Explosion in branching processes and applications to epidemics in random graph models

Júlia Komjáthy

joint with Enrico Baroni and Remco van der Hofstad
Eindhoven University of Technology

Advances on Epidemics in Complex Networks
31 Aug 2017

Class of processes on networks

Spreading processes

Class of processes on networks

Information diffusion

Class of processes on networks

Includes

Viruses


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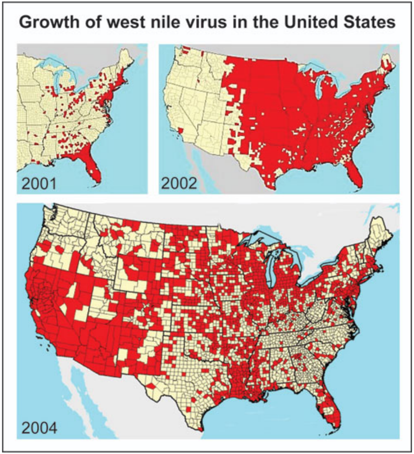
Viruses



The spread of the west-nile-virus

Viruses

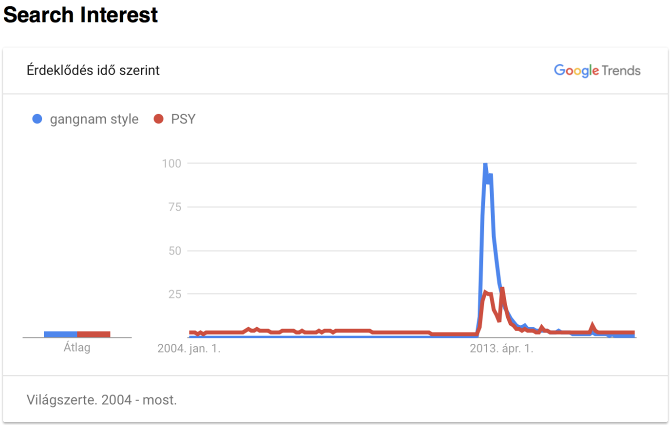
The spread of the Zika virus

Memes

Memes

Viral videos

Extremely fast spread

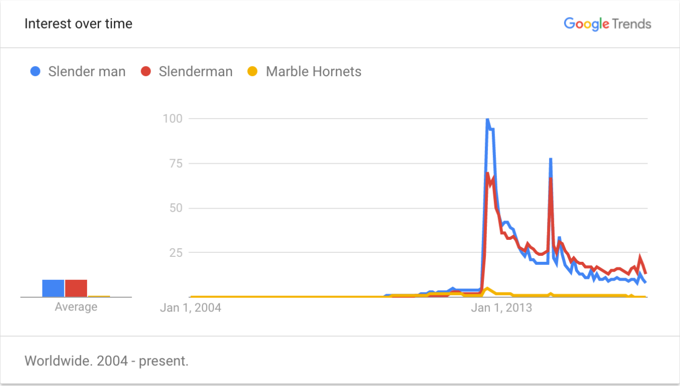


Search intensity of Gangnam style

from knowyourmeme.com

Extremely fast spread

Search Interest



Search intensity of the Slenderman meme

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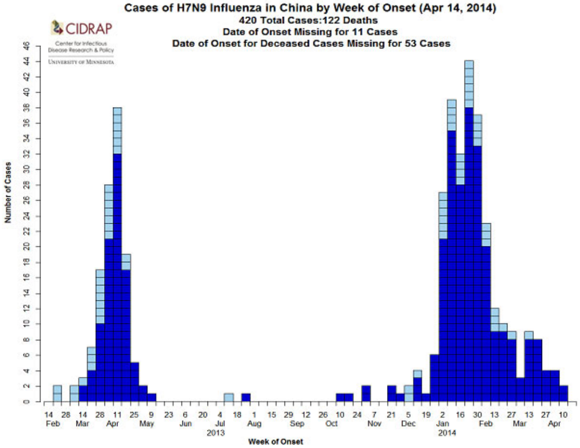

Epidemic curve of a flu from China

from Center for Infectious Disease Research and Policy

Modeling

We need models!

The scale-free property

Many real-life networks have *power-law degrees*.

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Power-law paradigm

For some $\tau > 2$, the degree of a uniformly chosen vertex satisfies

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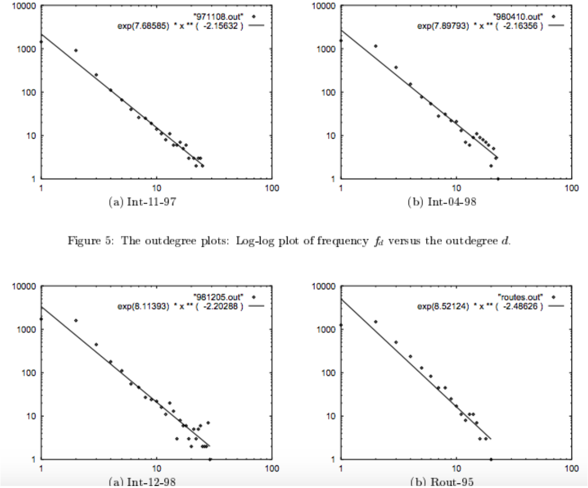
$$\mathbb{P}(\deg(v) = x) \asymp \frac{C}{x^{\tau}}$$

$$\log \mathbb{P}(\deg(v) = x) \asymp \log C - \tau \log x$$

log(proportion of degree x vertices) vs $\log x$ is a straight line.

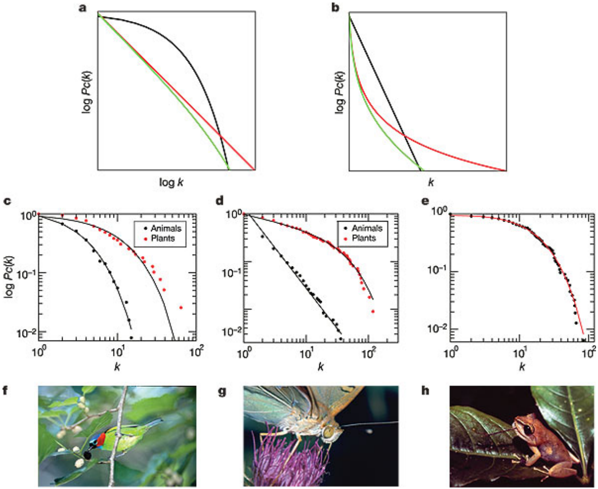
Power laws



Degree distribution of the router level internet network

from Faloutsos, Faloutsos, Faloutsos. 1999

Power laws



Degree distribution of ecological networks

from Montoya, Pimm, Polé. Nature 2006

Power laws

Note: $\tau \in (2,3)$ often!

When $\tau \in (2,3)$ then $\mathbb{V}ar_n[\deg(v)] \to \infty$ and $\mathbb{E}_n[\deg(v)] < \infty$.

Choice of model

Configuration model

The configuration model

Matches the degree sequence of the network you would like to model.

[Configuration model simulator by Robert Fitzner]

[Configuration model with power law degrees by Robert Fitzner]

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Power-law assumption

For some $\tau \in (2,3)$, the tail of the empirical degree distribution satisfies

$$\frac{c_1}{x^{\tau-1}} \le [1-F_n](x) = \mathbb{P}(\deg(v_n) \ge x) \le \frac{C_1}{x^{\tau-1}}$$

Weighted Configuration model

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The spreading time between two vertices u, v
= the weighted distance:

$$d_\sigma(u, v)$$

How does $d_\sigma(u, v)$ behave in terms of the degrees and the edge-weight distribution σ?

The epidemic curve

Epidemic curve

Considering vertex u as a (single) source of infection, σ_e as the transmission time of an infection through edge e, the *epidemic curve* is defined as

$$I_u(t) = \frac{1}{|V|} \sum_{v \in V} \mathbf{1}_{d_\sigma(u,v) \le t}.$$

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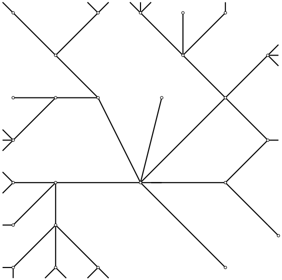
$$I_u(t) = \frac{1}{|V|} \sum_{v \in V} \mathbf{1}_{d_\sigma(u,v) \le t}.$$

How does $I_u(t)$ behave, in terms of the degree distribution, the edge-length distribution σ, and the source vertex u?

Locally tree-like structure

Local neighborhoods look like random trees *with size biased degrees*.



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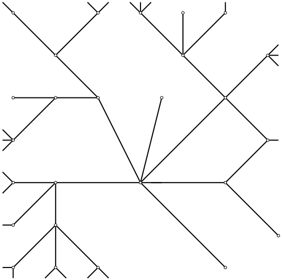

Size-biasing effect

A neighbor of a uniform vertex is more likely to have larger degree

$$\mathbb{P}\big(\deg(\text{neighbor}(v)) > x\big) \asymp \frac{C x}{x^{\tau-1}} = \frac{C}{x^{\tau-2}}$$

Preliminaries

Initial stage of the spreading in the graph looks like a *random tree with*

- power law degrees, tail exponent $\alpha := \tau - 2 \in (0, 1)$
- each edge has an independent 'length' or 'weight'

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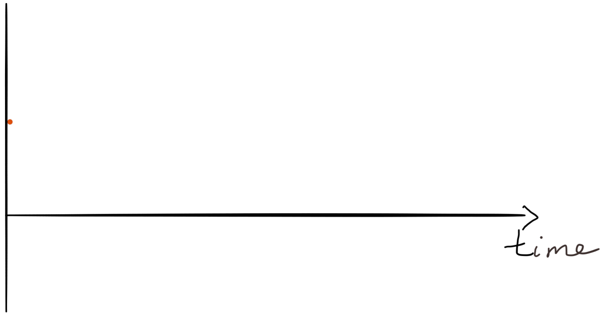

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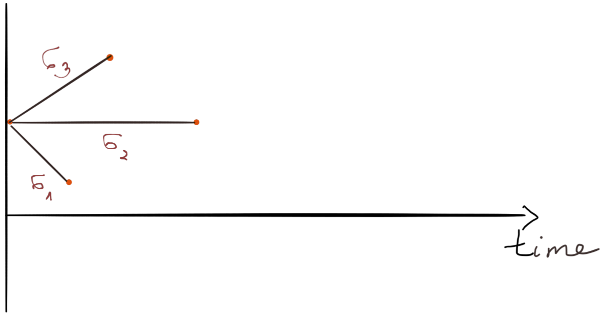

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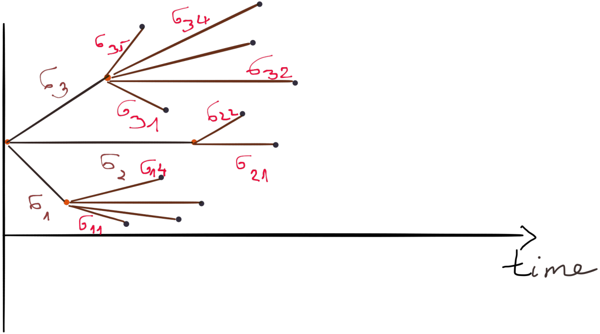

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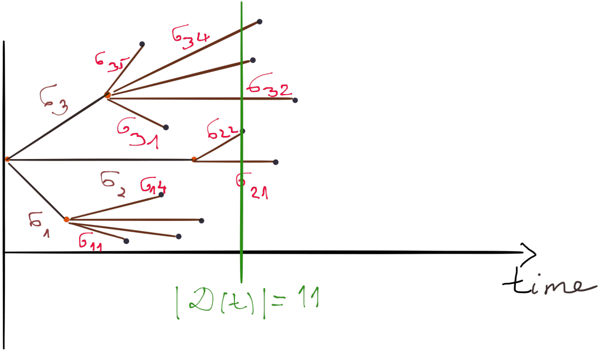

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Explosive vs conservative

When is a branching process $\mathrm{BP}(X, \sigma)$ explosive?

Explosion of BPs

Theorem (Amini, Devroye, Griffith, Olver)

Assume for x large enough and some $\varepsilon > 0$

$$\frac{1}{x^{\varepsilon}} > \mathbb{P}(X > x) > \frac{1}{x^{1-\varepsilon}}. \qquad \textbf{(P2)}$$

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The branching process $\mathrm{BP}(X, \sigma)$ is explosive if and only if for some $K > 0$

$$\sum_{K}^{\infty} F_{\sigma}^{(-1)}\left(\mathrm{e}^{-\mathrm{e}^{k}}\right) < \infty \qquad \textbf{(I)}$$

where $F_{\sigma}^{(-1)}$ is the generalised inverse of the distribution function of σ.

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Corollary

*If a distribution σ satisfies **(I)** then it explodes for all X satisfying **(P2)** (including all power law degrees with $\tau \in (2, 3)$).*

Explosive σ-s

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- Boundary case: $F_\sigma(t) = \exp\{-\exp\{\frac{1}{t^\beta}\}\}$. Explosive for $\beta < 1$, conservative for $\beta \geq 1$.
- F_σ does not have to be continuous to satisfy **(I)**: e.g. put point-mass $c_1^k/(1-c)$ to points at c_2^k, for $c_1, c_2 < 1$.

Application to epidemics in random graphs

Explosive weights on the configuration model

Theorem (Baroni, van der Hofstad, K)

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If the branching process $\mathrm{BP}(D^\star, \sigma)$ *is explosive,*

$$\lim_{n\to\infty} d_\sigma(u, v) = V^{(u)} + V^{(v)}$$

in the distributional sense. $V^{(u)}, V^{(v)}$ *explosion times of two copies of* $\mathrm{BP}(D^\star, \sigma)$*, with* $D^\star$*=size biased degree,* u, v *two uniformly chosen vertices.*

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Otherwise $d_\sigma(u,v) \to \infty$.

This was first shown for exponential edge weights by Bhamidi, Hofstad & Hooghiemstra.

Corrollary: Epidemic curve in the explosive case

Recall $I_u(t) = \frac{1}{|V|} \sum_{v \in V} \mathbf{1}_{d_\sigma(u,v) \le t}$ is the epidemic curve.

Convergence of the epidemic curve

Consider an epidemic started at a single, uniformly chosen vertex $u \in V$. Then

$$I_u(t) \xrightarrow{\mathbb{P}} f(t - V^{(u)}) = \mathbb{P}(V^{(u)} + V^{(v)} \le t \mid V^{(u)})$$

A deterministic curve with a random but constant shift $V^{(u)}$.

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If $\mathrm{BP}(D^\star, \sigma)$ *is conservative, then for all* $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{d_\sigma(u,w)}{2\sum_{k=1}^{\log\log n/|\log(\tau-2)|}} \in (1-\varepsilon, 1+\varepsilon)\right) \to 1.$$

Gives back the main term graph distances by setting $\sigma \equiv 1$.

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Not enough...

For an epidemic curve one would need *distributional convergence* of the *fluctuations* of $d_\sigma(u, v)$ around $2\sum_{k=1}^{\log\log n/|\log(\tau-2)|} F_\sigma^{(-1)}\Big(\exp\big(-(\frac{1}{\tau-2})^k\big)\Big)$ which is not known/not possible to show with our current methods.

When $\tau > 3$

$\tau \in (2, 3)$

Dichotomy: bounded average distance for explosive weight distributions, non-bounded average distance for conservative weight distributions

Theorem (Bhamidi, Hofstad, Hooghiemstra)

Universally, for all σ that have a density,

$$d_\sigma(u, v) = \frac{1}{\lambda} \log n + \textit{tight},$$

where λ is the Malthusian parameter (exponential growth rate) of the embedded BP.

Generalisation to spatial models

Two scale free spatial models

- Geometric Random Inhomogeneous Random Graphs
  vertices $= n$ uniform points in $[0, n^{1/d}]^d$
- Scale free percolation: vertex set is $\mathbf{Z}^d$.

In both models, each vertex v gets a weight W_v and two vertices are connected

$$\mathbb{P}(u \leftrightarrow v \mid W_u, W_v) = \Theta\left(\min\{1, \frac{W_u W_v}{\|u - v\|^\alpha}\}\right)$$

Theorems [K&Lodewijks, v/d Hofstad&K]

Both the explosive and conservative results carry through for these models.

Epidemics with contagious intervals

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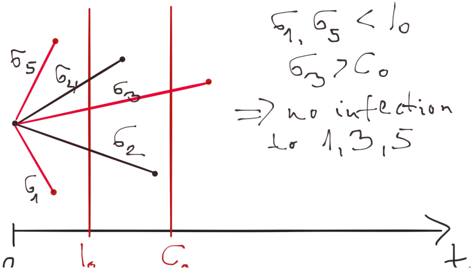

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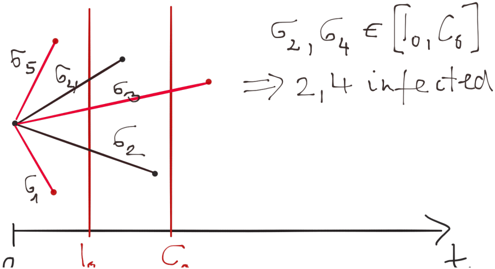

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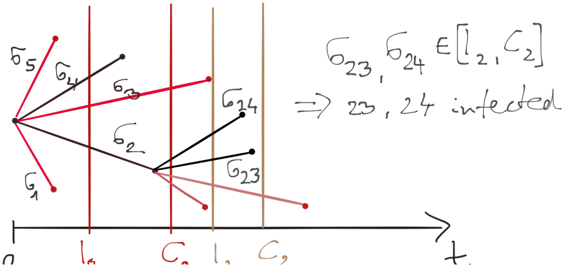

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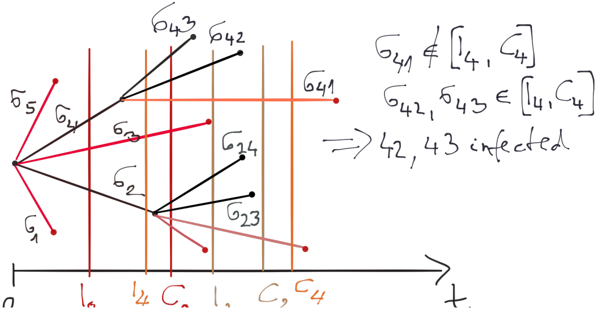

Explosion in this case?

Epidemics with contagious intervals

When is a triplet $(X, \sigma, [I, C])$ explosive?
Can the explosion of $\mathrm{BP}(X, \sigma)$ be stopped by adding $[I, C]$ to it?

Getting rid of end of interval

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Theorem (K)

Suppose X satisfies **(P2)** *and $[I, C]$ satisfies $\exists t_0, \delta > 0$*

$$\mathbb{P}(C > t | I = i) \geq \delta \qquad \forall i < t < t_0. \tag{$\star$}$$

Then $(X, \sigma, [I, C])$ explosive $\Leftrightarrow$ $(X, \sigma, [I, \infty])$ explosive.

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Natural condition

Condition $(\star)$ on $[I, C]$ is satisfied if

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- It means that the support of I, L is not concentrated on a 'slented wedge' separating the support from the L axes.

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- Other way round trickier...

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For the infection started from u*, the time it takes to infect* v*:*

$$d_{epi}(u,v) \xrightarrow{d} V^{(u)} + V_{bw}^{(v)}$$

if and only if $(D^\star, \sigma, [I,C])$ *is explosive,*

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$V^{(u)}$ *explosion time,* $V_{bw}^{(w)}$ *explosion time of the backward epidemics.*

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The epidemic curve of u:

$$f_{epi}(t) = \frac{1}{n} \sum_{w=1}^{n} \mathbb{1}_{\{w \text{ infected before } t\}} \xrightarrow{d} \mathbb{P}(V_{bw}^{(w)} \le t - V^{(u)} | V^{(u)})$$

a deterministic curve with a random shift, conditioned that $\mathrm{Epi}^{(u)}$ *survives.*
$V^{(u)}$ *explosion time,* $V_{bw}^{(w)}$ *explosion time of the backward epidemic.*

Picture-proof of explosion

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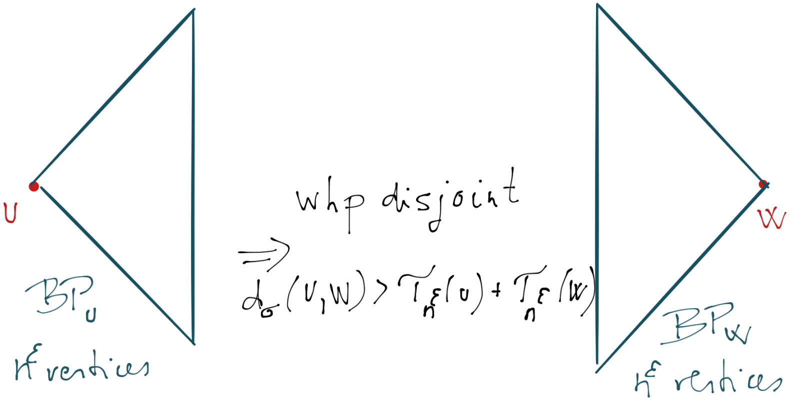

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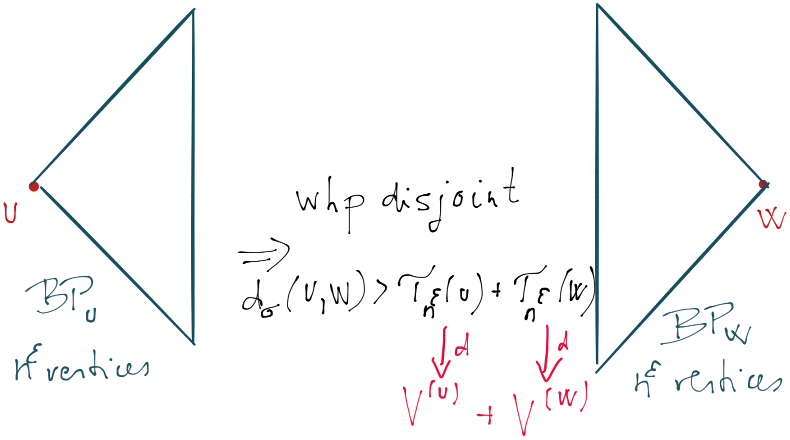

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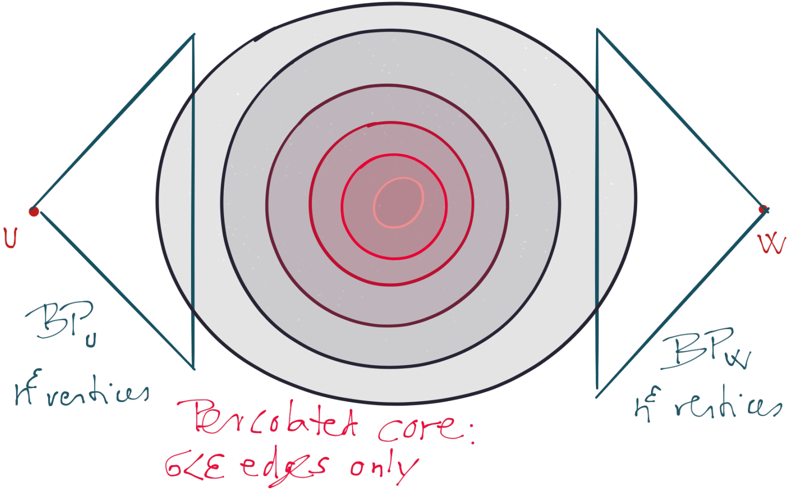

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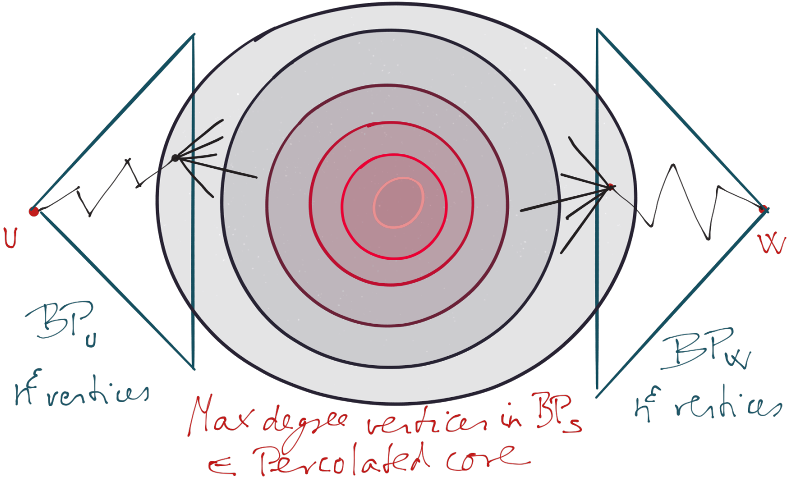

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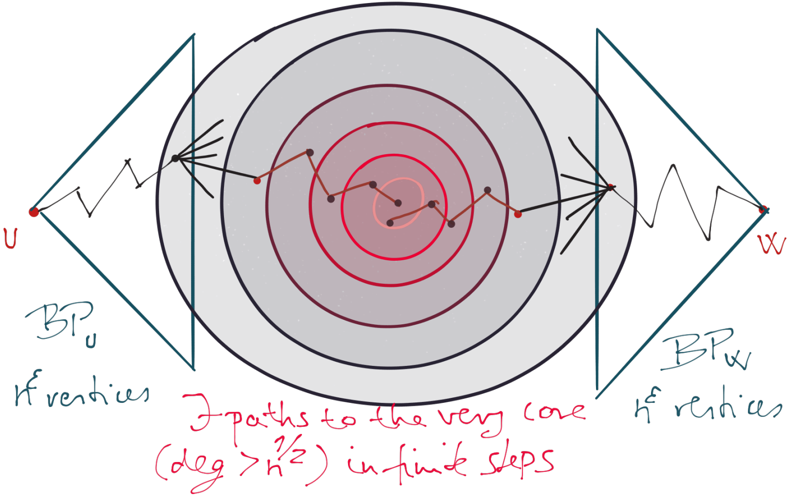

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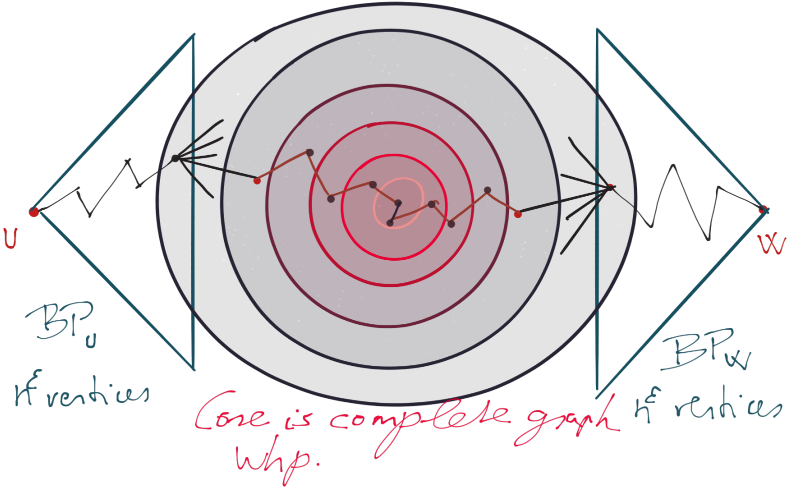

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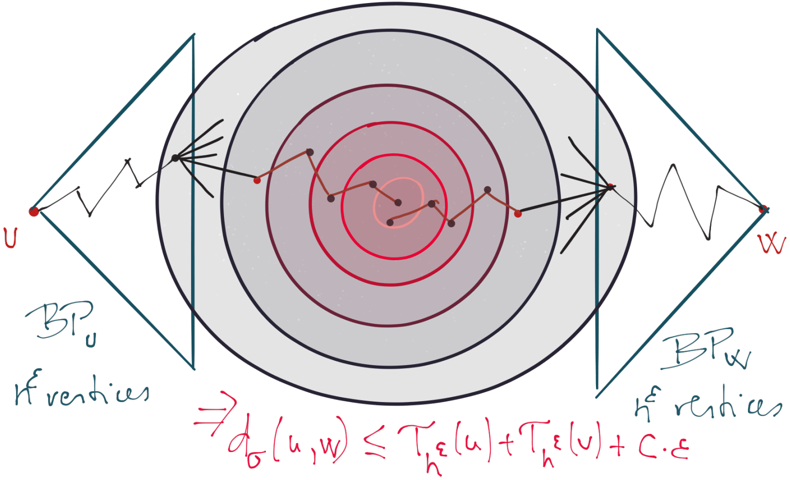

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Step 4: Show that in the percolated subgraph, there is a nested layering starting with degree K_n with the property that a vertex in layer i is connected to at least one vertex in layer $i+1$, and the degrees $\deg v$ in layer i is $\approx K_n^{1/(\tau-2)^i}$.

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Step 5: Show that $\widetilde{u}$, $\widetilde{v}$ falls into layer 1. Thus

$$d_L(u,v) \geq d_L(u,\widetilde{u}) + d_L(v,\widetilde{v}) + 2 \sum_{i=1}^{\#\text{ layers}} F_\sigma^{-1}\left(tr(K_n^{1/(\tau-2)^i}, K_n^{1/(\tau-2)^{i+1}}) \right).$$

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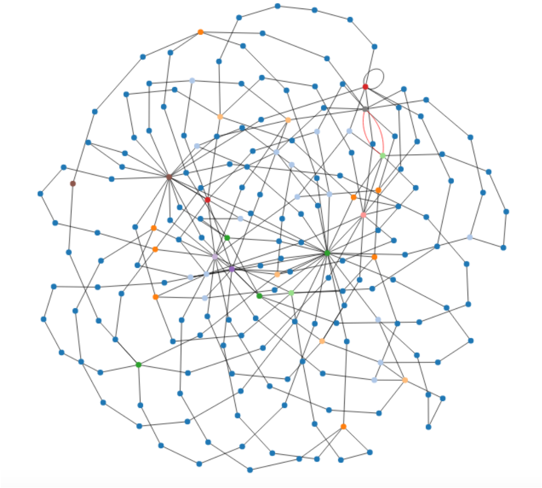
Step 6: Show that the first two terms can be chosen to be negligible and the second term is

$$(1+\varepsilon)2 \sum_{i=1}^{\log\log n/|\log(\tau-2)|} F_\sigma^{-1}\left(\exp\{-(\tau-2)^{-i}\}\right).$$

Thank you for the attention!



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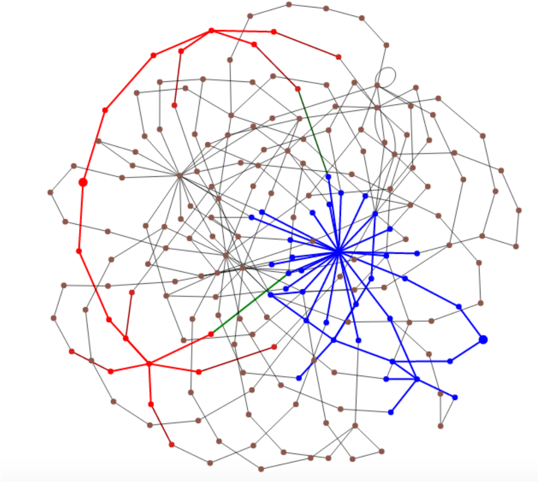

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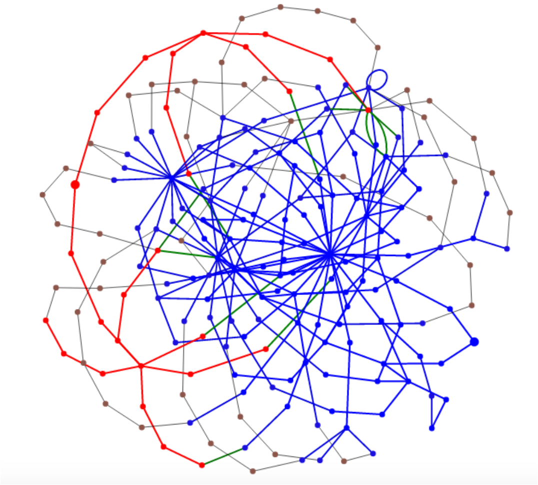

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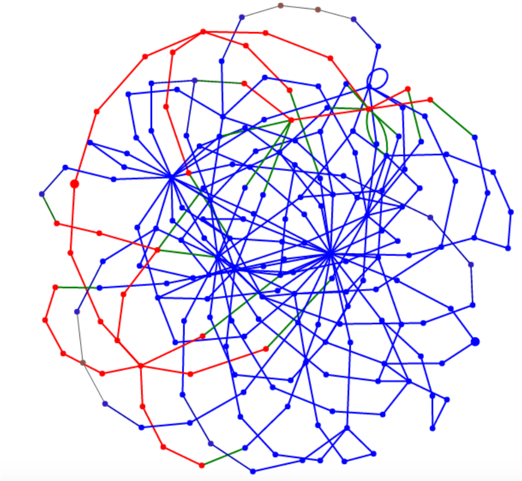

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X_n is a tight sequence of random variables if the tail probabilities decay uniformly in n:
$\forall \varepsilon > 0, \exists K_\varepsilon$ such that $\forall n : \mathbb{P}(|X_n| \geq K_\varepsilon) \leq \varepsilon$.

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- σ non-explosive: $d_\sigma(u,v) \to \infty$ by the previous theorem
- As $n \to \infty$,

$$d_{1+\sigma}(u,v) - \frac{2 \log \log n}{|\log(\tau - 2)|} = O(1) + d_\sigma(u,v) \to \infty$$

so the sequence cannot be tight.