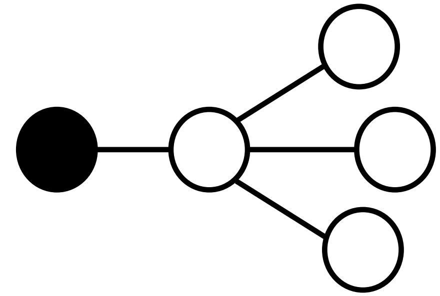
- Example: Neuronal dynamics model of Friedman et al.

$$v_{jk} = \frac{\phi_{max}}{2}$$

Effective offspring distribution

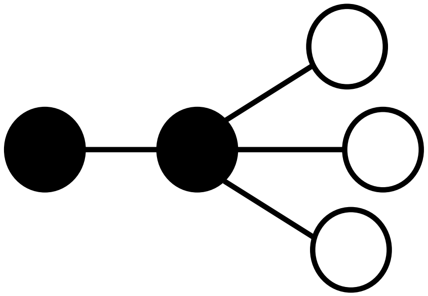


- Undirected network:

$$q_k = \frac{k+1}{z} p_{k+1} v_{k+1}$$

A node with degree $k+1$ (i.e., with k inactive neighbours) is *vulnerable* with probability v_{k+1}: This is the probability that the activation of a single neighbour (at time t_1) will lead to the activation of the node at some time $t > t_1$, assuming that no other neighbour of the node becomes active by time t.

Effective offspring distribution



- Undirected network:

$$q_k = \frac{k+1}{z} p_{k+1} v_{k+1}$$

A node with degree $k+1$ (i.e., with k inactive neighbours) is *vulnerable* with probability v_{k+1}: This is the probability that the activation of a single neighbour (at time t_1) will lead to the activation of the node at some time $t > t_1$, assuming that no other neighbour of the node becomes active by time t.

- Example: Threshold model

$$v_{k+1} = C(1)$$

CDF of thresholds

node of degree $k+1$ is activated by a single active neighbour iff its threshold is less than 1

Effective branching number

- The effective branching number is the mean of the offspring distribution:

$$\xi = \sum_{k} k\, q_k$$

A process is

- critical if $\xi = 1$
- subcritical if $\xi < 1$
- supercritical is $\xi > 1$

- Define the probability generating function (PGF) $f(x)$ by

$$f(x) = \sum_{k=0}^{\infty} q_k x^k$$

so that the effective branching number is $\xi = f'(1)$

Overview

1. Average avalanche shape functions (and beyond…)

2. Analytical results

3. Numerical simulations

Calculating the average avalanche shape using Markovian branching process theory

- Given: avalanche duration T and the offspring distribution q_k (and hence PGF $f(x)$)

- First, solve the ordinary differential equation

$$\boxed{\frac{dQ}{dt} = f(Q) - Q \quad \text{for} \quad Q(t) \ \text{with} \ \ Q(0) = 0}$$

- Then, using the solution $Q(t)$ and the PGF gives the average avalanche shape function as

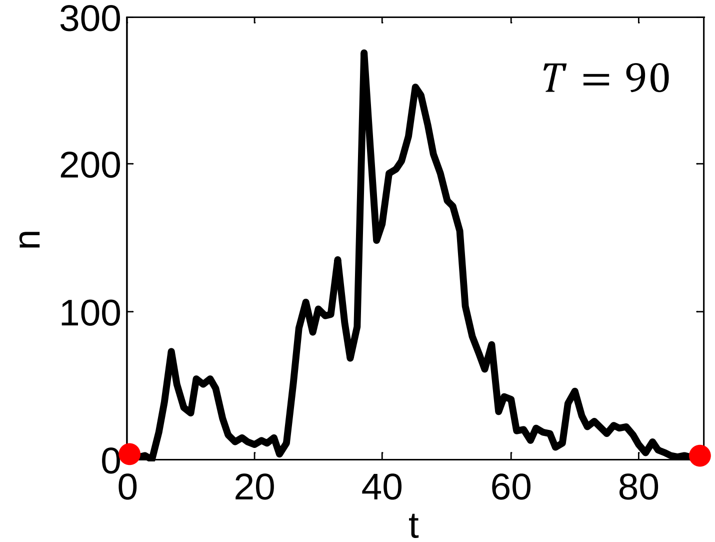
$$\boxed{A(t) = Q(T-t)\frac{f'(Q(T)) - f'(Q(T-t))}{f(Q(T-t)) - Q(T-t)}}$$

Derivation of the average avalanche shape result

- Continuous-time Markov branching processes

$$P_{1j}(t) = P\{Z(\tau + t) = j \mid Z(\tau) = 1\}$$

$$F(s,t) = \sum_k P_{1k}(t) s^k$$



- Probability of extinction by time t:

$$Q(t) = P\{Z(t) = 0 \mid Z(0) = 1\} = F(0,t)$$

- Avalanche path probability:

$$\pi_n(t) = P\{Z(t) = n \mid Z(0) = 1 \text{ and } Z(T) = 1\}$$

$$\pi_n(t) \propto P_{1n}(t) P_{n1}(T - t)$$

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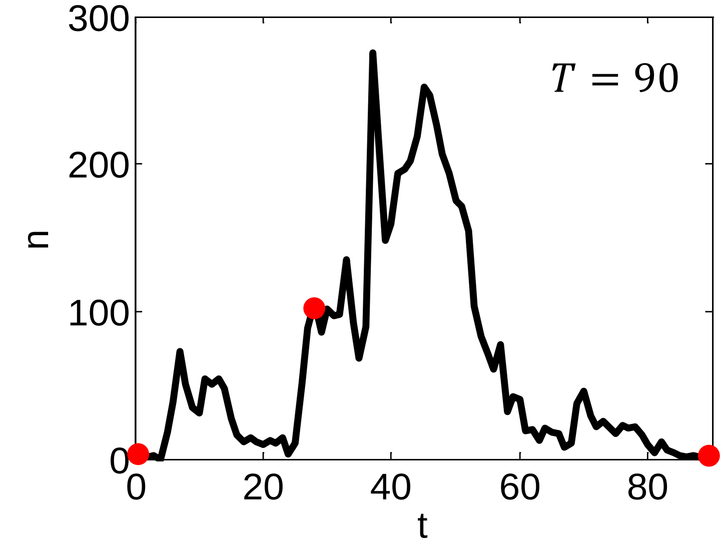

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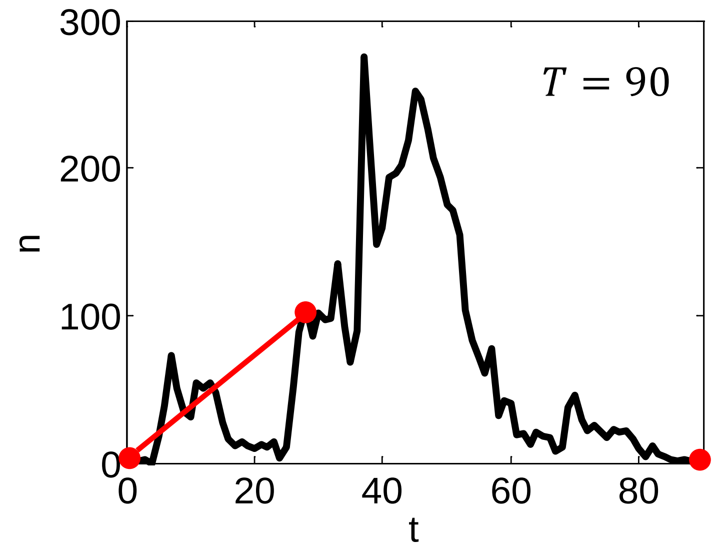

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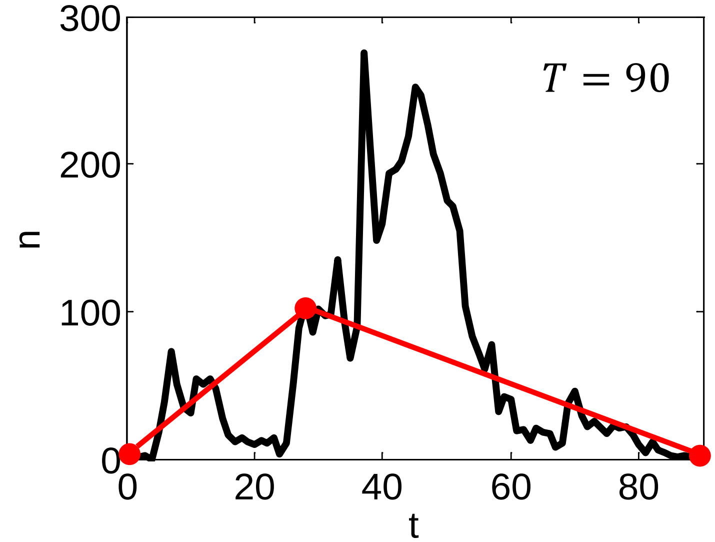

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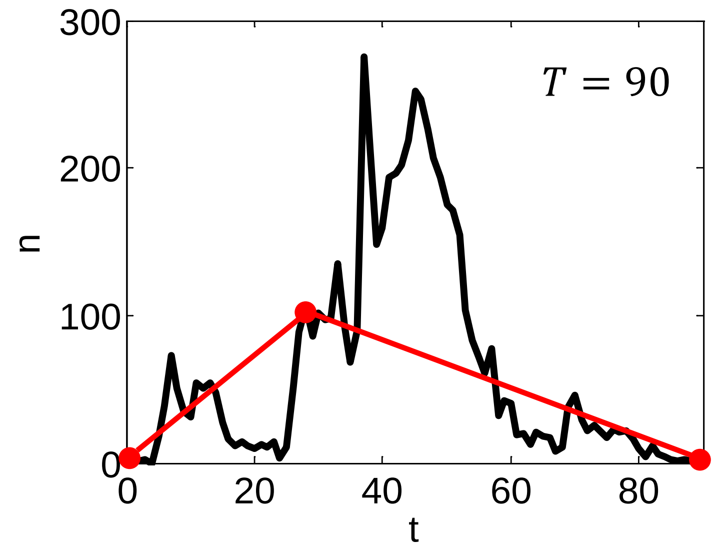

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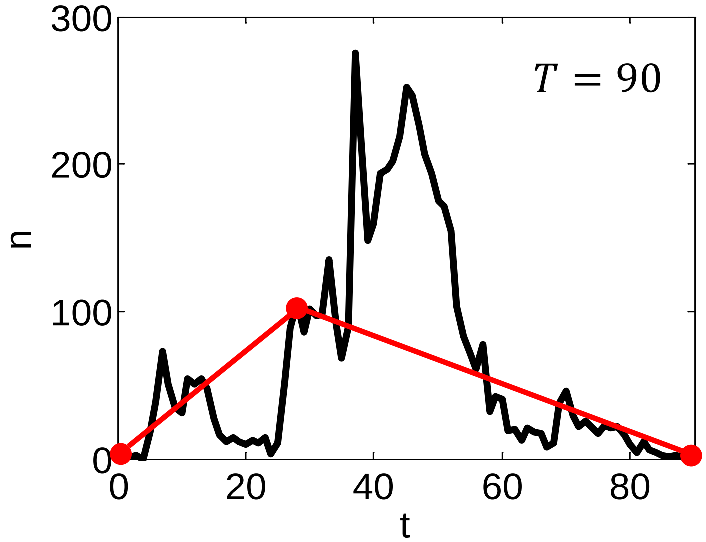

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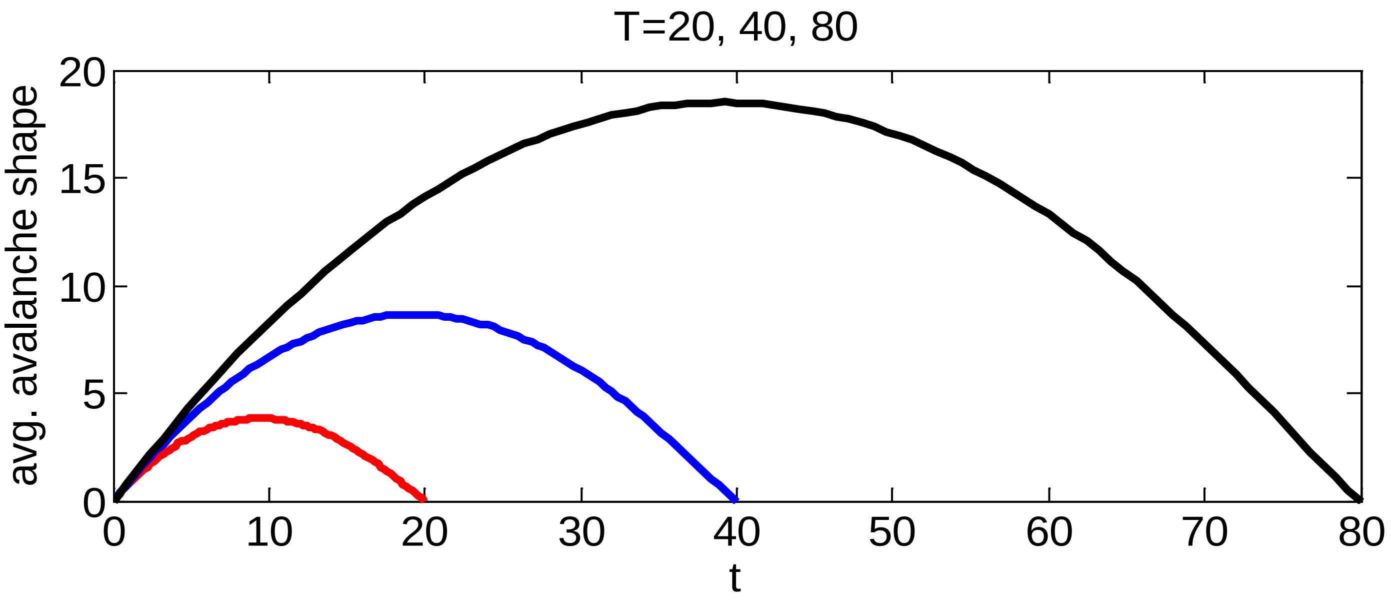
Examples of average avalanche shape functions

$$A(t) = Q(T-t)\,\frac{f'(Q(T)) - f'(Q(T-t))}{f(Q(T-t)) - Q(T-t)}$$

q_k Poisson, $\xi = 1$ (critical)



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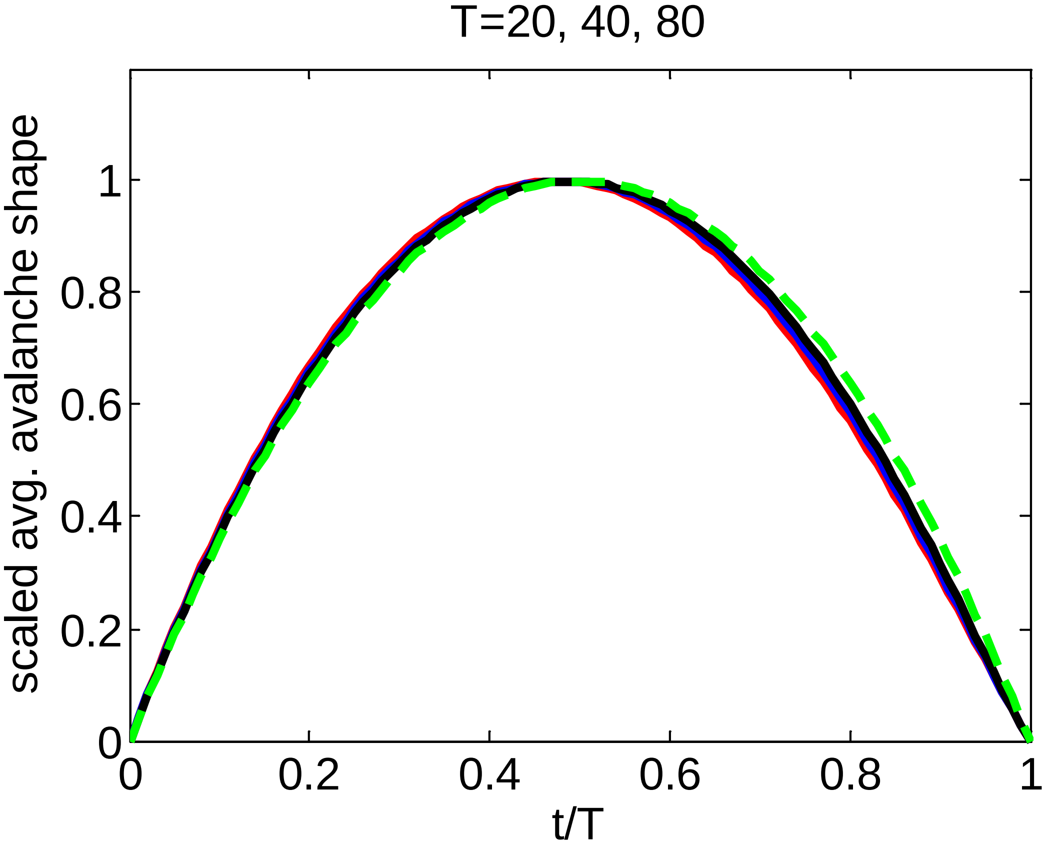

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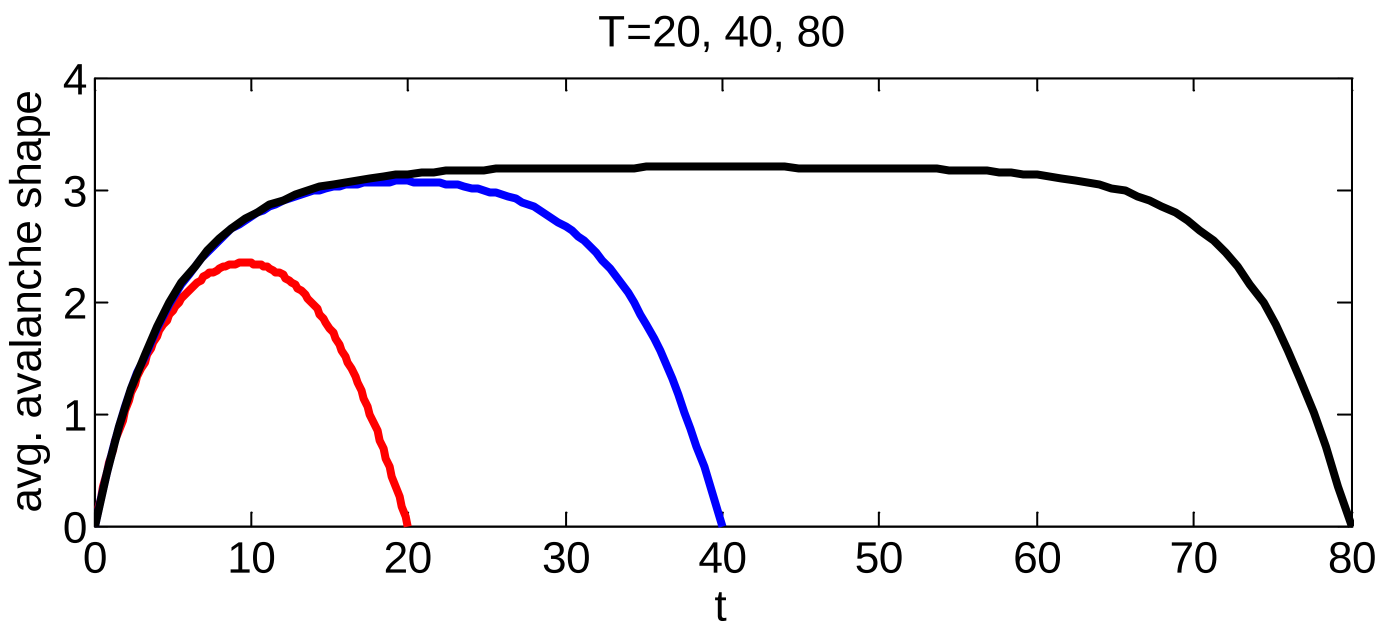

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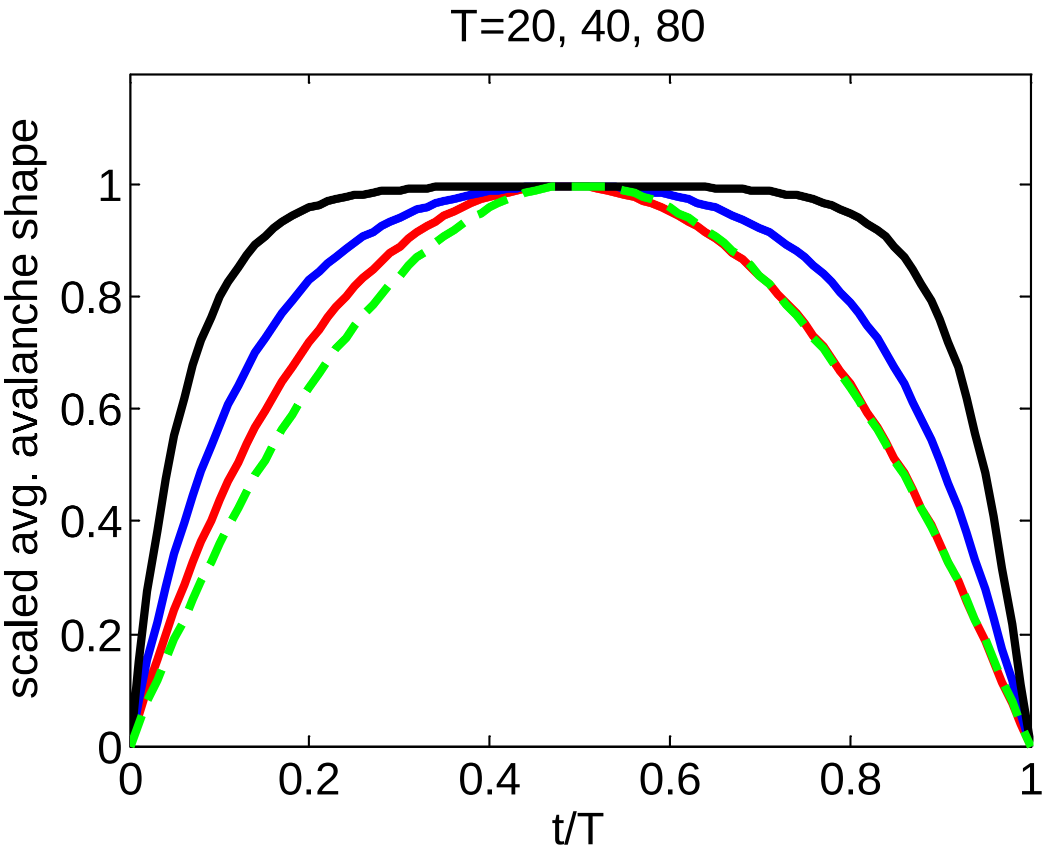

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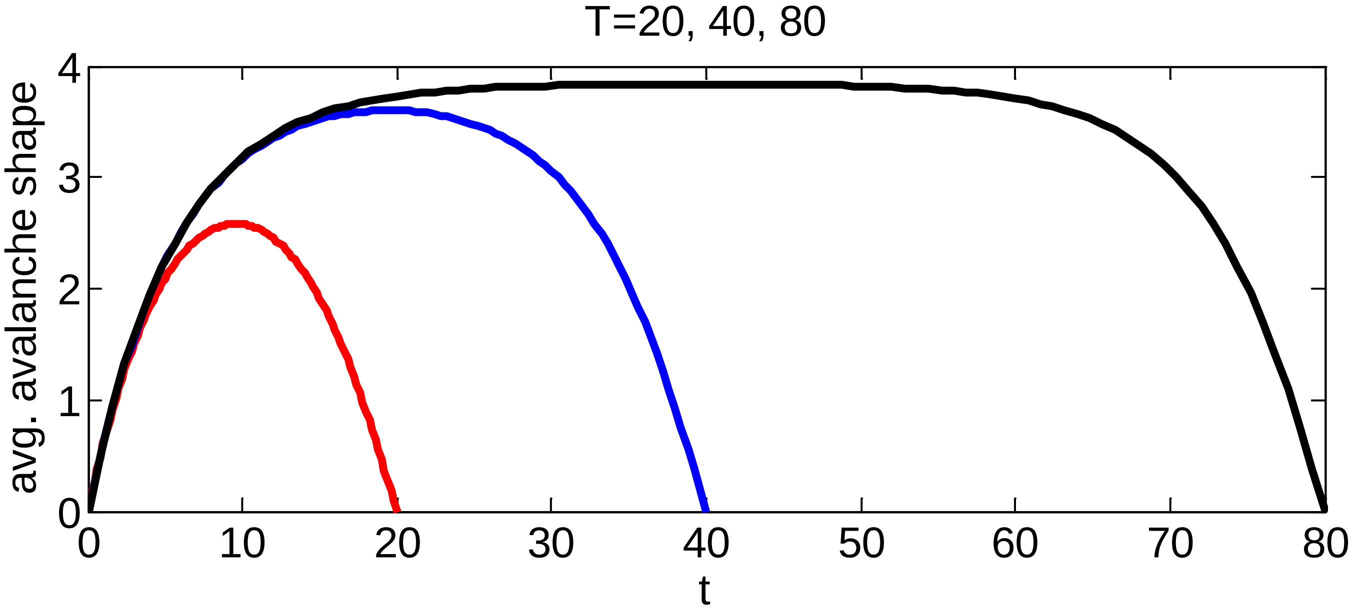

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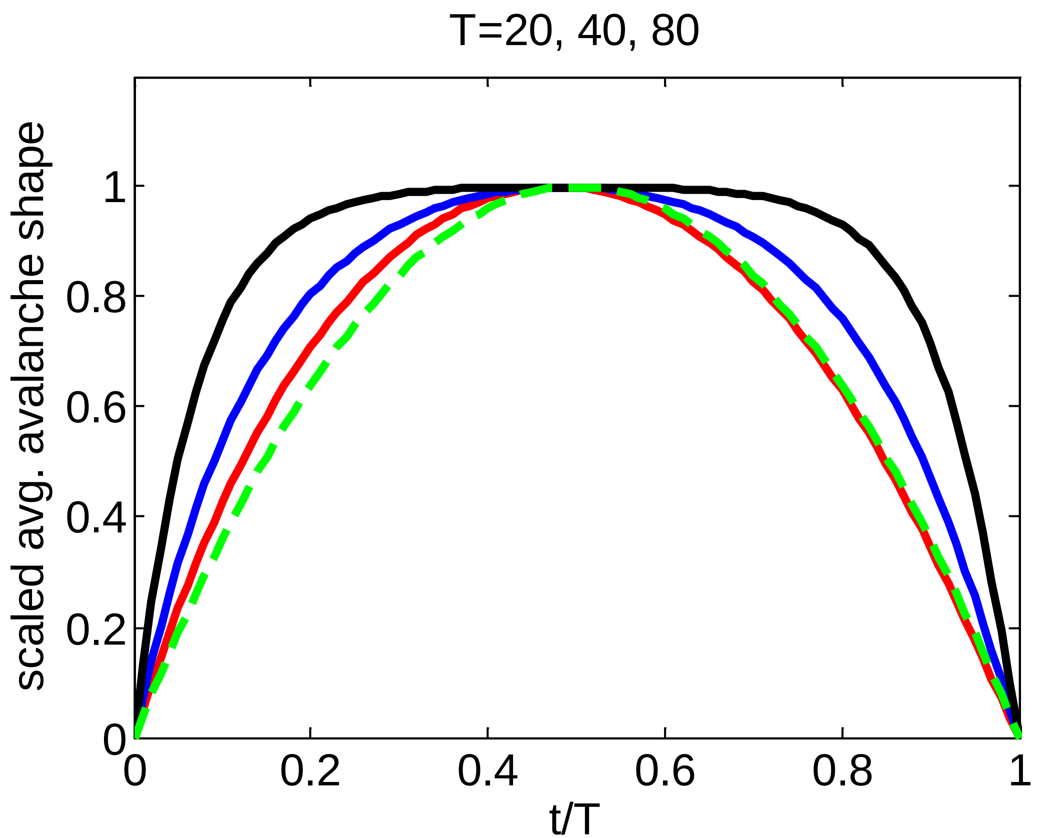

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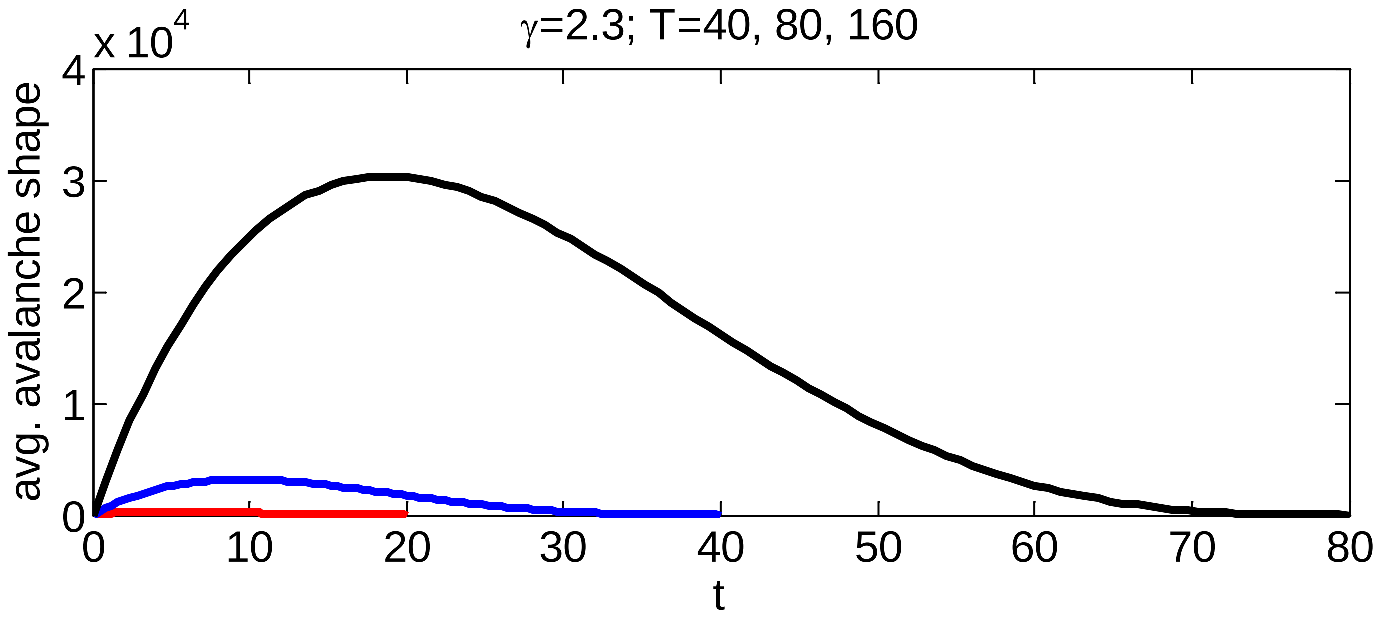

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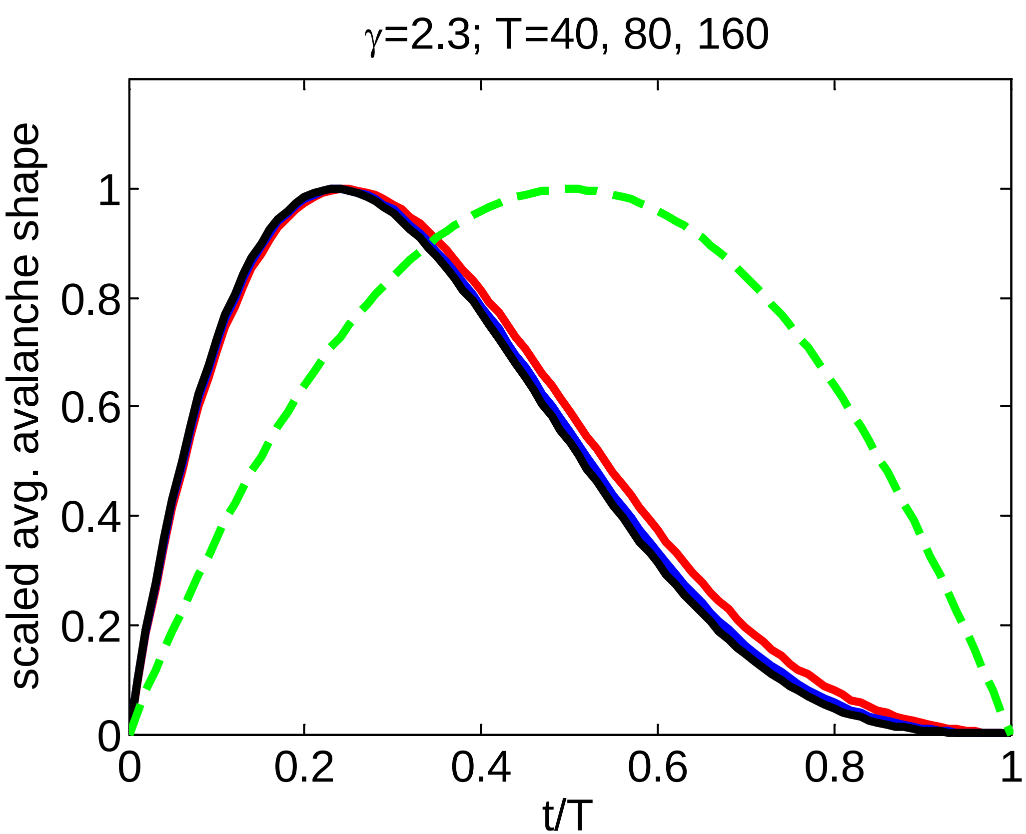

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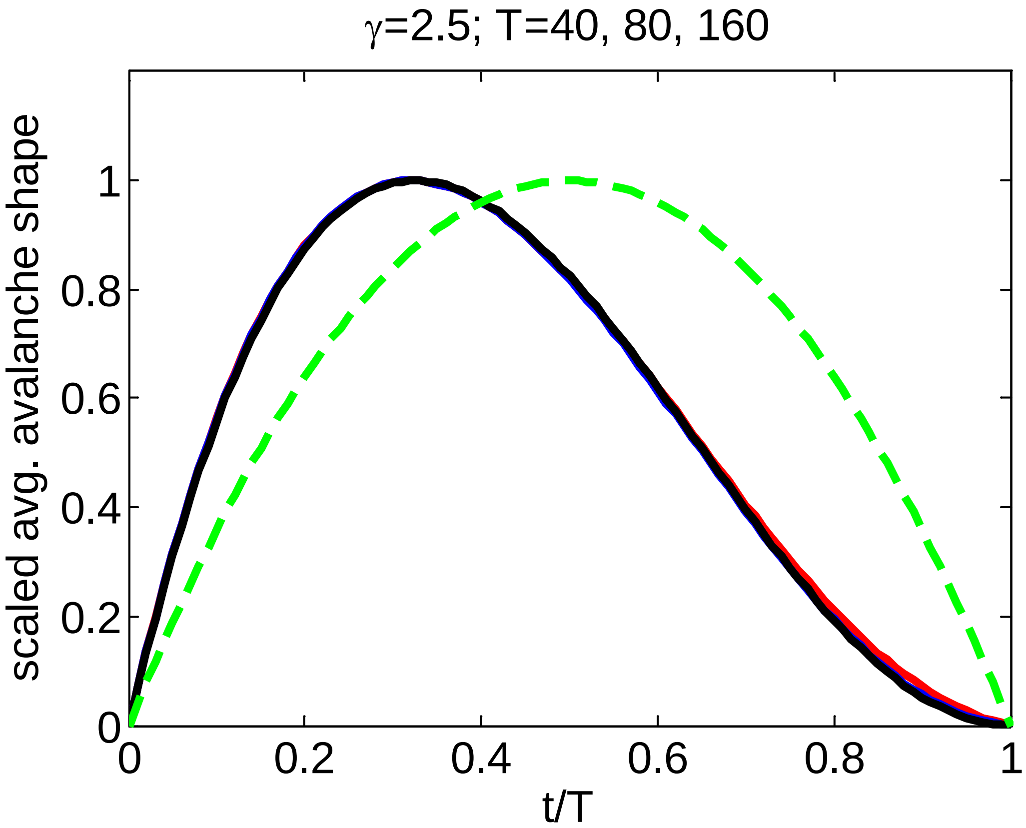

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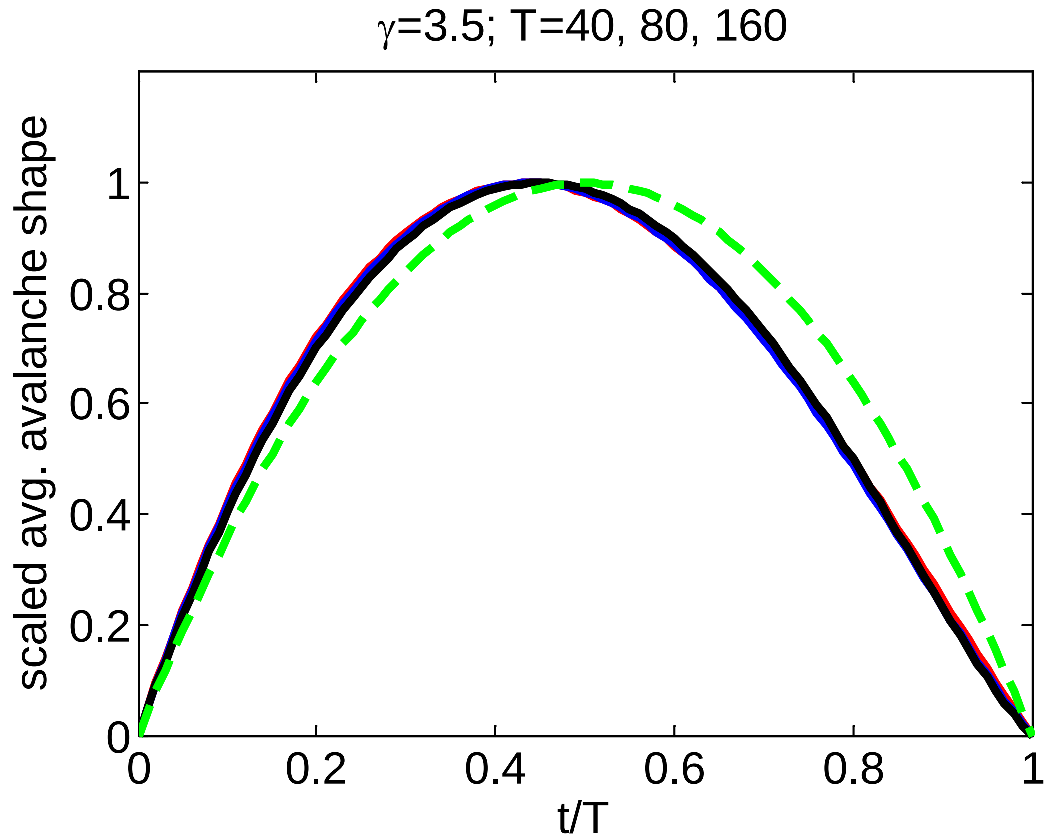

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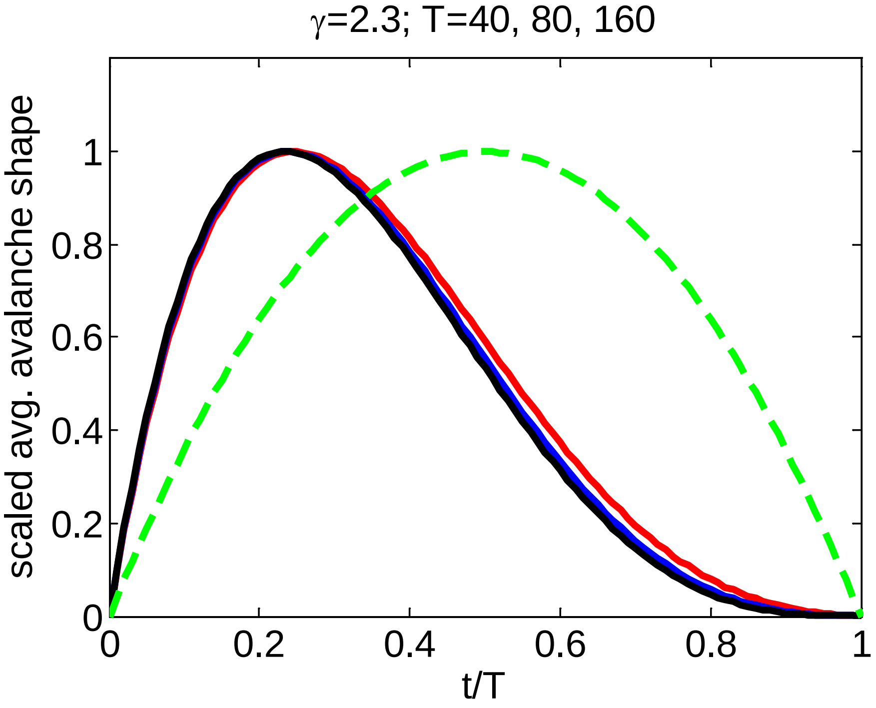

Nonsymmetric avalanche shape functions occur when the offspring distribution has a power-law tail: $q_k \sim k^{-\gamma}$ with $2 < \gamma < 3$

Asymptotics for average avalanche shape functions

$$A(t) = Q(T-t)\frac{f'(Q(T)) - f'(Q(T-t))}{f(Q(T-t)) - Q(T-t)}$$

Large-T asymptotics for critical case ($\xi = 1$):

$$A(t) \sim \begin{cases} \frac{t}{T}\left(1-\frac{t}{T}\right) & \text{if } q_k \text{ has finite second moment} \\ \frac{t}{T}\left(1-\frac{t}{T}\right)^{\frac{1}{\gamma-2}} & \text{if } q_k \sim k^{-\gamma} \text{ with } 2 < \gamma < 3 \end{cases}$$

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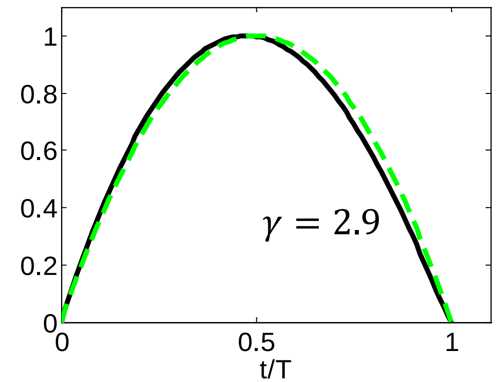

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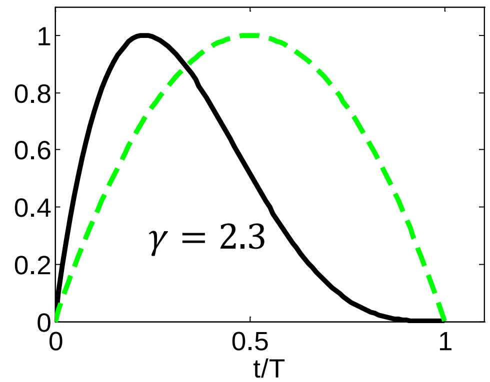

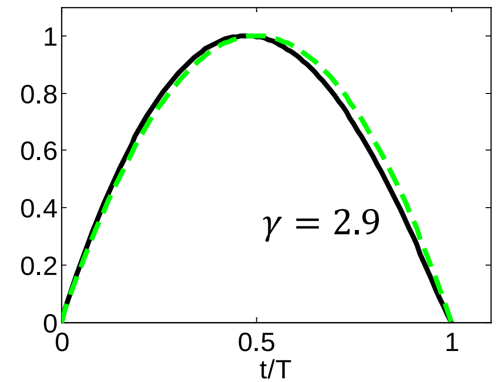

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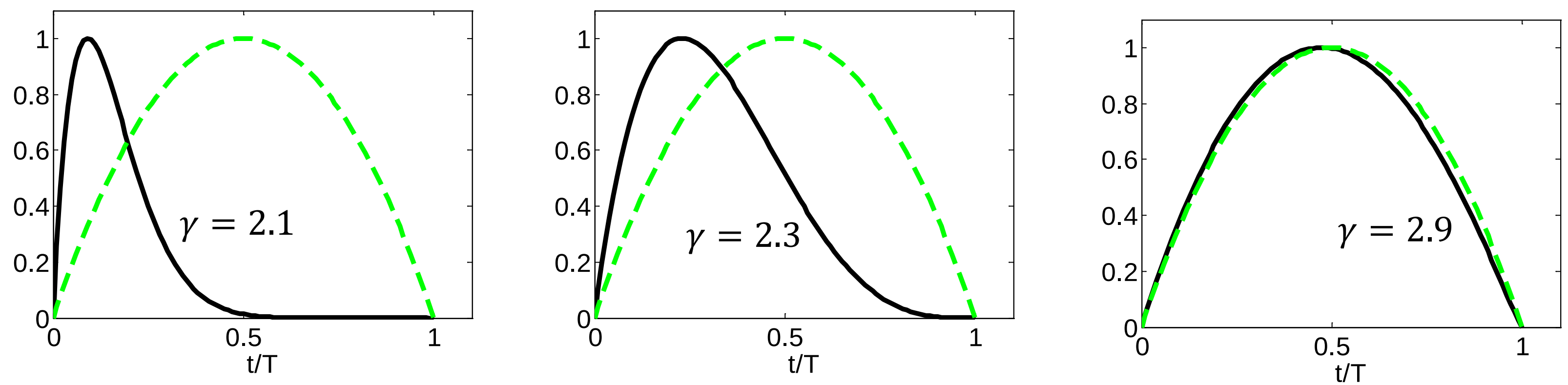

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Derivation of the average non-terminating avalanche shape

- Continuous-time Markov branching processes

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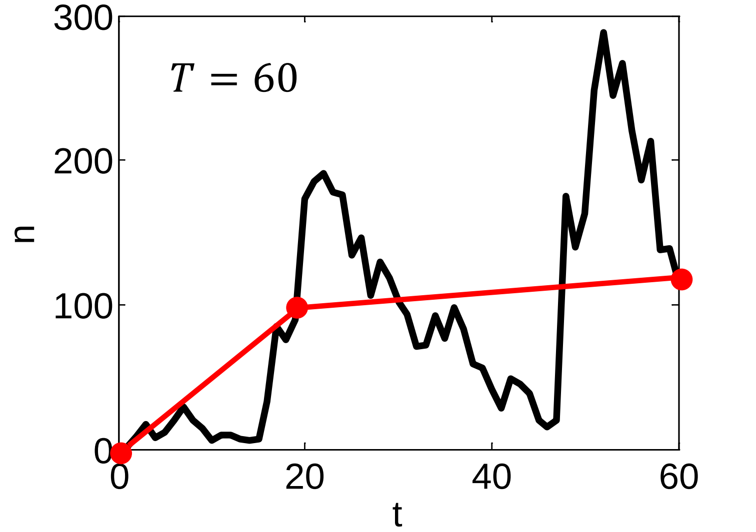

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$$\frac{\partial}{\partial t} F(s,t) = f\big(F(s,t)\big) - F(s,t)$$

- Average non-terminating avalanche shape:

$$A_{NT}(t) = \sum_n n\, \pi_n(t) = \frac{e^{(f'(1)-1)t} - Q(T - t)\dfrac{f(Q(T)) - Q(T)}{f(Q(T - t)) - Q(T - t)}}{1 - Q(T)}$$

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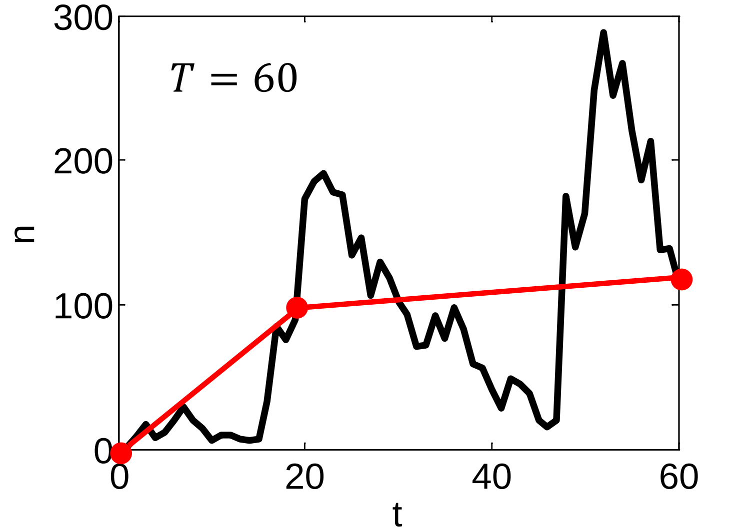

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$$\pi_n(t) = \frac{P_{1n}(t)[1 - P_{10}(T - t)^n]}{\sum_m P_{1m}(t)[1 - P_{10}(T - t)^m]}$$

$$\frac{\partial}{\partial t} F(s,t) = [f(s) - s] F'(s,t)$$

$$\frac{\partial}{\partial t} F(s,t) = f\big(F(s,t)\big) - F(s,t)$$

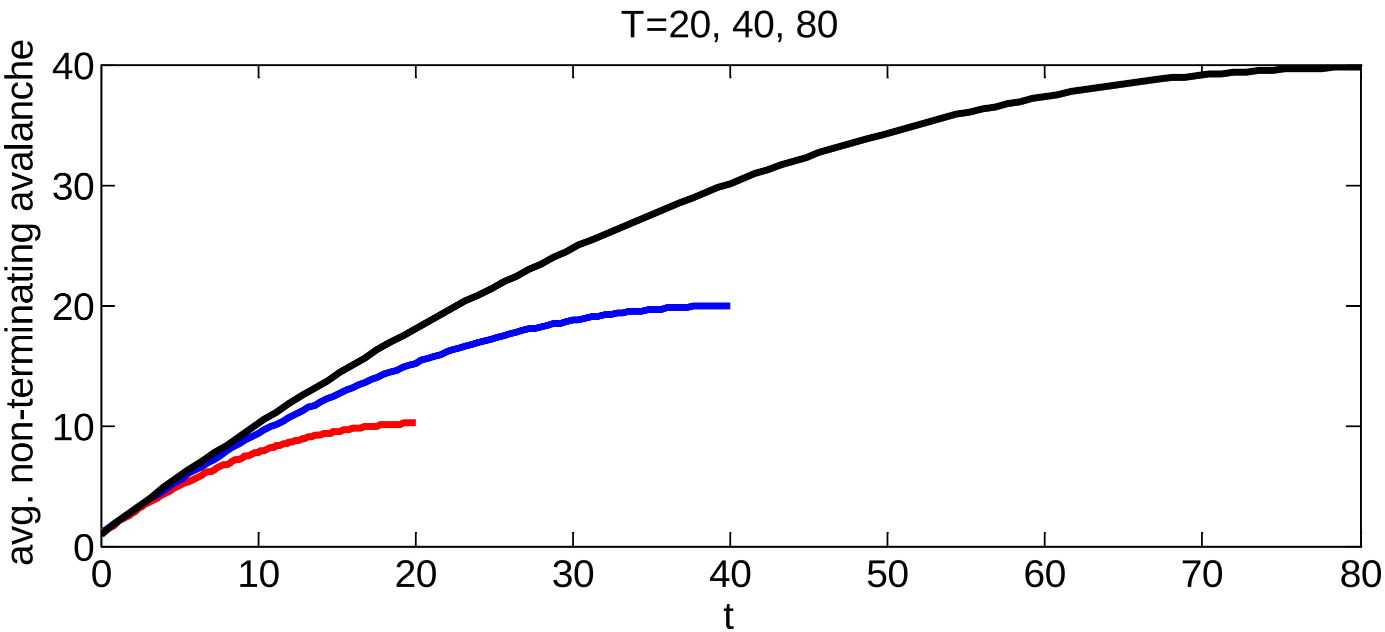
- Average non-terminating avalanche shape:

$$A_{NT}(t) = \sum_n n\, \pi_n(t) = \frac{e^{(f'(1)-1)t} - Q(T-t)\dfrac{f(Q(T)) - Q(T)}{f(Q(T-t)) - Q(T-t)}}{1 - Q(T)}$$

Average non-terminating avalanche shape function

$$A_{NT}(t) = \frac{e^{(\xi-1)t} - Q(T-t)\,\dfrac{f(Q(T)) - Q(T)}{f(Q(T-t)) - Q(T-t)}}{1 - Q(T)}$$

q_k Poisson, $\xi = 1$ (critical)



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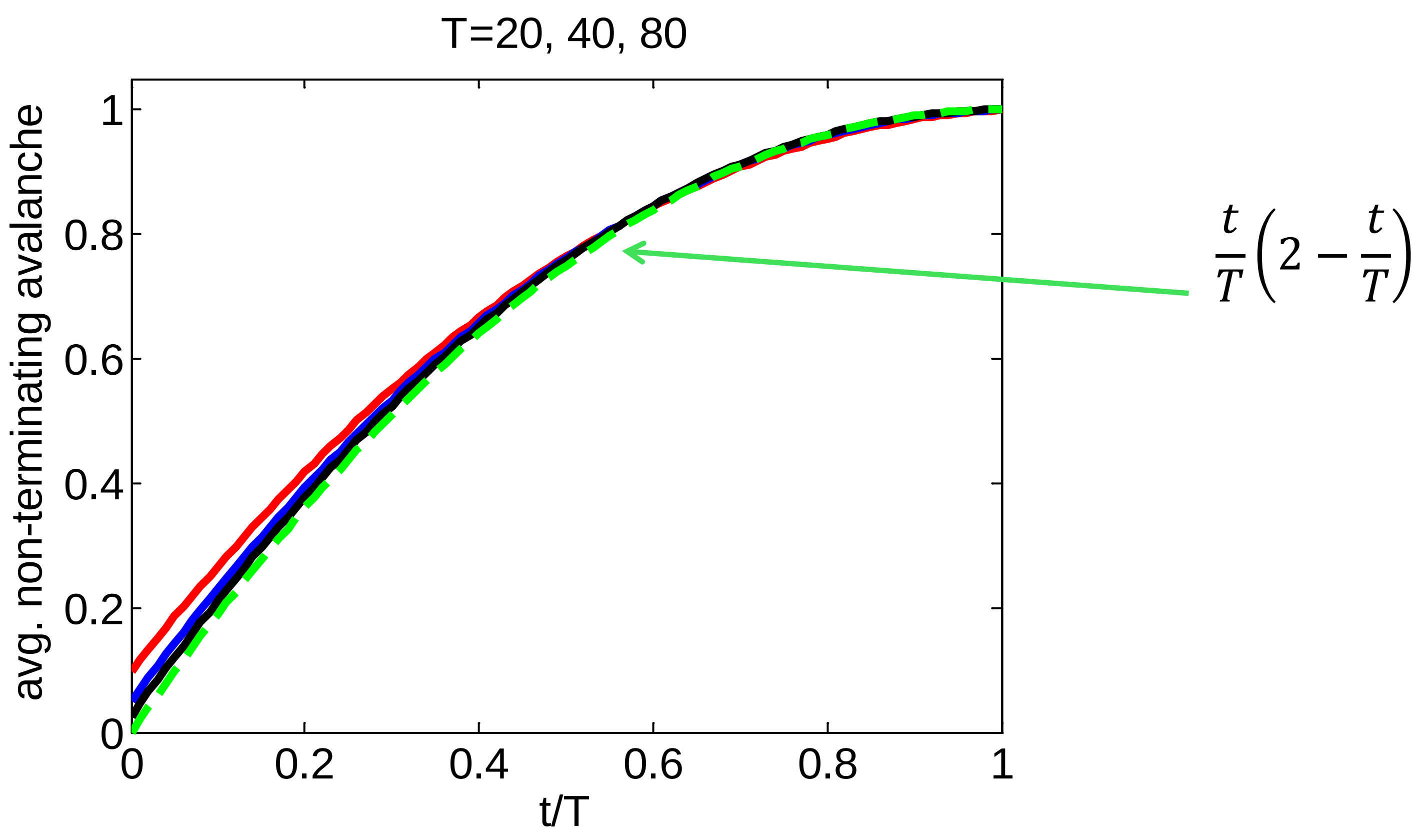

Average non-terminating avalanche shape function

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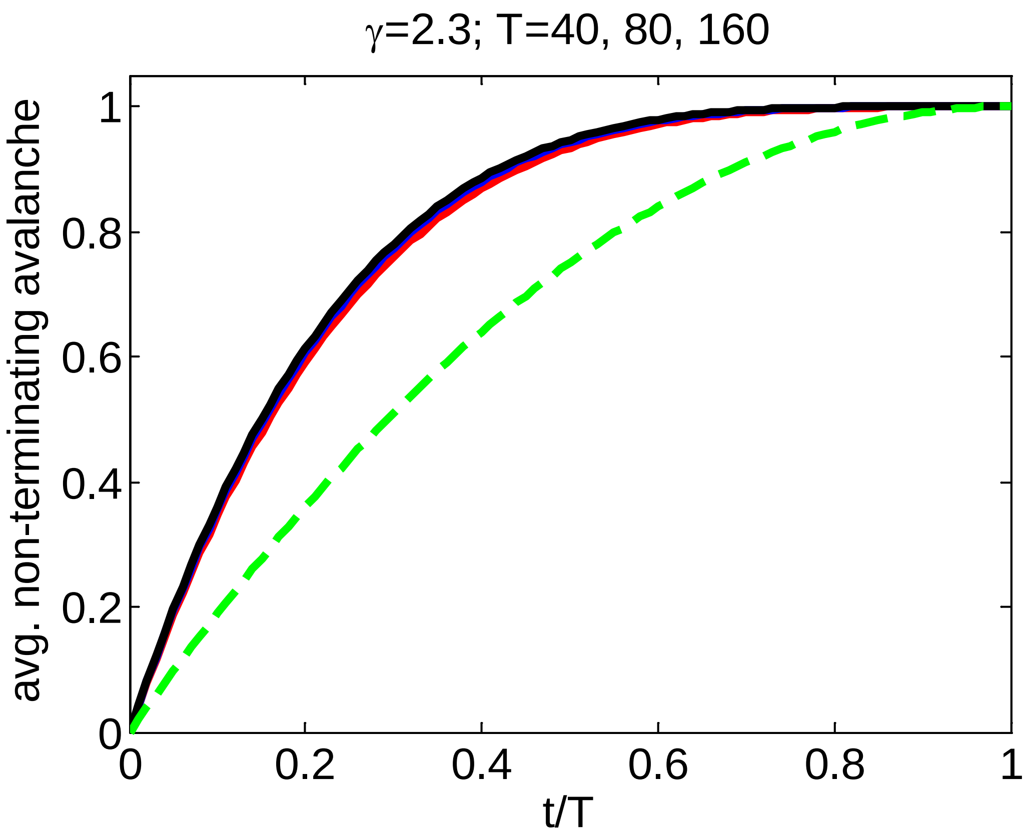
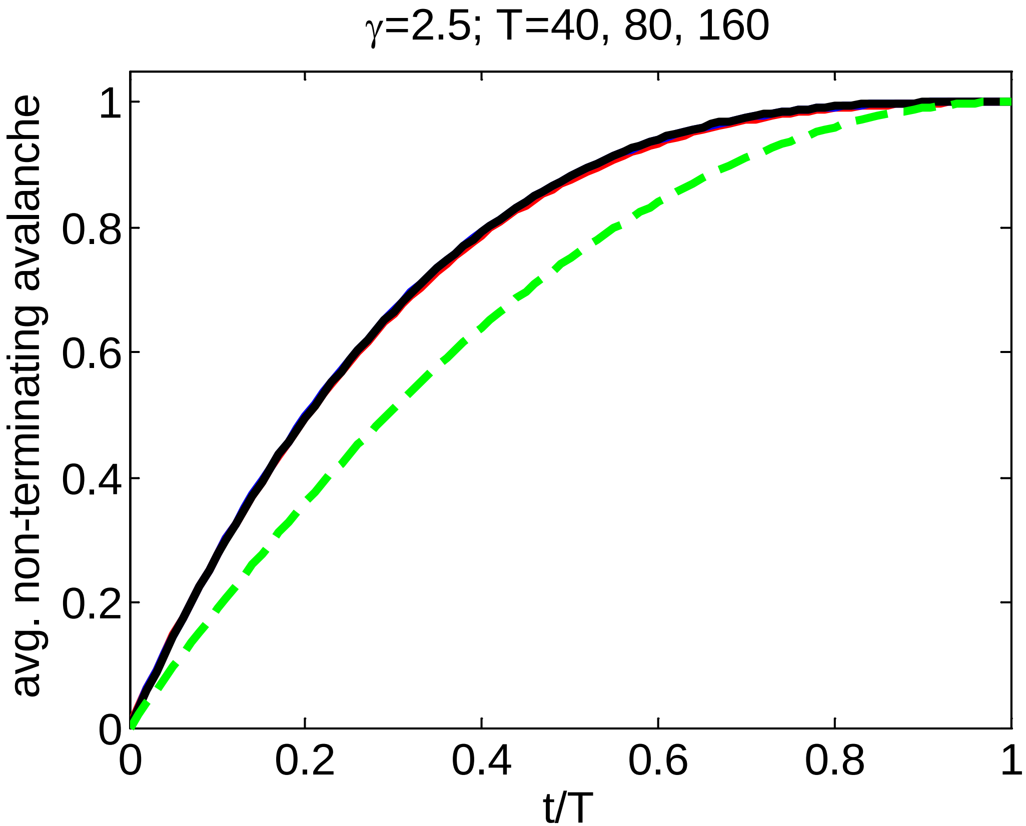
$q_k \sim k^{-\gamma}$ with $\gamma = 2.3, \xi = 1$ (critical)



Average non-terminating avalanche shape function

$$A_{NT}(t) = \frac{e^{(\xi-1)t} - Q(T-t)\,\dfrac{f(Q(T)) - Q(T)}{f(Q(T-t)) - Q(T-t)}}{1 - Q(T)}$$

$q_k \sim k^{-\gamma}$ with $\gamma = 2.5$, $\xi = 1$ (critical)

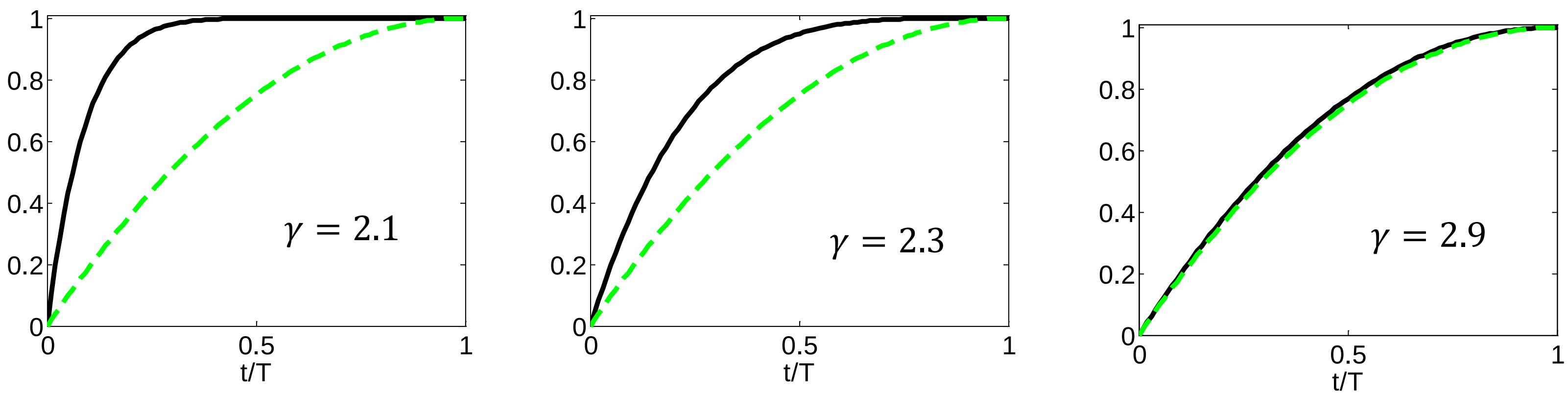


Asymptotics for average non-terminating avalanche shape

$$A_{NT}(t) = \frac{e^{(\xi-1)t} - Q(T-t)\,\dfrac{f(Q(T)) - Q(T)}{f(Q(T-t)) - Q(T-t)}}{1 - Q(T)}$$

Large-T asymptotics for critical case ($\xi = 1$):

$$A_{NT}(t) \sim \begin{cases} \frac{t}{T}\left(2 - \frac{t}{T}\right) & \text{if } q_k \text{ has finite second moment} \\ 1 - \left(1 - \frac{t}{T}\right)^{\frac{\gamma-1}{\gamma-2}} & \text{if } q_k \sim k^{-\gamma} \text{ with } 2 < \gamma < 3 \end{cases}$$



Non-parabolic shape functions occur when the offspring distribution has a power-law tail: $q_k \sim k^{-\gamma}$ with $2 < \gamma < 3$

Overview

1. Average avalanche shape functions (and beyond…)

2. Analytical results

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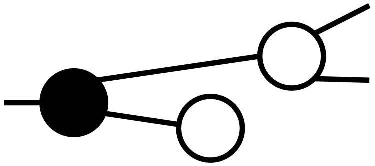
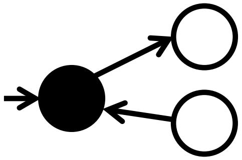
Cascade models: examples

- Threshold model (undirected network)



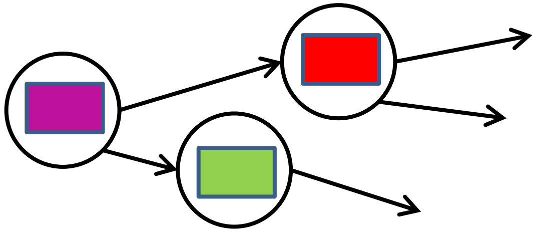
Each node has a threshold r that is assigned randomly from a distribution. All nodes are initially inactive, except for one seed node. When an inactive node is updated, it becomes active if the number m of its active neighbours exceeds its threshold r.

- Neuronal dynamics model of Friedman et al. (directed network)



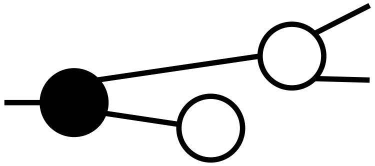
The weight ϕ_{ij} of each directed edge from neuron i to neuron j is assigned randomly from a uniform distribution on $[0, \phi_{max}]$. When neuron i fires (becomes active), it causes neuron j to become active (in the next discrete time step) with probability ϕ_{ij}. After a neuron fires, it returns to the inactive state in the next time step.

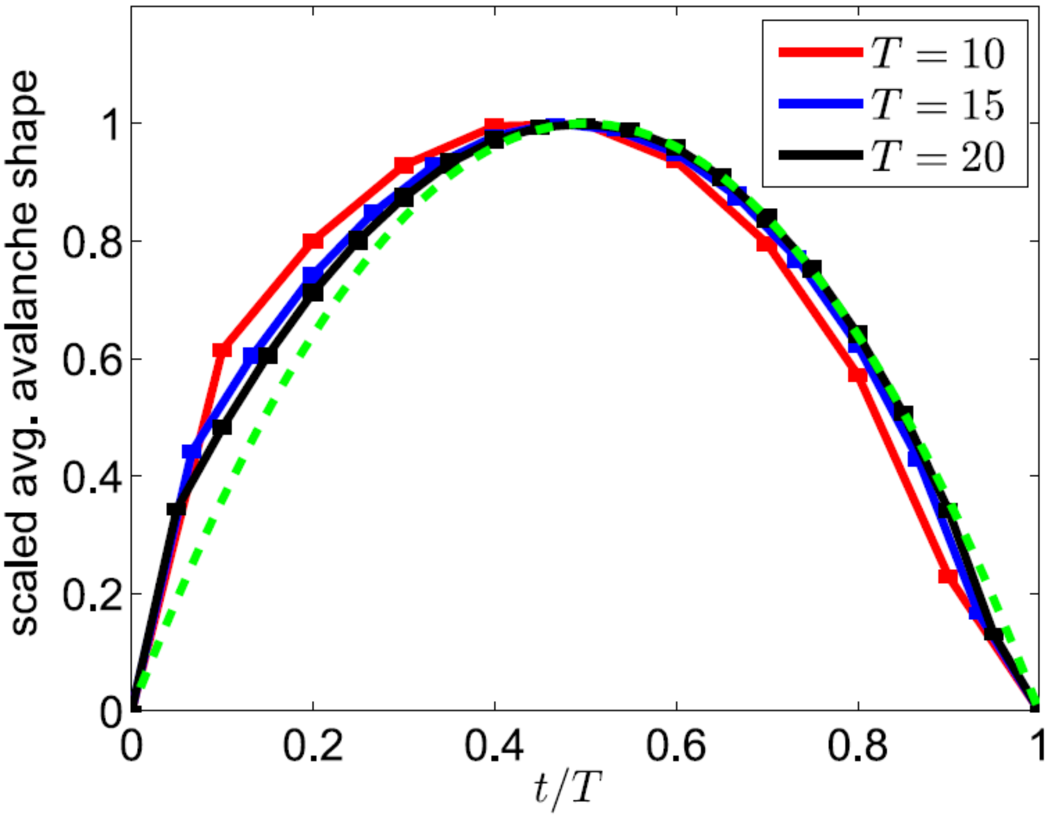
- Meme diffusion model (directed network)



Numerical simulation: threshold model ($\xi = 1$)

- random z-regular network, $z = 3$

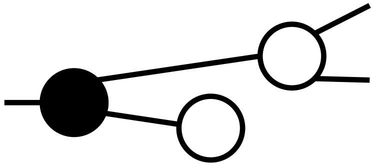


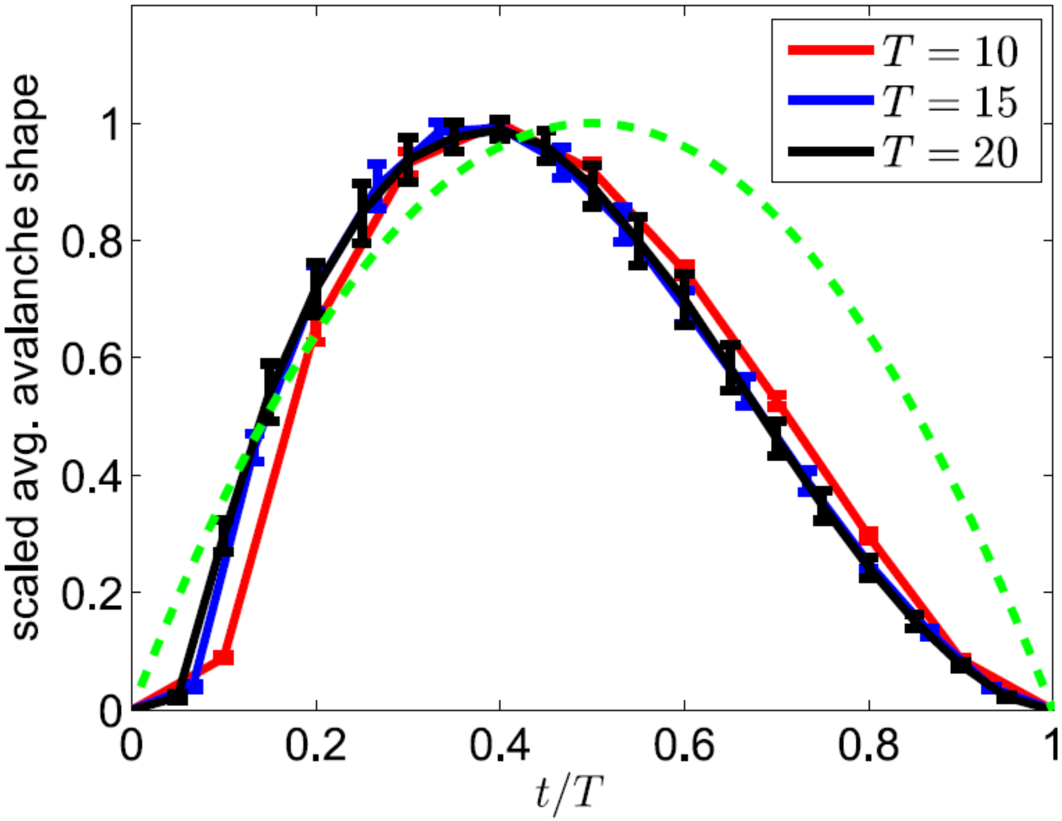


Number of nodes: $N = 10^5$; number of avalanches: $n_A = 10^6$

Numerical simulation: threshold model ($\xi = 1$)

- scale-free network, $p_k \sim k^{-\alpha}$ with $\alpha = 3.3$

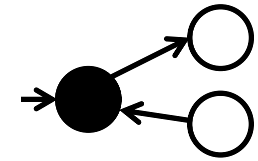


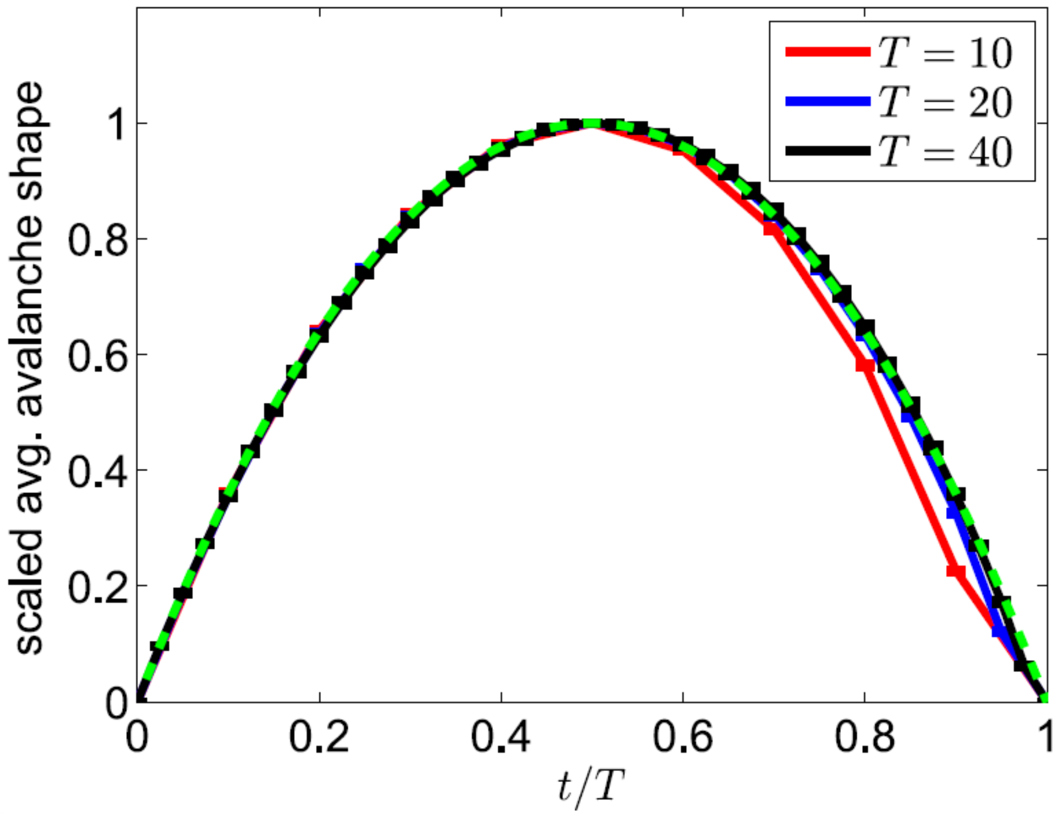


Number of nodes: $N = 10^6$; number of avalanches: $n_A = 10^6$

Numerical simulation: neuronal dynamics model ($\xi = 1$)

- $p_{jk} = p_j p_k$ with p_j Poisson, p_k z-regular, $z = 10$

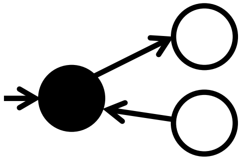


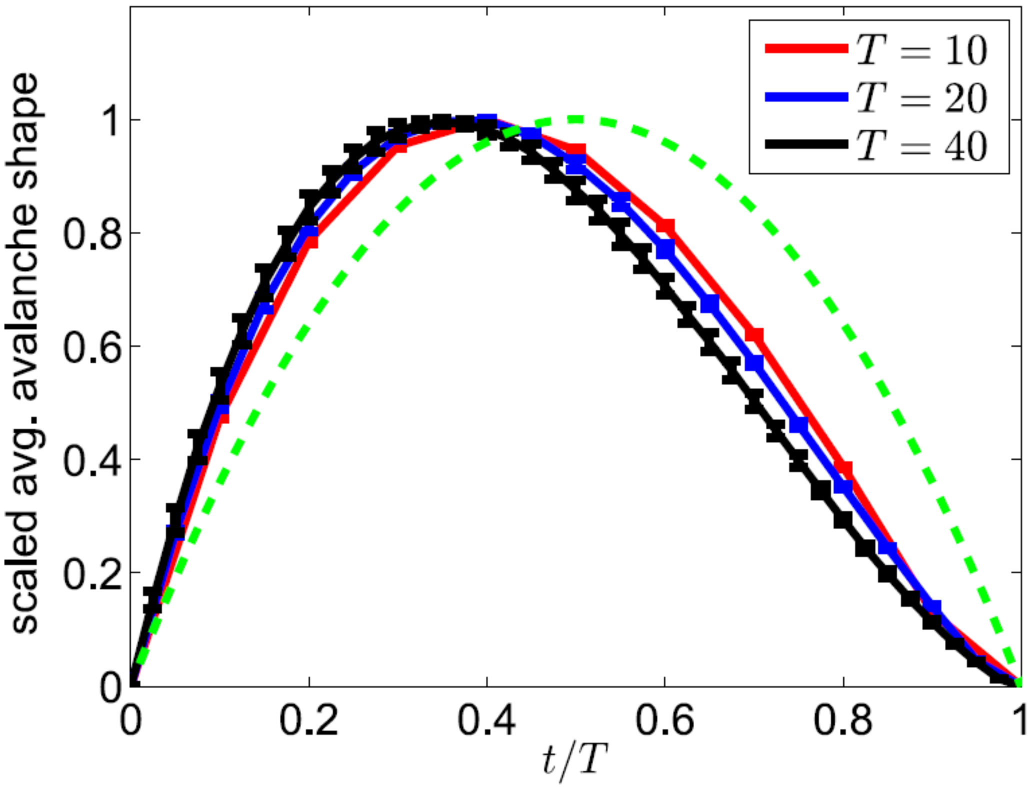


Number of nodes: $N = 10^5$; number of avalanches: $n_A = 10^7$

Numerical simulation: neuronal dynamics model ($\xi = 1$)

- $p_{jk} = p_j p_k$ with p_j Poisson, $p_k \sim k^{-\alpha}$, $\alpha = 2.5$

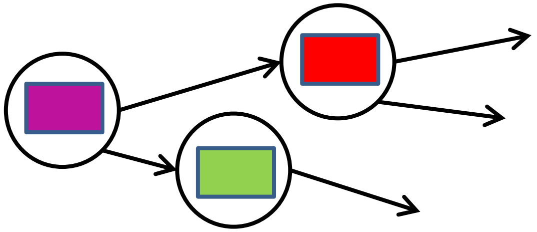


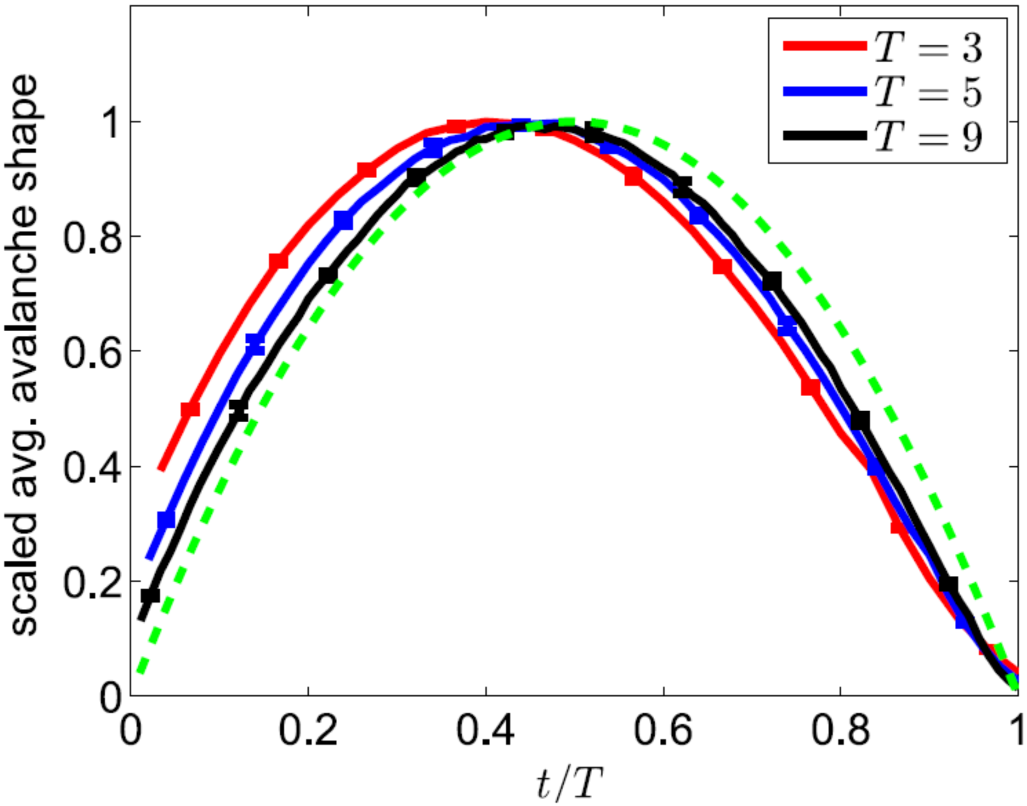


Number of nodes: $N = 10^5$; number of avalanches: $n_A = 10^7$

Numerical simulation: Twitter model ($\xi = 1$)

- $p_{jk} = p_j p_k$ with p_j Poisson, p_k z-regular, $z = 10$





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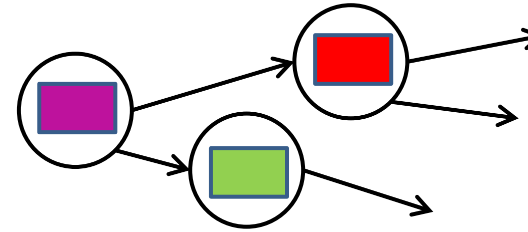
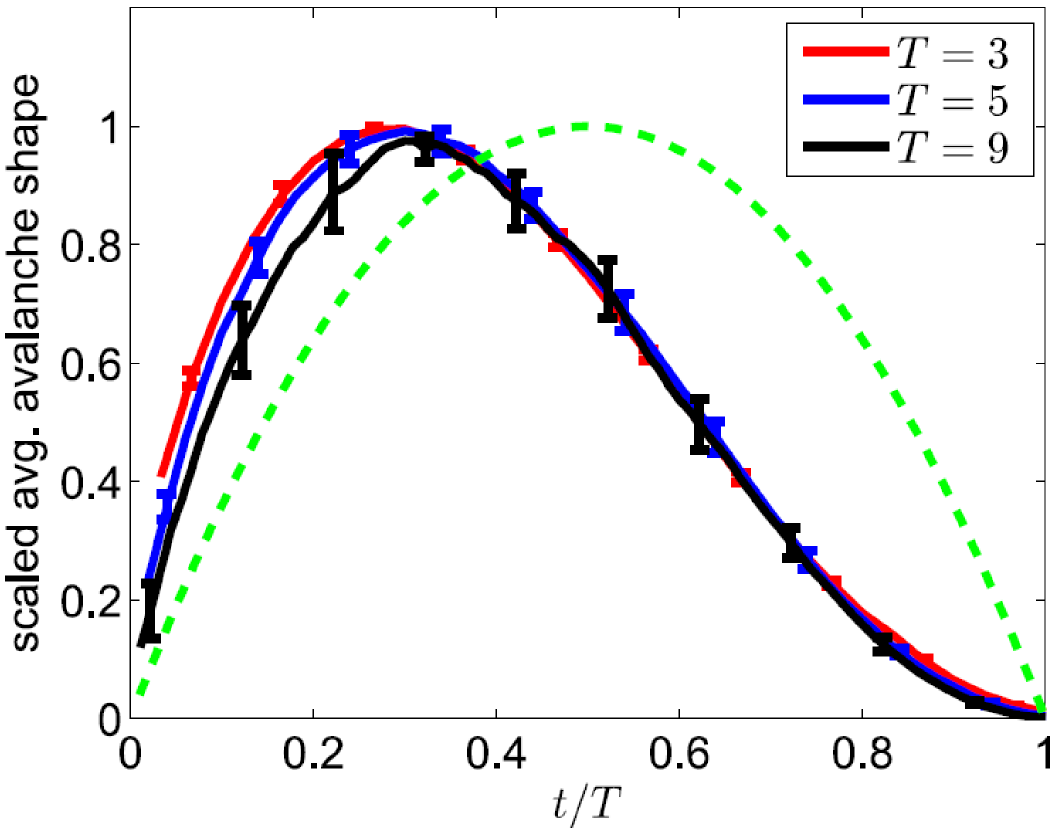
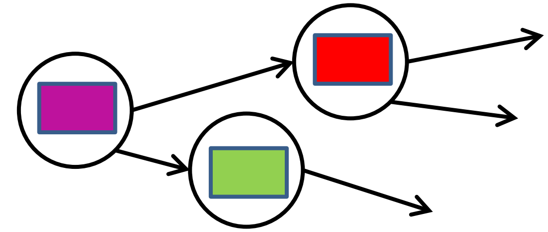


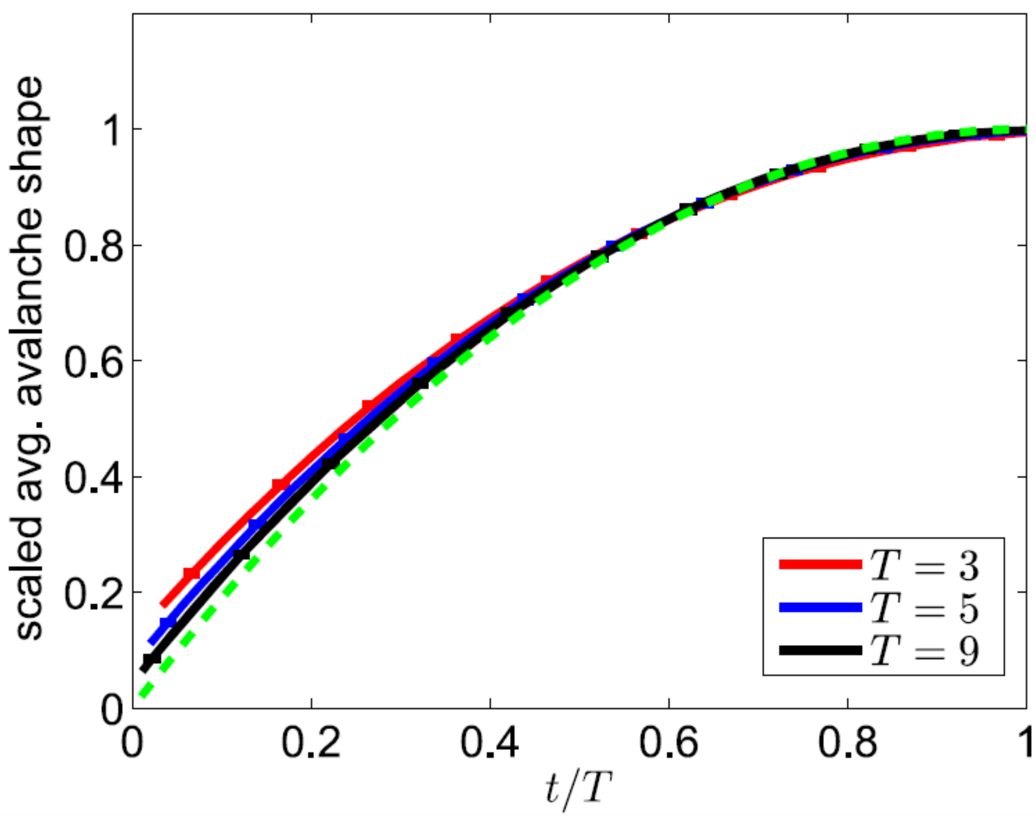


Number of nodes: $N = 10^5$; number of avalanches: $n_A = 10^7$

Average non-terminating avalanche shapes: Twitter model

- $p_{jk} = p_j p_k$ with p_j Poisson, p_k z-regular, $z = 10$





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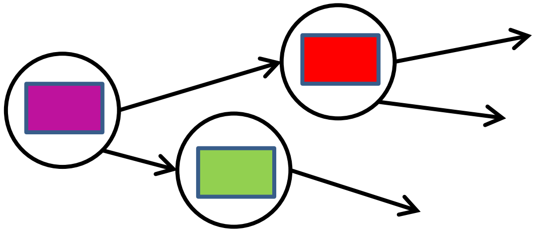
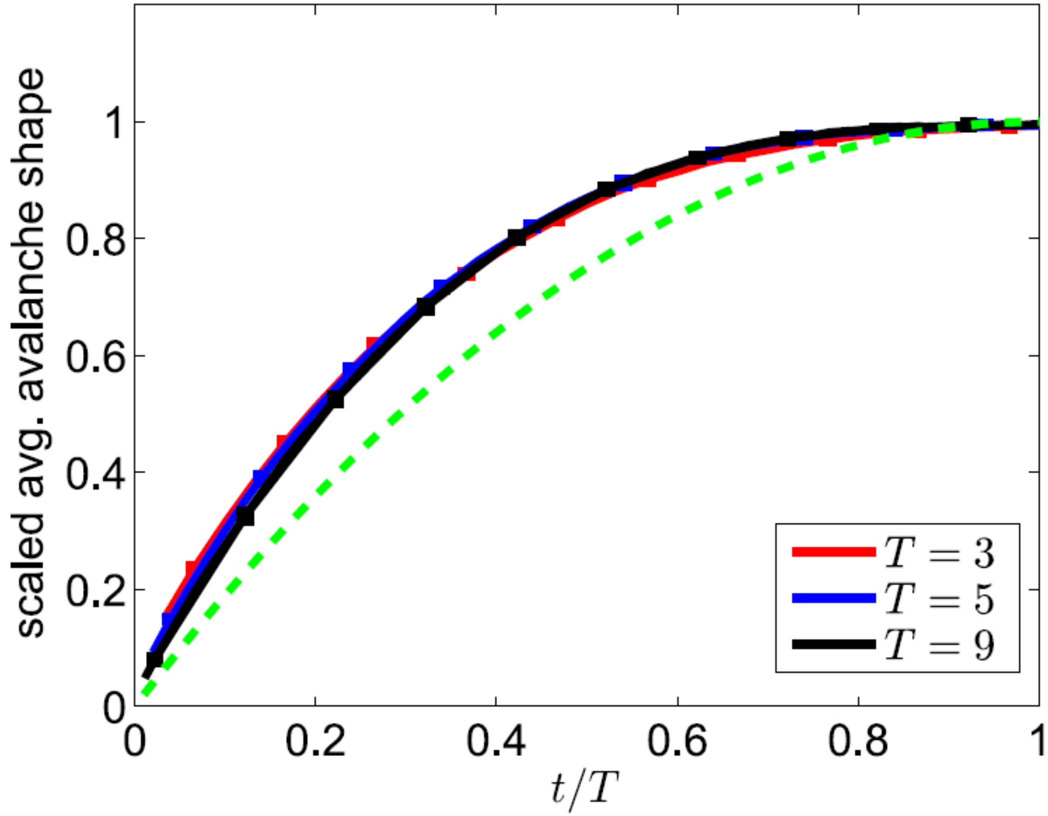


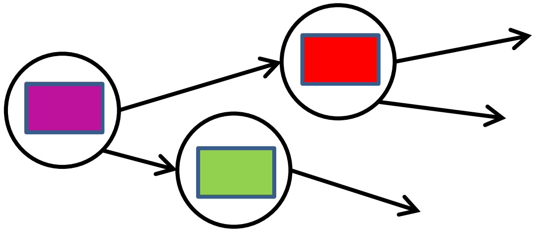


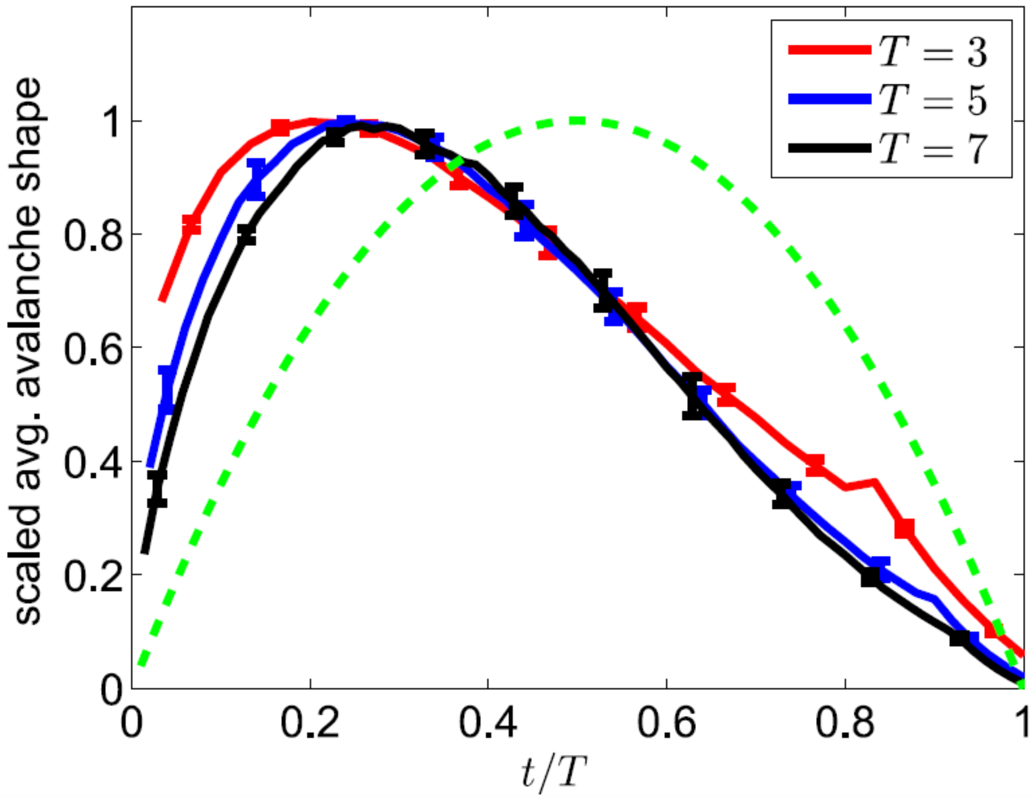
Number of nodes: $N = 10^5$; number of avalanches: $n_A = 10^7$

Twitter model on real Twitter network ($\xi = 1$)

- SNAP Twitter social circles network

  http://snap.stanford.edu/data/egonets-Twitter.html





Number of nodes: $N = 81{,}306$; number of avalanches: $n_A = 1.4 \times 10^6$

Twitter model on real Twitter network ($\xi = 1$)

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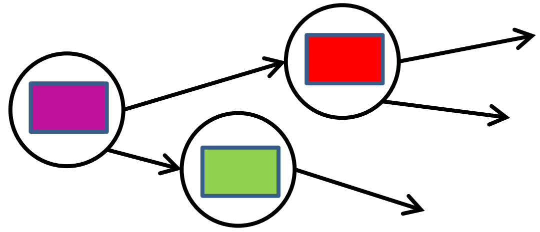

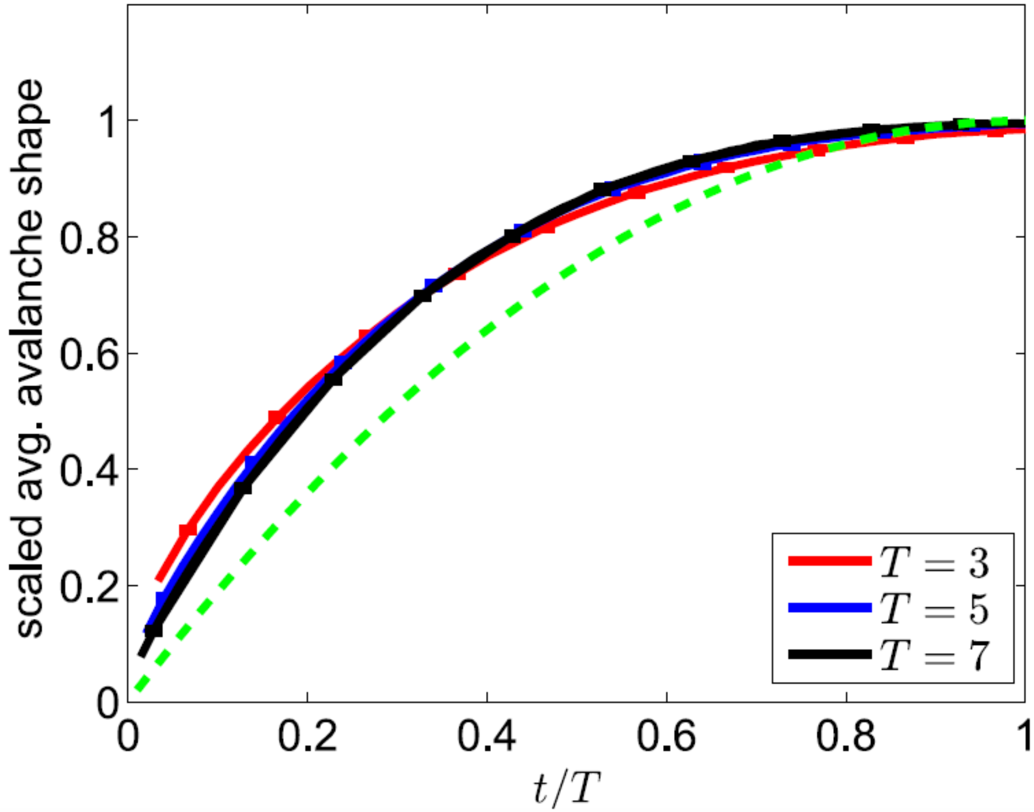

Number of nodes: $N = 81{,}306$; number of avalanches: $n_A = 1.4 \times 10^6$

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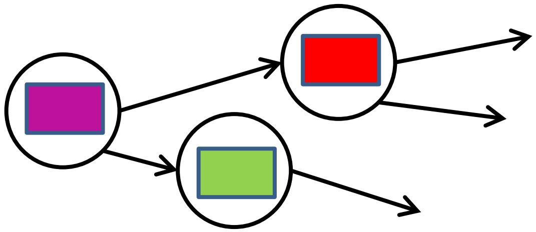
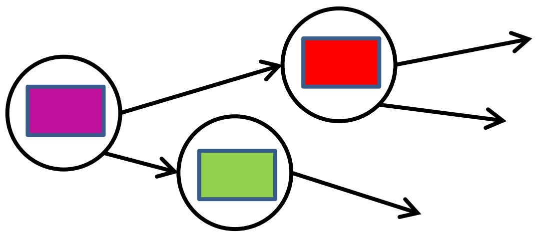


Competition-induced criticality in a model of meme diffusion

Summary



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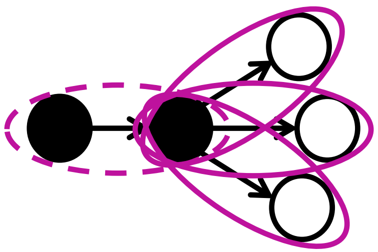
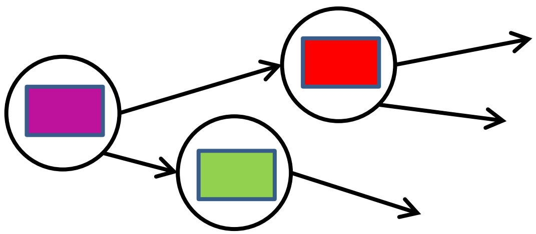
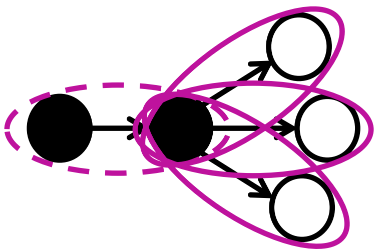


“active edges” (“exposed” nodes) in cascades play the role of particles in branching processes

Summary



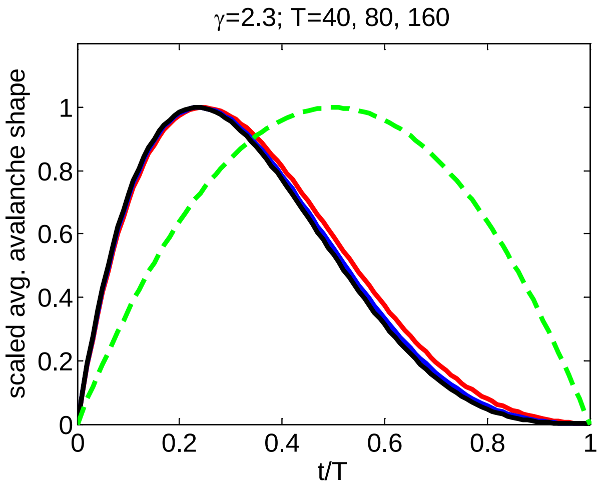
Competition-induced criticality in a model of meme diffusion



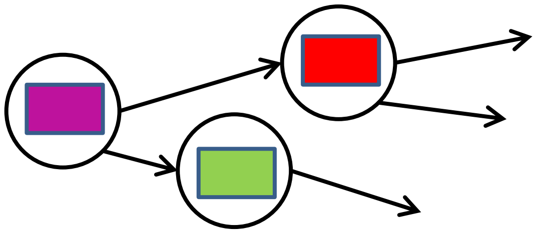
"active edges" ("exposed" nodes) in cascades play the role of particles in branching processes

$$\frac{dQ}{dt} = f(Q) - Q, \quad Q(0) = 0$$

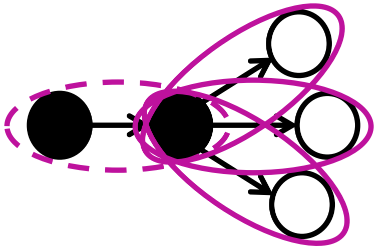
$$A(t) = Q(T-t)\frac{f'(Q(T)) - f'(Q(T-t))}{f(Q(T-t)) - Q(T-t)}$$



Summary



Competition-induced criticality in a model of meme diffusion



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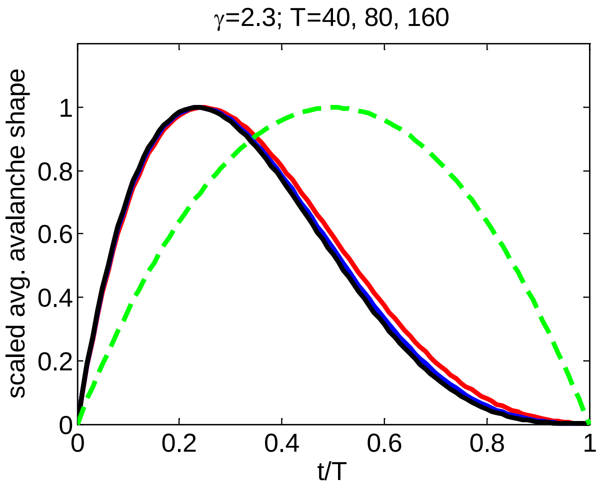
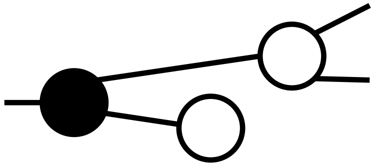
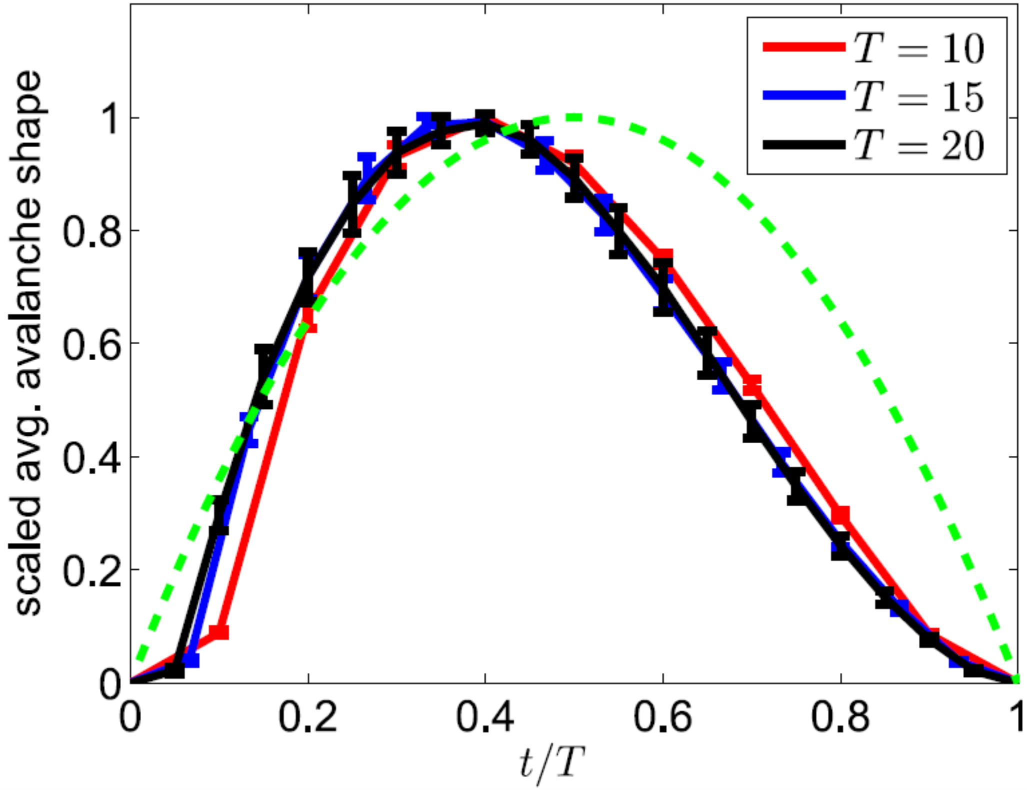


Nonsymmetric avalanche shape functions occur when the offspring distribution has a power-law tail: $q_k \sim k^{-\gamma}$ with $2 < \gamma < 3$

Numerical simulation: threshold model ($\xi = 1$)

- scale-free network, $p_k \sim k^{-\alpha}$ with $\alpha = 3.3$





$$q_k = \frac{k+1}{z} p_{k+1} v_{k+1}$$

Nonsymmetric avalanche shape functions occur when the **offspring distribution** has a power-law tail: $q_k \sim k^{-\gamma}$ with $2 < \gamma < 3$

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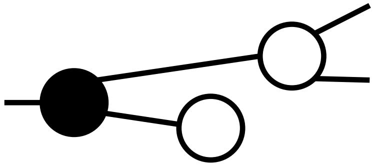

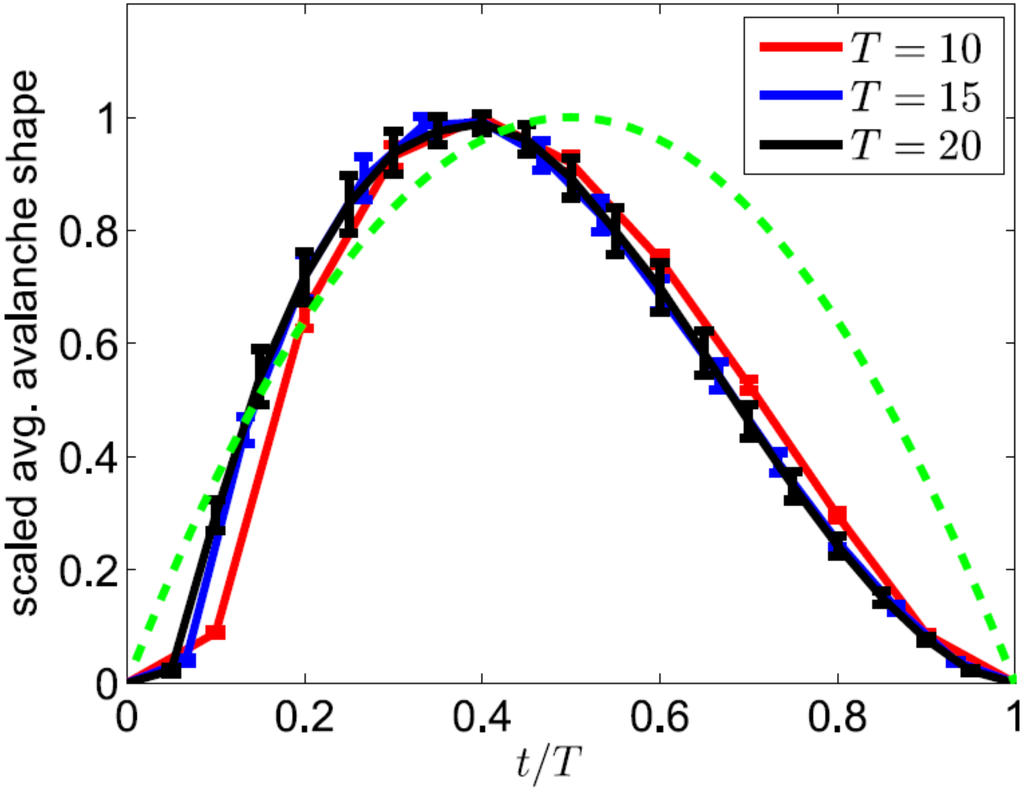

$$q_k = \frac{k+1}{z} p_{k+1} v_{k+1}$$

Nonsymmetric avalanche shape functions occur when the **offspring distribution** has a power-law tail: $q_k \sim k^{-\gamma}$ with $2 < \gamma < 3$

Conclusions

- Average avalanche shapes can be predicted from the network structure and the dynamics, via q_k

$$q_k = \frac{k+1}{z} p_{k+1} v_{k+1} \qquad\qquad q_k = \sum_j \frac{j}{z} p_{jk}\, v_{jk}$$

- Nonsymmetric avalanche shape functions occur when the offspring distribution has a power-law tail: $q_k \sim k^{-\gamma}$ with $2 < \gamma < 3$

- Other scaling functions (e.g. non-terminating avalanche shapes) can be defined

- Depending on the dynamics, nonsymmetric avalanche shapes can occur on scale-free networks with various power-law exponents $p_k \sim k^{-\alpha}$ (i.e., γ may not be equal to α)

The theoretical tools developed here should be useful for analysing the criticality (or otherwise) of a range of cascading dynamics on networks

Temporal profiles of avalanches on networks

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University of Limerick

Rick Durrett

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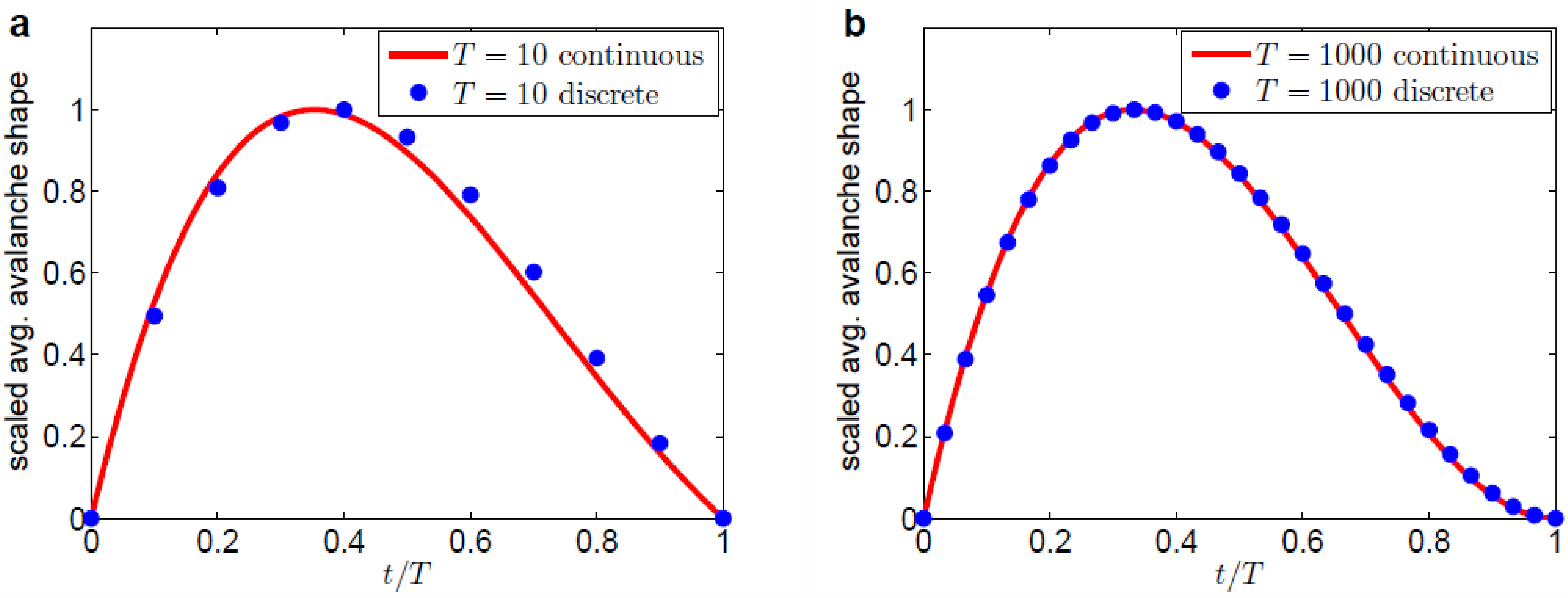
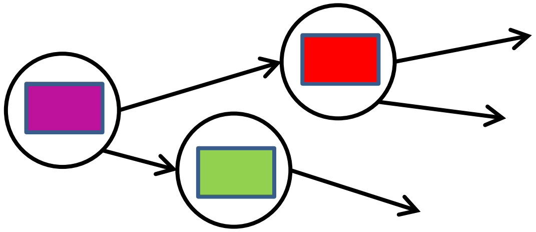
Extra slides
(Supplementary material of arXiv: 1612.06477)

Comparing discrete-time and continuous-time results

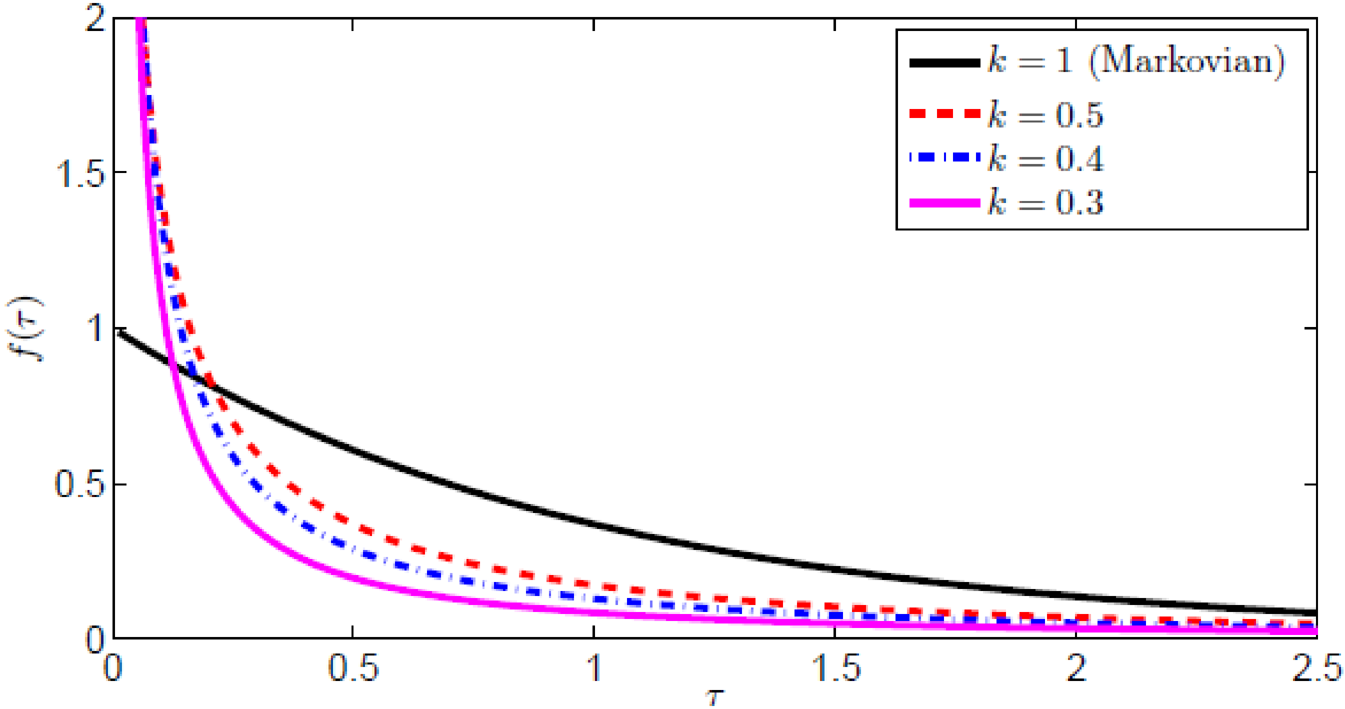


Non-Markovian meme diffusion model

- Weibull inter-event time distribution



$$f(\tau) = \frac{k}{\lambda}\left(\frac{\tau}{\lambda}\right)^{k-1} e^{-(\tau/\lambda)^k}$$



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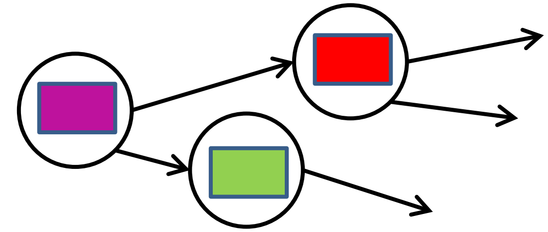
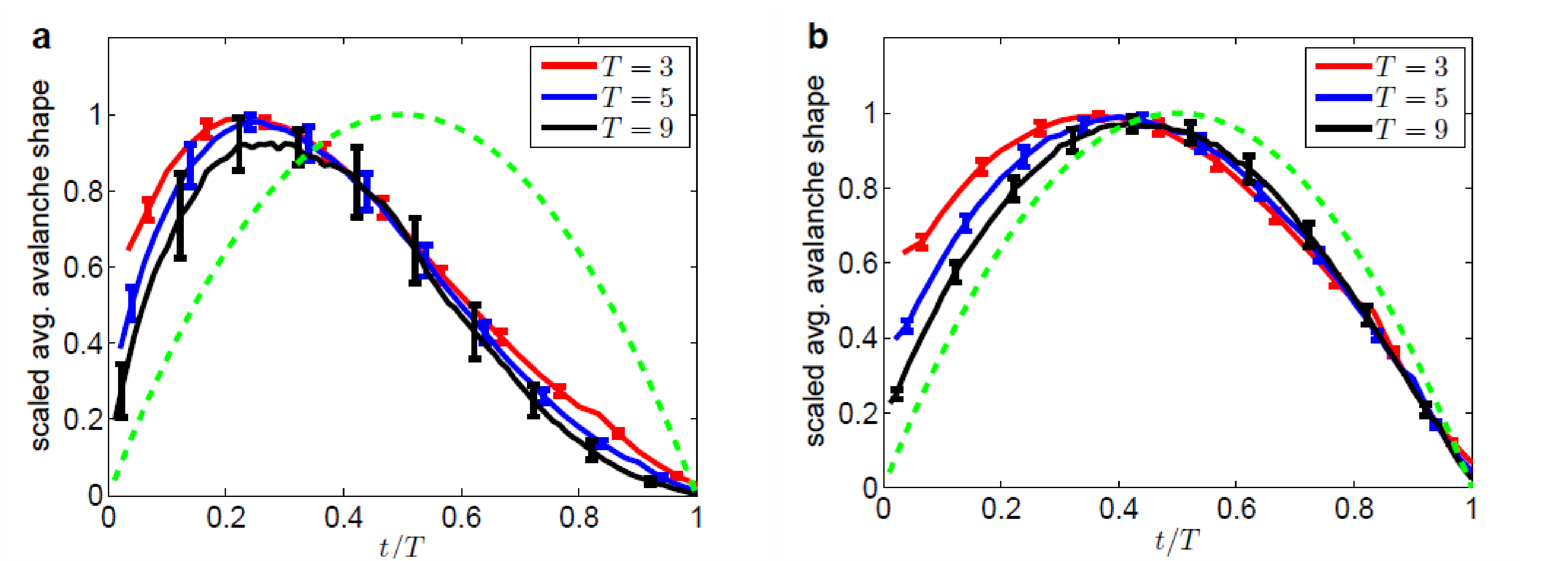


$k = 0.5$



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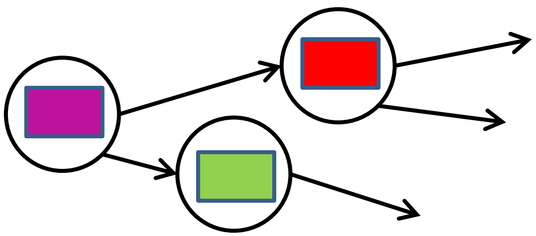

$k = 0.4$