Using the effective graph resistance as a robustness metric

Massimo Achterberg 7 July 2022
Outline for today

• Robustness
• Effective graph resistance
• Greedy algorithms
• Submodularity
Network = service + graph

What is robustness?

“The ability of a network to withstand failures and/or attacks”

Key question: How to measure it?
Effective graph resistance

All resistors $1\Omega$

Effective resistance between two nodes

Considers all paths between the nodes

Eff. graph resistance:
Sum over all effective resistances

Other names:
resistance distance, Kirchhoff index

Given a graph $G$, where to add $k$ links such that the eff. graph resistance is optimal?

- Method 1: Try all possible combinations

Number of options: $\binom{L^c}{k} \approx \mathcal{O}\left((L^c)^k\right)$

where $L^c$ is the number of non-existent links

Infeasible!
Given a graph $G$, where to add $k$ links such that the eff. graph resistance is optimal?

• Method 2: Greedy algorithm
  For every link $k$, check all $L^c$ options and choose the best one

**Number of options**: $O(k \ L^c)$

How accurate is it?
Definition Submodularity

- Consider two sets \( S \subseteq R \).
- Consider a function \( f \).
- The function \( f \) is submodular if
- Adding an element \( v \) to \( S \) has a larger impact than adding \( v \) to \( R \).

\[
f(S \cup \{v\}) - f(S) \geq f(R \cup \{v\}) - f(R)
\]

Why submodularity?

• If $f$ is sub-modular and monotone, then

\[ f_{\text{greedy}} \geq \left(1 - \frac{1}{e}\right) \cdot f_{\text{optimal}} \]

\[ \approx 0.63 \cdot f_{\text{optimal}} \]

Is the eff. graph resistance submodular? No, see [4]

Weak Submodularity

• Consider two sets $S \subseteq R$.
• Consider a function $f$.
• The function $f$ is weakly $\gamma$-submodular iff
• Adding an element $v$ to $S$ has a larger impact than adding $v$ to $R$.

$$f(S \cup \{v\}) - f(S) \geq \gamma [f(R \cup \{v\}) - f(R)], \quad \gamma \in (0,1]$$

Why weak submodularity?

- If $f$ is weakly $\gamma$-submodular and monotone, then

$$f_{\text{greedy}} \geq (1 - e^{-\gamma}) \cdot f_{\text{optimal}}$$

Is the eff. graph resistance weakly submodular?
Example

Clique $N$ nodes

\[ f(S \cup \{v\}) - f(S) = \frac{4}{N - 2} \]

\[ f(R \cup \{v\}) - f(R) \to 1 \]

\[ \gamma \to \frac{4}{N - 2} \]
Conclusion

• Effective graph resistance is NOT weakly $\gamma$-submodular

![Graph showing $f_{\text{greedy}}$, $f_{\text{optimal}}$, and $(1 - \frac{1}{e})$.]
Open questions – Eff. graph resistance

- Are there other algorithms with guaranteed performance?
- How well does the greedy algorithm perform in practice?
  - Seems to perform rather well
- Other types of extensions of submodularity
Wrap-up

• Robustness metric: Effective graph resistance
Wrap-up

- Submodularity
- Weak submodularity

- Eff. graph resistance is NOT submodular
Using the effective graph resistance as a robustness metric

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Money ↔ Eff. Graph resistance

- The function $f$ is **submodular** if
- Adding an element $v$ to $S$ has a larger impact than adding $v$ to $R$.

$$f(S \cup \{v\}) - f(S) \geq f(R \cup \{v\}) - f(R)$$

<table>
<thead>
<tr>
<th>Item</th>
<th>Money</th>
<th>Robustness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function $f$</td>
<td>Measure of happiness</td>
<td>Eff. graph resistance</td>
</tr>
<tr>
<td>Set $V$</td>
<td>All money in the world</td>
<td>All non-existing links in the graph</td>
</tr>
<tr>
<td>Set $S$</td>
<td>A small amount of money</td>
<td>Subset of $V</td>
</tr>
<tr>
<td>Set $R$</td>
<td>A large amount of money</td>
<td>Larger subset of $V$</td>
</tr>
<tr>
<td>Element $v$</td>
<td>Receive €1000</td>
<td>Add a link</td>
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